IV. CONCLUSION

In conclusion, we have reported on a setup for demonstrating the storage of macroscopic particles in a "Paul trap." The simplicity of the experiment allows direct applications in lectures, laboratories, etc. which demonstrate features of the confinement of particles. Finally, we note that we also have succeeded in establishing a "Paul-storage-ring" by similar experimental procedures. In this ring, the lining up of particles can be visualized nicely, a property that one hopes will be achieved in future storage rings for fast and cooled ions.

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Motion of a leaky tank car

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A solution is presented to the following problem: Describe the motion of a tank car initially at rest once an off-center drain opens. The tank car rolls without friction on a horizontal surface, and the water flows out of the drain vertically in the rest frame of the car.

I. INTRODUCTION

The motion of a leaky tank car is surprisingly complex. We approach a solution in four steps: a brief introductory discussion of the motion, a discussion of the forces that cause the motion, a general analysis, and last, two detailed examples. (This problem has appeared in recent years on qualifying exams in Russia.)

There are no external horizontal forces on the system of tank car + water (including the water that has drained out), so the momentum is conserved and the center of mass (c.m.) of the whole system must remain fixed. As the water drains out of the off-center hole the tank car initially moves opposite to the direction of the drain to keep the c.m. fixed. But if this motion persisted until all the water drained out, then both the car and the fallen water would have momen-

tum in the same direction. Rather, the car reverses its direction of motion at some time and has a final velocity toward the drain.

II. FORCES THAT CAUSE MOTION

The water leaves the tank with zero relative horizontal velocity. On recalling the usual rocket problem, there is no propulsion due to the rocket exhaust unless the exhaust has a nonzero velocity relative to the rocket. So, in the present problem, there is no rocket action in the usual sense. However, in both problems the forces are due to momentum transfers between the walls of the rocket or tank and the working substance inside. In a rocket, the walls absorb momentum from the gas molecules moving outward from the region of combustion; because of the hole at the rear of the rocket this leads to a net forward force.

The case of the leaky tank car is more subtle. The motion begins as a transient response in which the walls of the tank push on the water to move it toward the drain once the latter is opened. The reaction on the walls of the tank gives it an initial velocity opposite to the direction from the c.m. to the drain.

If the flow rate were constant in time the tank would roll with constant velocity (see Sec. IV). But if the flow rate decreases as a function of time the tank walls (or floor) must slow down the flow toward the drain, which in turn reduces the velocity of the tank. Since the tank + water now has less mass than before, the momentum reabsorbed from the flowing water can actually reverse the initial velocity!

Indeed, the final velocity of the tank car must be opposite to its initial velocity. If not, then the empty tank car + water that has fallen out would all have nonzero momentum in the same direction, in contradiction to the state before the drain was opened.

III. GENERAL ANALYSIS

We present a solution that avoids the need for a detailed description of the internal motion of the water in the tank, even though this is where the "real physics" resides.

To discuss the motion, let $x(t)$ be the horizontal coordinate of the center of the tank car whose mass (with no water) is m, and suppose that the tank car starts from rest at the origin at $t = 0$. The drain is located a distance D in the $+ x$ direction from the center of the tank. The center of mass of the entire system must remain at the origin:

$$0 = [m + M(t)]x(t) + \int dM(t')X(t,t'),$$

where $M(t)$ is the mass of the water remaining in the tank, $dM(t')$ is the amount of water that drained out in the interval dt' centered on an earlier time t', and $X(t,t')$ is the horizontal coordinate at time t of the water that drained out at time t'. In writing this we have assumed that the surface of the water in the tank is always horizontal so that the center of mass of that water is at the center of the tank.

In the interval dt' at an earlier time t', mass $- \dot{M}(t')dt'$ of water drains out with horizontal velocity $\dot{x}(t')$ in the lab frame. At time t' the drain was at $x(t') + D$, so at time t the element dM is at $X(t,t') = x(t') + D + \dot{x}(t')(t - t')$. Thus the c.m. of the whole system obeys

$$0 = [m + M(t)]x(t) - t \int_0^t dt' \, \dot{M}(t')\dot{x}(t')$$

$$- \int_0^t dt' \, \dot{M}(t')[x(t') + D - t'\dot{x}(t')]. \quad (1)$$

While an integral equation is not the usual starting point in mechanics problems, it has the advantage here of integrating over the unknown transient forces that occur when the drain is first opened.

