$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\Phi = 0. \quad (2)$$

The present experimental upper bounds on such masses are quite small, e.g., any hypothetical mass of the photon is constrained to be less than[8] 3×10^{-33} MeV/c^2 (the c here would still be c_{SR} if the mass were nonzero).

In summary, the propagation speeds of electromagnetic and gravitational waves are equal within the presently accepted theories, and equal to the limiting speed of special relativity, because Maxwell's electrodynamics and Einstein's general relativity are Lorentz invariant theories describing the propagation of massless waves.

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Answer to Question #49. Why c for gravitational waves?

The question[1] on why both gravity waves and electromagnetic waves travel at c might be turned around. For example, consider someone who studies gravitational waves "first," and finds (in vacuum and in the correct gauge) the wave equation

$$\nabla^2 h_{\mu\nu} - \frac{1}{c^2}\frac{\partial^2 h_{\mu\nu}}{\partial t^2} = 0 \quad (1)$$

for the gravitational perturbation. This is the wave equation for gravitational waves.[2] Plane wave solutions are of the form

$$h_{\mu\nu} = \epsilon_{\mu\nu} e^{i(kx-\omega t)} \quad (2)$$

and one concludes, using (2) in (1), that gravitational waves travel at the speed $\omega/k = c$. In fact, there are many fields that are postulated to exist, and the wave equation often has the form of (1). For example, the ubiquitous (from a theoretical standpoint) massless scalar field ϕ obeys

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \quad (3)$$

and so it too has wave solutions that travel at the speed of light. There are other less familiar fields that also travel at the speed of light.[2] And now comes electromagnetism, the potential of which obeys *in Gaussian units*

$$\nabla^2 A_\mu - \frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2} = 0 \quad (4)$$

and obviously has wave solutions that travel at the speed of light. This fact may be somewhat obscured in SI units, where c^2 is replaced by $1/\mu_0\epsilon_0$, but the choice of units is not the essence.

The underlying physical similarity between all of these fields is that they are of so-called infinite range. This means that in the particle-like solutions the force is of the $1/r^2$ form (for the higher multipole solutions the force is, of course, of a higher inverse order).

From another point of view, in the quantized version, the exchange particles are massless. This is really the underlying physical similarity that ensures that these waves propagate at the universal speed c.

An equally important view, and one that is equivalent to the masslessness of the quanta, is that of gauge invariance. Gauge invariance in electromagnetism requires that the photon is massless, which in turn is equivalent to the statement that the velocity is c. In gravity, the gauge invariance results from the general covariance of the theory and gives rise to an invariance in the quantity $h_{\mu\nu}$.

Thus as a general underlying physical principle, fields of infinite range, or those with massless quanta, have wave solutions that travel at c. From this view, there is no surprise that electromagnetic and gravitational waves both travel at c.

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Answer to Question #49. Why c for gravitational waves?

Keeports's question ["Question #49. Why c for gravitational waves?," Am. J. Phys. **64**(9), 1097 (1996)] prompts me to make some observations on little-known work by Heaviside.

The speed of light emerges as the square root of the ratio of Newton's gravitational constant G to the little-known constant H that arises in the gravitational equivalent of magnetostatic effects.

J. C. Maxwell ended his great 1864 paper "A Dynamical Theory of the Electromagnetic Field" with remarks on Newtonian gravity as a vector field theory. He was dissatisfied with his results because the potential energy of a static gravitational configuration is always negative but he felt this should be re-expressible as an integral over field-energy density which, being the square of the gravitational field, is positive.

Nonetheless, in 1893 O. Heaviside pursued the topic further and speculated on the role of a gravitomagnetic field as outlined below. See pp. 455–465 of his book, *Electromagnetic Theory* ("The Electrician" Printing and Publishing Co., London, 1894).

The static gravitational force on a mass m due to a mass density ρ can be written in close analogy to electrostatics as

$$\mathbf{F}=m\mathbf{g},$$

where the static gravitational field $\mathbf{g}$ obeys

$$\nabla\cdot\mathbf{g}=-4\pi G\rho,\quad \nabla\times\mathbf{g}=0,$$

where G is Newton's gravitational constant.

