fraction solutions has found nontrivial and extremely elegant applications in analyzing the dynamics of an anisotropic Heisenberg model on a lattice of infinite dimensions (the so-called spin van der Waals model),[11] a quasi-two-dimensional electron gas,[12] spin-dynamics problems,[10] etc. In a very well defined sense, *each of these problems is best analyzed in terms of continued fractions*. Finally, the method of recurrence relations in many-body dynamics reveals a subtle intellectual achievement, exciting realistic hopes that, in the words of M. Howard Lee, "there are simple solutions lurking amidst impossibly complicated problems."

[1]Barry R. Holstein, "Answer to Question #4," Am. J. Phys. **65** (12), 1133–1135 (1997); F. J. Lopez-Lopez, "Answer to Question #4," *ibid.* **63** (7), 583–584 (1995); for a summary of earlier Answers to Question #4, see Ref. 2.

[2]Dwight E. Neuenschwander, "Responses to Question #4," Am. J. Phys. **63** (7), 586 (1995).

[3]S. Lovesey, *Dynamic Correlations* (Benjamin, Reading, MA, 1980).

[4]See Refs. 2 and 3 in the "Answer to Question #4" by Thomas J. Pickett, Am. J. Phys. **65** (6), 462–463 (1997).

[5]H. Mori, "A continued-fraction representation of the time-correlation functions," Progr. Theor. Phys. **34**, 399–416 (1965).

[6]R. Zwanzig, "Time-correlation functions and transport coefficients in statistical mechanics," Annu. Rev. Phys. Chem. **16**, 67–102 (1965).

[7]C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer, Berlin, 1997), 2nd ed., 4th printing, Chap. 1, pp. 1–20.

[8]M. H. Lee, "Orthogonalization process by recurrence relations," Phys. Rev. Lett. **49**, 1072–1075 (1982); "Solutions of the generalized Langevin equation by a method of recurrence relations," Phys. Rev. B **26**, 2547–2551 (1982).

[9]R. Kubo, "Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems," J. Phys. Soc. Jpn. **12**, 570–586 (1957); see also R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II* (Springer, Berlin, 1991), 2nd ed., Chap. 4, pp. 146–202.

[10]V. S. Viswanath and G. Müller, *The Recursion Method: Applications to Many-Body Dynamics* (Springer, Berlin, 1994).

[11]M. H. Lee, I. M. Kim, and R. Dekeyser, "Time-dependent behavior of the spin-$\frac{1}{2}$ anisotropic Heisenberg model in infinite lattice dimensions," Phys. Rev. Lett. **52**, 1579–1582 (1984).

[12]M. H. Lee and J. Hong, "Dynamic form factor, polarizability, and intrinsic conductivity of a two-dimensional dense electron gas at zero temperature," Phys. Rev. Lett. **48**, 634–637 (1982).

Yonko T. Millev
Max Planck Institute of Mathematics in the Sciences
Inselstr. 22-26
D-04103 Leipzig
Germany

Answer to Question #43. The Fokker–Planck equation

Neuenschwander's question [Am. J. Phys. **64** (6), 681 (1996)] seeks a convincing application of the Fokker–Planck equation for presentation to undergraduates. Its use to describe the motion of positrons in a medium in the presence of an impressed electric field is suitable for students at this level; where to introduce it in the syllabus may be a bit problematic. I use it in a course on computational physics.

Three distinct mechanisms can be identified which affect the local density of particles in a time Δt: *diffusion*, $\Delta\rho = \mathrm{div}(D\ \mathrm{grad}\ \rho)\Delta t$, where the diffusion current is $\mathbf{j} = -D\ \mathrm{grad}\ \rho$, and D is the local diffusion constant; *convection*, which superimposes a net directed motion on this drift with $\Delta\rho = -\mathrm{div}(\mathbf{v}\rho)\Delta t$, where $\mathbf{v}$ is the local drift velocity of the particles; and *annihilation* of electrons in the medium, which serves to remove particles from the population with $\Delta\rho = -\mu\rho\Delta t$, where μ is the probability of annihilation per unit time or the reciprocal of the local lifetime against annihilation. By combining these three contributions we find for the local time rate of change of ρ, in one dimension,

$$\frac{\partial\rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x)\,\frac{\partial\rho(x,t)}{\partial x}\right) - \frac{\partial}{\partial x}\left(v(x)\rho(x,t)\right) - \mu(x)\rho(x,t) + S(x,t).$$

The great generality of the equation, which can accommodate an inhomogeneous medium and/or field, for which D, $\mathbf{v}$, and μ depend on position, can be seen, as well as the possibility of a term corresponding to the spontaneous creation of particles at x at a rate $S(x,t)$ per unit volume per unit time—such as could occur with a positron population in the presence of energetic γ rays.