We take time derivatives of Eq. (1) to find the equation of motion. The first derivative yields

$$0 = (m + M)\dot{x} - \int_0^t dt' \, \dot{M}(t')\dot{x}(t') - \dot{M}D. \quad (2)$$

We recognize (2) as stating that the total momentum of the system is always zero: The first term is the momentum in the tank + water supposing there is no relative motion of the tank and water; the second term is the momentum of the water that has left the tank; and (hence) the third term is the momentum of the water in the tank as measured in the rest frame of the tank.

We can use Eq. (2) to examine the "initial" condition on $\dot{x}$ at an arbitrarily small but positive time:

$$\dot{x}(0) = D\dot{M}(0)/[m + M(0)]. \quad (3)$$

Whatever form of the flow rate $\dot{M}(t)$ could be arranged, $\dot{M}(0)$ is negative so long as the water is draining out, and $\dot{x}(0)$ is in the opposite direction from the c.m. to the drain.

Since Eq. (3) is the limit of (2) as $t \to 0$ from positive values, it contains the result of the transient at $t = 0$. So while the velocity of the tank is zero before the drain is opened, it has a finite value just after the drain is opened and the flow of water inside the tank has been established. In reaction to the forces from the tank on the water, there is an impulse from the water on the tank that creates the initial velocity at the value given by (3).

On taking the derivative of (2) we find:

$$0 = (m + M)\ddot{x} - \ddot{M}D. \quad (4)$$

This can be interpreted as indicating that the force on the tank + water is just the reaction force $\ddot{M}D$ of the acceleration of the water relative to the tank. Because the water leaves the tank with zero relative velocity, the momentum $(m + M)\dot{x}$ is thereby reduced without any reaction force; hence the simplification of (4) compared to (2).

While $\dot{M}$ is always less than zero, $\ddot{M}$ is positive for any realistic flow out of a constant-sized drain hole, and there will be a force in the $+ x$ direction. This force arises as the walls of the tank arrest the motion of the water toward the drain so the water can leave the tank with zero relative velocity. To guarantee the latter condition, the drain might have to be placed in a sump whose vertical walls can assist in absorbing the horizontal momentum of the water flow.

The interpretations of (2) and (4) given above have taken the tank + water inside as the subsystem of interest. The reader might prefer to give emphasis to the tank alone. Then (2) should be rewritten as

$$m\dot{x} = - \left((M\dot{x} - \dot{M}D) - \int_0^t dt' \, \dot{M}(t')\dot{x}(t') \right),$$

where the term in parentheses is the momentum of the water relative to the lab frame. Taking the time derivative:

$$m\ddot{x} = - M\ddot{x} - \dot{M}\dot{x} + \ddot{M}D + \dot{M}\dot{x}.$$

There are four forces on the tank: $- M\ddot{x}$ is the inertial force

of the water in the tank back on the walls of the tank that are accelerating the water; $-\dot{M}\dot{x}$ is a correction to the inertial force of the water because the amount of water inside the tank is changing; $\dot{M}D$ is the force from the water onto the tank near the drain hole where the water is slowed down until it leaves with zero relative velocity; $\dot{M}\dot{x}$ is the reaction force of the water that leaves the tank back on the tank. As previously noted, because the water leaves the tank with zero relative velocity, the reaction force of the water leaving the tank exactly cancels the correction to the inertial force of the water left in the tank.

On writing Eq. (4) as

$$\ddot{x} = D\ddot{M}/(m + M),$$

we can integrate this as

$$\dot{x} = \dot{x}(0) + D\int_0^t dt'\,\frac{\ddot{M}(t')}{m + M(t')}$$

$$= D\left[\frac{\dot{M}(t)}{m + M(t)} + \int_0^t dt'\left(\frac{\dot{M}(t')}{m + M(t')}\right)^2\right], \qquad (5)$$

where we have integrated by parts to obtain the second form. When all the water runs out, $\dot{M} \to 0$ but the integral is positive definite, so the final velocity must be positive. For any finite mass m of the car it will return to the origin after some time, and continue moving in the $+x$ direction.

IV. EXAMPLES

If energy is conserved in the water flow, and if we can ignore the vertical velocity of the water in the tank, then the water drains out with a vertical velocity v given by $v^2 = 2gh$, where h is the height of the water remaining in the tank car. The mass $M(t)$ then varies according to

$$-\dot{M} = \rho v A_{drain} = \rho A_{drain}\sqrt{2gh}\,.$$

If the tank has a constant vertical cross section A_{tank} then $h = M/\rho A_{tank}$ and

$$\dot{M} = -A_{drain}\sqrt{2g\rho M/A_{tank}}\,.$$

This can be integrated to give

$$M(t) = (\sqrt{M_0} - St)^2, \qquad (6)$$

where M_0 is the mass of the water when the drain is first opened, and S is a constant given by $A_{drain}\sqrt{g\rho/2A_{tank}}\,$.

