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E-144: Analytic Approximation to the Constrained Solution for η^2 as a Function of Monitors $N_1,\ N_2$ and N_3

We desire to determine an effective value of $\eta^2 = (eE_{\rm rms}/m\omega_0 c)^2$ for each e-laser interaction via the three monitors, $N_1,\ N_2$ and N_3, of first-, second- and third-order Compton scattering. In principle we could extract a value of η^2 from each of the three monitor values. However, we believe the result will be much more stable against various experimental uncertainties such as e-laser timing jitter if we extract η^2 only from ratios of the monitor values.

For $\eta^2 \ll 1$ the approximate relations are

$$\eta^2 = k_1 \frac{N_2}{N_1}, \qquad \text{and} \qquad \eta^2 = k_2 \frac{N_3}{N_2}. \tag{1}$$

Christian and Tomas have made more detailed approximations, each being a quadratic function of the monitor ratios.

In this note we show how eq. (1) can be used to obtain a best-fit estimate of η^2 via an analytic procedure involving the solutions to a quartic equation. This may be of use in checking the iterative numerical fit that uses the more detailed versions of (1).

In the following we distinguish between the measured values N_i of the monitors and corrected (fit) values n_i. The best-fit estimate of η^2 will be based on the corrected n_i rather than the measured N_i. The key to the procedure is that both expressions in (1) must give the same value of η^2 when the corrected n_i are used. That is, we enforce the constraint that

$$n_2^2 = k n_1 n_3, \qquad \text{where} \qquad k = \frac{k_2}{k_1}. \tag{2}$$

The corrected n_i are deduced from the measured N_i by a χ^2-minimization. We assign errors σ_i to the measured N_i based on our understanding of the monitor performance. The χ^2 includes the constraint (2) via a Lagrange multiplier λ.

$$\chi^2 = \sum_{i=1}^{3} \frac{(n_i - N_i)^2}{\sigma_i^2} + 2\lambda(n_2^2 - k n_1 n_3). \tag{3}$$

The four parameters, n_i and λ are adjusted to minimize the χ^2. Formally we set the derivatives of the χ^2 to 0.

$$\frac{1}{2}\frac{\partial\chi^2}{\partial n_1} = \frac{n_1 - N_1}{\sigma_1^2} - \lambda k n_3 = 0, \quad (4)$$

$$\frac{1}{2}\frac{\partial\chi^2}{\partial n_2} = \frac{n_2 - N_2}{\sigma_2^2} + 2\lambda n_2 = 0, \quad (5)$$

$$\frac{1}{2}\frac{\partial\chi^2}{\partial n_3} = \frac{n_3 - N_3}{\sigma_3^2} - \lambda k n_1 = 0, \quad (6)$$

and

$$\frac{1}{2}\frac{\partial\chi^2}{\partial \lambda} = n_2^2 - k n_1 n_3 = 0. \quad (7)$$

We recognize eq. (7) as the requirement that the constraint (2) be satisfied.

I rewrite (4)-(6) as

$$n_1 - \lambda k \sigma_1^2 n_3 = N_1, \quad (8)$$

$$n_2 = \frac{N_2}{1 + 2\lambda\sigma_2^2}, \quad (9)$$

$$-\lambda k \sigma_3^2 n_1 + n_3 = N_3. \quad (10)$$

Equations (8) and (10) can be combined to yield n_1 and n_3 as functions of λ:

$$n_1 = \frac{N_1 + \lambda k \sigma_1^2 N_3}{1 - \lambda^2 k^2 \sigma_1^2 \sigma_3^2}, \quad (11)$$

and

$$n_3 = \frac{N_3 + \lambda k \sigma_3^2 N_1}{1 - \lambda^2 k^2 \sigma_1^2 \sigma_3^2}, \quad (12)$$

We now have all the n_i expressed as functions of λ only. Inserting these expressions, (9), (11) and (12), into the constraint (7) we obtain a quartic equation for λ. After solving this we then have determined all four parameters n_i and λ.