Heaviside argued that in analogy to the remaining Maxwell equations we should expect a gravitomagnetic field $\mathbf{h}$ that obeys

$$\nabla\cdot\mathbf{h}=0,\quad \nabla\times\mathbf{h}=-4\pi H\rho\mathbf{v},$$

where $\mathbf{v}$ is the velocity of the mass that causes field $\mathbf{h}$ and H is a constant (that should be called Heaviside's constant) which characterizes the strength of the gravitomagnetic interaction. The force on mass m is now

$$\mathbf{F}=m\mathbf{g}+m\mathbf{v}\times\mathbf{h},$$

where $\mathbf{v}$ is the velocity of mass m in this expression, in analogy to the Lorentz force law (actually first written down by Heaviside in 1889).

If the constant H (and also G) were larger, there might have been an experimental measurement of its value. Then, it would have been noted that

$$\sqrt{\frac{G}{H}}=c,$$

the speed of light, providing a positive answer to Question #49.

In the preceding I have chosen different units for the field $\mathbf{h}$ than those recommended by Heaviside to emphasize how, if observed, $\mathbf{h}$ might have been interpreted initially as quite distinct from the field $\mathbf{g}$ and having nothing to do with the speed of light. For a discussion on how special relativity requires a field $\mathbf{h}$ given the field $\mathbf{g}$, see "On Relativistic Gravitation" by D. Bedford and P. Krumm, Am. J. Phys. **53**, 889–890 (1985).

Lacking evidence of gravitomagnetostatic effects, Heaviside proceed by analogy to the full Maxwell equations and inferred that the time-dependent equations of the gravitational field would be

$$\nabla\cdot\mathbf{g}=-4\pi G\rho,\quad \nabla\times\mathbf{g}=-\frac{\partial\mathbf{h}}{\partial t},$$

and

$$\nabla\cdot\mathbf{h}=0,\quad \nabla\times\mathbf{h}=-4\pi H\rho\mathbf{v}+\frac{H}{G}\frac{\partial\mathbf{g}}{\partial t}.$$

Heaviside then noted that there should be gravitational waves which propagate with velocity

$$v=\sqrt{\frac{G}{H}}.$$

As an example, Heaviside considered that the propagation velocity might well be the speed of light. From this assumption, the constant H has the value 7.3×10^{-28} m/kg.

Heaviside then noted that the gravitational field of the Sun, taken as moving relative to the "ether" defined by the fixed stars, would be modified by terms in $(v_{\rm Sun}/c)^2$ exactly as is the case for the field of a rapidly moving electric charge (which result he had been the first to derive correctly to all orders in v/c in 1888). He then calculated the resulting precession of the Earth's orbit around the Sun and concluded that this effect was small enough to have gone unnoticed thus far, and therefore offered no contradiction to the hypothesis that gravitational effects propagate at the speed of light.

Heaviside also considered the effect of the dipole gravitomagnetic field of the rotating S, finding the dipole moment to be $-H\mathbf{L}/2$ where $\mathbf{L}$ is the angular momentum of the Sun. However, the effect of this moment on the precession of a planet's orbit has the opposite sign to the observed effect, and is too small in magnitude by a factor $L_{\rm Sun}/L_{\rm orbital,\,planet}$. (Surprisingly, Heaviside seemed to be unaware of the long history of measurements of the precession of Mercury's orbit.)

It appears that the first confrontation between experiment and new predictions of gravitational field theory occurred some 20 years before Einstein's celebrated work.

From Heaviside's habit of recording the date on which sections of his book first appeared as short articles in *The Electrician* magazine I infer that gravitation occupied his attention for only three weeks in 1893 and that he never returned to the subject.

Heaviside's work could be called a low-velocity, weak-field approximation to general relativity. This topic was revived in an interesting paper by R. L. Forward, "General Relativity for the Experimentalist," that is perhaps insufficiently well-known due in part to its place of publication: Proc. Inst. Radio Eng. **49**, 892–904 (1961).

Additional discussions can be found in Sec. III of "Laboratory Experiments to Test Relativistic Gravity" by V. B. Braginsky, C. M. Caves, and K. S. Thorne, Phys. Rev. D **15**, 2047 (1977) and in the article, "Gravitomagnetism, Jets in Quasars and the Stanford Gyroscope Experiment" by K. S. Thorne in *Near Zero: New Frontiers of Physics*, edited by J. D. Fairbank *et al.* (Freeman, New York, 1988).

The precession of planetary orbits is not a good test of gravitomagnetism; that precession is due to corrections of order v^2/c^2 to the field $\mathbf{g}$ that are "post-Maxwellian." (The term "post-Newtonian" typically used in the literature is perhaps not sufficiently precise in this regard.) However, gravitomagnetism provides a useful insight for understanding the precession of orbiting gyroscopes that will hopefully be observed in experiments now under construction.

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