A similar analysis could be applied to cosmic rays in supernovae envelopes, or in interstellar space, but may be somewhat remote from the undergraduate syllabus.

P. Kevin MacKeown
Department of Physics
The University of Hong Kong
Hong Kong

Answer to Question #52. Group velocity and energy propagation

Awati and Howes [Am. J. Phys. **64** (11), 1353 (1996)] ask for a general proof of the relationship between group velocity and the velocity of energy propagation.

It seems that the concept of group velocity was first enunciated by Hamilton in 1839 in published abstracts of works that never appeared. Hamilton considered a wave $\cos(kx - \omega t)$ defined only for negative x at $t = 0$ and incident on a dispersive medium that occupies the region $x > 0$. He concluded "that the velocity with which such vibration spreads into those portions of the vibratory medium which were previously undisturbed is in general different for the velocity of passage of a given phase from one particle to another within that portion of the medium which is already fully agitated; since we have velocity of transmission of phase $= \omega/k$ but velocity of propagation of vibrating motion $= d\omega/dk$." However, these results were largely ignored.

The group-velocity concept became widely known after being (re)introduced by Stokes in 1876 in a hydrodynamic context, and the greater generality of the concept emphasized by Rayleigh in 1877 in Sec. 191 of his book *The Theory of Sound*. The early history of the group-velocity concept is well summarized in the book *The Propagation of Disturbances in Dispersive Media* by T. H. Havelock (Cambridge U.P., Cambridge, 1914).

I give two answers to the question of how one knows that wave energy propagates with the group velocity, both of which are "standard" and sufficient to my taste. The discussion will be restricted to wave motion along the x axis for brevity.

(1) The total energy E associated with a wave of amplitude $f(x,t)$ at a time t can in general be written

$$E(t)=\int (Af^2+B\dot{f}^2)dx,$$

where $\dot{f}=\partial f/\partial t$ and either of A or B might be zero depending on the physical system. Typically, the term Af^2 is associated with energy stored in the wave medium due to the strain of the wave while $B\dot{f}^2$ is the kinetic energy of the medium due to the wave motion. For example, $B=0$ for an electromagnetic wave while $A=0$ for the granular systems recently studied by Swinney *et al.* [Nature **382**, 793 (1996)].

Question #52 arises when one wishes to interpret the quantity $Af^2+B\dot{f}^2$ as an energy density. As the wave changes with time it is possible that the energy moves in space. If the wave amplitude has the form of a travelling wave, $f(x-vt)$, then both f^2 and $\dot{f}^2$ are functions of a single variable, $x-vt$, and the energy can be said to be propagating with velocity v.

The concept of group velocity arises when a waveform is Fourier analyzed into a set of harmonic waves

$$f(x,t)=\int F(k)e^{i(kx-\omega t)}dk,$$

characterized by wave number k and frequency $\omega(k)$ where the latter relation can be nontrivial due to dispersion in the wave medium. The harmonic wave of frequency ω has phase velocity $v_p=\omega/k$ which is not necessarily equal to the velocity v of the localized waveform. (In this discussion only the real part of f has physical significance.)

The spectral function $F(k)$ can be determined by the Fourier inverse relation for the wave at a fixed time, say $t=0$:

$$F(k)=\frac{1}{2\pi}\int f(x,0)e^{-ikx}dx.$$

However, we don't need to use this result in the present case.

The usual argument asks us to restrict our attention to waveforms whose spectral function $F(k)$ is narrow enough that the dispersion relation can be approximated as

$$\omega=\omega(k_0)+\frac{d\omega(k_0)}{dk}(k-k_0),$$

the leading terms in a Taylor expansion about some central wave number k_0. (The sign of k_0 determines whether the pulse moves in the $+x$ or $-x$ direction.) Certainly this approximation breaks down for very short pulses in highly dispersive media. In this approximation we have

$$f(x,t)=e^{i[k_0(d\omega(k_0)/dk_0-\omega_0]t}\int F(k)e^{ik[x-(d\omega(k_0)/dk)t]}dk$$

$$=e^{i[k_0(d\omega(k_0)/dk_0-\omega_0]t}f(x-(d\omega(k_0)/dk)t,0).$$

That is, to within a phase factor of unit modulus, the waveform $f(x,t)$ is a function of a single variable, $x-(d\omega(k_0)/dk)t$, and so can be said to propagate with velocity

$$v_{\text{group}}=\frac{d\omega(k_0)}{dk},$$

the group velocity. As argued above, the energy then propagates with this velocity as well.