Some steps for the record:

$$n_2^2 = \frac{N_2^2}{(1 + 2\lambda\sigma_2^2)^2} = k n_1 n_3 = \frac{k(N_1 + \lambda k \sigma_1^2 N_3)(N_3 + \lambda k \sigma_3^2 N_1)}{(1 - \lambda^2 k^2 \sigma_1^2 \sigma_3^2)^2}, \quad (13)$$

$$\begin{aligned} &N_2^2(1 - 2\lambda^2 k^2 \sigma_1^2 \sigma_3^2 + \lambda^4 k^4 \sigma_1^4 \sigma_3^4) = \\ &\quad k(1 + 4\lambda\sigma_2^2 + 4\lambda^2\sigma_2^4)(N_1 N_3 + \lambda k(\sigma_1^2 N_3^2 + \sigma_3^2 N_1^2) + \lambda^2 k^2 \sigma_1^2 \sigma_3^2 N_1 N_3), \end{aligned} \quad (14)$$

which yields the quartic equation

$$\begin{aligned} &\lambda^4[4k^3\sigma_1^2\sigma_2^4\sigma_3^2 N_1 N_3 - k^4\sigma_1^4\sigma_3^4 N_2^2] \\ &+ \quad \lambda^3[4k^2\sigma_2^4(\sigma_1^2 N_3^2 + \sigma_3^2 N_1^2) + 4k^3\sigma_1^2\sigma_2^2\sigma_3^2 N_1 N_3] \\ &+ \quad \lambda^2[4k\sigma_2^4 N_1 N_3 + 4k^2\sigma_2^2(\sigma_1^2 N_3^2 + \sigma_3^2 N_1^2) + 2k^2\sigma_1^2\sigma_3^2 N_2^2 + k^3\sigma_1^2\sigma_3^2 N_1 N_3] \\ &+ \quad \lambda[4k\sigma_2^2 N_1 N_3 + k^2(\sigma_1^2 N_3^2 + \sigma_3^2 N_1^2)] \\ &+ \quad k N_1 N_3 - N_2^2 = 0. \end{aligned} \quad (15)$$

The procedure for solving a quartic equation is given, for example, on pp. 17-18 of the *Handbook of Mathematical Functions* by Abramowitz and Stegun. A quartic equation will in general have 0, 2 or 4 real roots (including double roots in special cases). We infer that there will be real roots in our case, so some care must be made in chosing the proper root.

Some qualitative insights may be deduced from the above.

If the Lagrange multiplier λ is 'small' then the linear approximation to eq. (15) tells us

$$\lambda \approx \frac{N_2^2 - kN_1N_3}{4k\sigma_2^2 N_1 N_3 + k^2(\sigma_1^2 N_3^2 + \sigma_3^2 N_1^2)}. \tag{16}$$

Thus we see that λ is a measure of how badly the raw data fails to satisfy the constraint, divided by a factor involving products of squares of the data and squares of the errors. Hence we expect both positive and negative values of λ, and smaller absolute values when the data is better.

Equations (9), (11) and (12) then tell us that when the fitted n_2 is smaller than the data N_2, both n_1 and n_3 will be higher than the corresponding N_1 and N_3. In a further approximation we ignore the denominators of eqs. (11) and (12) and take $k \approx N_2^2/N_1N_3$ as holds when λ is 'small'. Then

$$n_1 \approx N_1 \left(1 + \lambda N_2^2 \frac{\sigma_1^2}{N_1^2}\right), \tag{17}$$

$$n_2 \approx N_2 \left(1 - 2\lambda N_2^2 \frac{\sigma_2^2}{N_2^2}\right), \tag{18}$$

and

$$n_3 \approx N_3 \left(1 + \lambda N_2^2 \frac{\sigma_3^2}{N_3^2}\right). \tag{19}$$

Somewhat surprisingly, the relative changes in the N_i are proportional to the square rather than the first power of the relative errors. Then the monitor with the smallest relative error will change by only extremely small amounts, and will contribute very little to the χ^2. This seems to be borne out by the numerical fit, in which N_1 has the smallest relative error and indeed changes by much less than one σ.