No matter what the shape of the tank we will have

$$\ddot{M} \propto -\dot{h}/\sqrt{h} > 0,$$

so long as energy is conserved and the drain hole does not change with time. A constant flow rate could be arranged in principle by enlarging the hole as time progresses.

From Eq. (3) the initial velocity is

$$\dot{x}(0) = -2DS\sqrt{M_0}/(m + M_0). \qquad (7)$$

From Eqs. (4) and (6) the acceleration of the tank car is

$$\ddot{x} = 2DS^2/[m + (\sqrt{M_0} - St)^2],$$

which is always positive. We integrate this to find

$$\dot{x} = 2DS\left[-\frac{\sqrt{M_0}}{m + M_0} + \frac{1}{\sqrt{m}}\right.$$

$$\left. \times\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \tan^{-1}\frac{\sqrt{M_0} - St}{\sqrt{m}}\right)\right],$$

using $\dot{x}(0)$ from Eq. (7). When the tank is empty at time $t = \sqrt{M_0}/S$ this becomes

$$\dot{x}_{empty} = 2DS\left(-\frac{\sqrt{M_0}}{m + M_0} + \frac{1}{\sqrt{m}}\tan^{-1}\sqrt{\frac{M_0}{m}}\right).$$

This is positive for any combination of masses m and M_0, but for large tank masses m, $\dot{x}_{empty} \to (4/3)DSM_0^{3/2}/m^2$, which approaches zero.

We can verify by direct integration that momentum is conserved in the lab frame:

$$\dot{x}_{empty} = \frac{1}{m}\int dP = \frac{1}{m}\int_0^{\sqrt{M_0}/S} dt\,\dot{M}\dot{x}.$$

We integrate $\dot{x}$ to find

$$\frac{x}{D} = -2\frac{\sqrt{M_0}St}{m + M_0} + \ln\frac{m + M_0}{m + (\sqrt{M_0} - St)^2} - 2\frac{\sqrt{M_0} - St}{\sqrt{m}}$$

$$\times\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \tan^{-1}\frac{\sqrt{M_0} - St}{\sqrt{m}}\right).$$

All three terms vanish at $t = 0$; the first term is just $\dot{x}(0)t$. When the tank is empty at time $t = \sqrt{M_0}/S$ its position is

$$\frac{x_{empty}}{D} = -2\frac{M_0}{m + M_0} + \ln\frac{m + M_0}{m}\,.$$

The critical value of the mass of the tank car that just returns to the origin when empty is $m = 0.255M_0$. For larger masses m the tank car is at negative x when the tank goes empty. Since $\dot{x}_{empty}$ is always positive, the tank car passes through the origin at some later time.

We also consider the limit of a massless tank car. Then

$$\ddot{x} = 2DS^2/(\sqrt{M_0} - St)^2,$$

$$\dot{x} = -4DS/\sqrt{M_0} + 2DS/(\sqrt{M_0} - St),$$

using $\dot{x}(0) = -2DS/\sqrt{M_0}$, and

$$\frac{x}{D} = -\frac{4St}{\sqrt{M_0}} + 2\ln\frac{\sqrt{M_0}}{\sqrt{M_0} - St}\,.$$

Here, the tank car reverses its direction at time $t = \sqrt{M_0}/2S$ (i.e., when it is still $\frac{3}{4}$ full) and at position $x_{min} = -0.61D$, passes the origin at some later time and moves toward large x as it empties.

It is instructive to contrast the above results with the less realistic case that the water drains out at a constant rate:

$$M(t) = M_0 - Rt, \quad \dot{M} = -R, \quad \text{and} \quad \ddot{M} = 0.$$

Because the flow rate is uniform no (horizontal) momentum is transferred from the flowing water to the tank walls once the flow has been established. Hence there are no further (horizontal) forces between the water and the tank, and the velocity of the tank is constant at its initial value. Then

$$\dot{x}(t) = \dot{x}(0) = -DR/(m + M_0)$$

holds until the tank goes empty at time $t = M_0/R$. Its position is then $x_{empty}/D = -M_0/(m + M_0) > -1$.