If the waveform is highly localized in space it will have a broad spectral content and the linear approximation to the dispersion relation may not suffice. If so, the waveform will change shape (disperse) as it propagates and the group velocity is not well defined.

This well-known argument appears to be due to Lord Kelvin [Proc. R. Soc. London **XLII**, 80 (1887)], and is reproduced in much the above form in Sec. 3 of the book by Havelock.

(2) In 1877 both Reynolds and Rayleigh published articles relating energy flow to group velocity. Reynolds's discussion is based on water waves and can be found in Sec. 273 of the book *Hydrodynamics* by H. Lamb, as well as in Sec. 26 of *Mechanics of Deformable Bodies* by A. Sommerfeld (1946).

Rayleigh's argument has been reprinted in the Appendix to Vol. 1 of his book *The Theory of Sound* and is based on the general observation that dispersion in a physical system is always accompanied by absorption. While the latter can often be ignored as a first approximation, it should not be left out of discussions of energy flow.

Here I repeat Rayleigh's argument, which seems to be less well known than Kelvin's.

A pure harmonic wave, $e^{i(kx-\omega t)}$, is incident on a dispersive medium that occupies the half space $x>0$. Because of absorption in the medium, this wave dies out over some characteristic distance d. To find that distance, we suppose that in the absence of absorption the harmonic solutions obey a known dispersion relation, $k=k(\omega)$. Then the equation of motion including absorption, taken to be velocity dependent, differs from that without absorption only by replacing the second time derivative $\partial^2/\partial t^2$ with the form

$$\frac{\partial^2}{\partial t^2}+\gamma\frac{\partial}{\partial t},$$

where γ, whose dimensions are 1/time, characterizes the absorption process. The new dispersion relation that results on inserting our trial harmonic solution into the wave equation differs only in the term ω^2 being replaced by $\omega^2+i\gamma\omega$. For weak absorption this is equivalent to replacing ω by $\omega+i\gamma/2$. The corresponding wave number is therefore $k(\omega+i\gamma/2)\approx k(\omega)+i(\gamma/2)(dk/d\omega)$, again ignoring terms of order γ^2. The wave solution in the presence of absorption is therefore approximately

$$e^{-(\gamma/2)(dk/d\omega)x}e^{i(kx-\omega t)}.$$

Thus the characteristic attenuation length is

$$d=\frac{2}{\gamma}\frac{d\omega}{dk}.$$

For a steady wave, energy is being transported into the medium at the same rate at which it is being absorbed, when averages are taken over a whole cycle of the wave. The power P absorbed by a mass m in the medium is $P=Fv$, where v is the velocity of the mass and $F=\gamma mv$ the dissipative force. Thus $P=\gamma mv^2$, and summing over all masses in some region, $P=2\gamma T$, where T is the kinetic energy. Taking the time average, $\langle P\rangle=2\gamma\langle T\rangle=\gamma E$, where E is the total energy. In writing $E=2\langle T\rangle$ we suppose that the wave motion is a small departure from equilibrium so the restoring forces can be described by a quadratic potential, for which $E=2\langle T\rangle$ according to the virial theorem of classical mechanics.

The time-averaged power absorbed for $x>x_0$ in the medium is then

$$\langle P\rangle(x_0)=C\gamma\int_{x_0}^{\infty}(e^{-x/d})^2dx=C\frac{d\omega}{dk}e^{-2x_0/d},$$

where C is a constant that relates the square of the nonoscillatory part of the wave amplitude to its time-averaged energy density. The virial theorem assures us that constant C exists.

The (time-averaged) rate of energy flow across the plane $x=x_0$ is the (time-averaged) energy density there times the desired velocity of energy flow, v_E. This product is just $Cv_Ee^{-2x_0/d}$, noting the meaning of constant C.

Hence the velocity of energy flow is

$$v_E=\frac{d\omega}{dk}=v_{\text{group}}.$$

An objection to this argument would be that it doesn't apply if the absorption is too strong (and not if it is too weak as implied in the statement of Question #52). It may be that the heroic efforts of Sommerfeld and Brillouin [Ann. Phys. (1914); see also, for instance, Max Born and Emil Wolf, *Principles of Optics* and Léon Brillouin, *Wave Propagation and Group Velocity*] to clarify signal propagation in the case of highly absorptive anomalous dispersion in optical media (where v_{group} exceeds the speed of light) have left the impression that the more ordinary case is similarly intricate.