However, we seem to have a paradox: at time $t = M_0/R$ the water has all emptied out of the tank and now has momentum $M_0\dot{x}(0)$, and the tank appears to have momentum $m\dot{x}(0)$, both of which are negative! The water actually does have the momentum as stated, and we infer that since the total momentum must be zero, the tank takes on a final ve-

locity

$$\dot{x}_{\text{final}} = -M_0 \dot{x}(0)/m.$$

This arises due to another transient force at time $t = M_0/R$. The momentum of the water in the tank relative to the tank has remained constant at $-MD$ even though the mass of water that contains this momentum approaches zero. At time $t = M_0/R$ all this momentum must suddenly be transferred to the tank, which leads to an impulse and hence the final velocity found above. In practice, viscous drag will prevent the water from acquiring the infinite velocity implied above, so the leak rate will decrease and the momentum in the water will be transferred to the tank over a short interval just before the water runs out.

In the present example it is also easy to keep track of the momentum of the water in the tank relative to the lab frame. The impulse at $t = 0$ gives the tank the velocity $\dot{x}(0)$ and hence momentum $m\dot{x}(0)$. The initial momentum of the water in the tank is consequently $-m\dot{x}(0)$. By time t water of total mass Rt has left the tank taking momentum $Rt\dot{x}(0)$ with it. Since the tank's momentum is unchanged the momentum of the water in the tank is $-(m + Rt)\dot{x}(0)$. Just before the tank goes empty, the water in it contains momentum $-(m + M_0)\dot{x}(0) > 0$ which must be transferred to the tank in the final impulse. Again, we arrive at the value of $\dot{x}_{\text{final}}$ found above.

As a final remark we consider whether the motion of a leaky tank car can be observed in practice. For a railroad tank car that is 20 m long with a 5×5 m^2 cross section and has a 10×10 cm^2 drain at one end, the initial velocity would be about 6 cm/s according to Eq. (7). The forces that produce this velocity are likely too small to overcome friction, and no motion would be observed. Instead, one might use an air track from a physics teaching lab and build a sliding tank of mass, say, 2 g that could hold 10 g of water. The drain hole could then be 1 cm off-center. If the hole has area of 1 mm^2 the initial velocity would be about 0.05 cm/s. This is rather small, but should be observable. Care must be given that external forces during the opening of the hole do not impart a comparable velocity.

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An inverted liquid demonstration

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With the help of a hand-held jigsaw and a tin of syrup, the little known dynamically stabilized inverted liquid experiment can be demonstrated to large classes. Lengthy effective potential arguments are short-circuited by a simple inverted U-tube theory accessible to most. The role of viscosity and wall effects is discussed. The first dynamic stabilization experiment with water is reported.

I. INTRODUCTION

In a previous article in this Journal,[1] we showed how the inverted pendulum phenomenon could be demonstrated with the help of a simple jigsaw. The saw vibrates the base of the pendulum along a vertical axis. We now show how this educational trick can be extended to inverted liquids. With the same hand-held piece of inexpensive equipment, both dynamic stabilization experiments can be easily and rapidly demonstrated to large groups.

As pointed out in the previous article,[1] dynamic stabilization has a wealth of generally little known but important applications ranging from control of the lateral dimensions of accelerator and laser beams to thermonuclear fusion experiments. It is our opinion that with the present growth of discoveries in nonlinear processes (e.g., in plasma physics and in high-power optics) an early introduction to two striking nonlinear phenomena is worthwhile. It is the purpose of this article to emphasize the similarities between the inverted pendulum and the inverted liquid.

The article is divided into four sections. In Sec. II we describe the experimental apparatus. In Sec. III we briefly review Kalmus'[2] theory of the inverted pendulum and show how it can be extended to a special type of inverted liquid experiment, the inverted U tube. In Sec. IV we show that the normal outflow of liquid from a container can be likened to that from a U tube and that the dynamic stabilization parameters are almost identical. We also apply the arguments of the previous article[1] to show that we never have a "perfectly inverted liquid." In Sec. V we compare experimental results with the theory of the earlier sections. We then report a novel dynamic stabilization experiment with water contained in a U tube.

No introduction to an article on inverted liquids would be complete without mentioning first and foremost the work of Wolf.[3,4] Anyone desirous of an advanced theoretical treatment of the inverted liquid problem rather than our simplified didactic treatment inspired by Kalmus should consult Wolf's early papers. Anyone of a quantum mechanical inclination would enjoy Pippard's inverted pendulum treatment[5] —which might easily be extended to liquids.