Kirk T. McDonald
Joseph Henry Laboratories
Department of Physics
Princeton University
P.O. Box 708
Princeton, New Jersey 08544

Answer to Question #52. Group velocity and energy propagation

K. M. Awati and T. Howes have asked for a general proof that the energy propagation in a dispersive medium is at the group, and not the phase, velocity. This is an interesting issue because it requires a broader understanding of wave energy than simply the electromagnetic component. It was originally pointed out by Max von Laue in 1905[1] that in a dispersive medium the kinetic energy of the oscillators as well as the field energy must be considered. The discussion was subsequently pursued by numerous authors leading to a general formulation of the total energy of a slightly damped wave in a dispersive medium, particularly a plasma.[2–7] Most of these treatments are based only on Maxwell's equations with a conductivity introduced to account for the particles. Without an explicit identification of the particle energy considered as coherent with the wave, however, the treatment is incomplete. Allis *et al.* do identify this energy for a cold plasma.[3] I carried out a complete description for a fully ionized hot plasma.[8] At that time, considerations of this problem were also being pursued by the group at Göteborg.[9]

If one does not ask about the actual identity of the coherent particle energy, to show that the total wave energy must propagate at the group velocity is fairly straightforward. A dispersive medium is one in which there is a time damping and spatial dispersion of an electromagnetic wave. From Maxwell's equations it is clear that such a medium must contain charges and be capable of producing currents. The presence of these charges is accounted for by introducing a conductivity, σ. The wave disturbances in such a medium are those with time and spatial dependence of the form

$$\text{Re}\{f(\vec{k},\omega)\exp[i\omega t-i\vec{k}\cdot\vec{r}]\},\tag{1}$$

where $f(k,\omega)$ is any of the field or particle quantities of interest. In an isotropic medium the (complex) conductivity, $\sigma(\vec{k},\omega)$, is a scalar. The necessary condition for the existence of such disturbances is

$$\det\{\overleftrightarrow{D}(\vec{k},\omega)\}=0,\tag{2}$$

with

$$\overleftrightarrow{D}(\vec{k},\omega)\equiv\vec{k}\vec{k}+(\omega^2\mu_0\epsilon_0-k^2)\overleftrightarrow{1}-i\omega\mu_0\sigma\overleftrightarrow{1}.\tag{3}$$

To be considered as propagating in the medium, these disturbances, Eq. (1), must exist over a large number of wavelengths. That is, the imaginary parts of k and ω must be small. If one performs an expansion in small imaginary parts, the time average of the field energy equation produces

$$\omega_iU+(-\vec{k}_i)[U\ \text{grad}_{\vec{k}}(\omega)]=-\sigma_rE^2,\tag{4}$$

where

$$U=U_{\text{em}}-\frac{\partial\sigma_i}{\partial\omega}E^2\tag{5}$$

is the total energy of the wave disturbance, and σ_r and σ_i are the real and imaginary parts of the conductivity. In (5) U_{em} is the electromagnetic energy density in the wave, and the second term in U is that identified as the coherent particle kinetic energy. The term multiplied by ω_i in (4) is the time derivative of the total energy and that involving the scalar product with $\vec{k}_i$ is the divergence of the flux of total energy. The right-hand side of (4) then represents the loss rate of this total energy to the background dispersive medium. This is the degradation of the coherent particle energy component of U into (noncoherent) thermal energy.

An important step in obtaining (4) is to consider the form of the electrical conductivity for the slightly damped wave. Expanding around the undamped condition, the conductivity is

$$\sigma(\vec{k},\omega)\approx(i\sigma_i)+i\omega_i\frac{\partial}{\partial\omega}(i\sigma_i)+i\vec{k}_i\cdot\text{grad}_{\vec{k}}(i\sigma_i)+\sigma_r.\tag{6}$$

It is easy to show that the conductivity must be purely imaginary at the propagation condition.

The form of the flux term in (4) provides an answer to the question asked. In a dispersive medium, a general (total) Poynting vector must be considered. This is

$$\vec{S}=U\ \text{grad}_{\vec{k}}(\omega),\tag{7}$$

where, of course, $\text{grad}_{\vec{k}}(\omega)$ is the group velocity of the wave.

As satisfying as this is, the door has only been opened a crack; the real problem is to identify the coherent particle energy. In Ref. 8 this coherent energy is identified as the standard hydrodynamic energy, quadratic in the current, plus a part of the thermal energy related to particle density variations. An equally interesting question is the origin of the damping represented by the real part of the complex conductivity. The form of this term depends on the thermodynamic state of the background medium. Landau's classic treatment of the damping and excitation of Coulomb waves in a Vlasov plasma is an example.[10] Perhaps a more interesting question