Galilean Transformation of Wave Velocity

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1 Problem

In Galilean relativity an object with velocity $\mathbf{u}$ in one inertial frame appears to have velocity

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \tag{1}$$

to an observer moving with velocity $\mathbf{v}$ with respect to the original frame.

What is the Galilean transformation for the phase velocity $\mathbf{u}_p = \omega\,\hat{\mathbf{k}}/k$ of a wave of angular frequency $\omega > 0$ and wave vector $\mathbf{k}$, for waves where the phase velocity is much less than the speed of light?

Show that the component of phase velocity perpendicular to velocity $\mathbf{v}$ is not invariant under the Galilean transformation, so that the sign of the transverse velocity can be opposite in different frames of reference.

Discuss also the Galilean transformation of group velocity, and of the wave equation. *Restrict your discussion to classical physics. The possibly paradoxical behavior of Galilean transformations of quantum theoretic wave functions is reviewed in [1].*

For completeness, consider also the Lorentz transformations of phase and group velocity.

2 Solution

2.1 Galilean Transformation of Phase Velocity

Much of the content of this section is also in sec. 11.2 of [2].

Consider a wave

$$f = \cos(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{2}$$

in the inertial frame with coordinates $(\mathbf{x}, t)$ in which the elastic medium that supports the wave is at rest. The phase velocity of this wave has magnitude $u = \omega/k$ and direction $\mathbf{k}$. That is, the phase velocity vector of the wave is

$$\mathbf{u}_p = \frac{\omega}{k}\hat{\mathbf{k}} = \frac{\omega}{k^2}\mathbf{k}. \tag{3}$$

An observer whose velocity is $\mathbf{v}$ with respect to the original frame uses coordinates $(\mathbf{x}', t')$ to describe an event $(\mathbf{x}, t)$ in the original frame obtained by the Galilean transformation

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t, \tag{4}$$

supposing that the spatial axes in the two frames are parallel. The moving observer sees the wave to have the same amplitude as described by eq. (2), which he describes in terms of $(\mathbf{x}', t')$ as the wave function f',

$$f' \equiv \cos(\mathbf{k}' \cdot \mathbf{x}' - \omega' t') = f = \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) = \cos[\mathbf{k} \cdot \mathbf{x}' - (\omega - \mathbf{k} \cdot \mathbf{v})t']. \tag{5}$$

From eq. (5) we see that the wave vector is the same in the moving frame as in the original frame,

$$\mathbf{k}' = \mathbf{k}, \tag{6}$$

so the wavelength is the same in both frames, and the direction of the wave vector is the same in both frames (or the direction of the wave in the moving frame is opposite to that in the original frame if $\mathbf{k} \cdot \mathbf{v} > \omega$). Similarly, the wave frequency in the moving frame is

$$\omega' = |\omega - \mathbf{k} \cdot \mathbf{v}|\,, \tag{7}$$

which is the well-known Doppler effect for a source at rest and a moving observer.

Thus, the phenomenon of the aberration of light (apparent change of direction a light ray depending on the motion of the observer, as discovered in 1728 by Bradley) would not exist if light waves obeyed Galilean transformations. However, if light consists of particles with a finite speed, as advocated by Newton, then aberration of light is expected in Galilean relativity. Advocates of the wave theory of light in the 1800's went to considerable lengths to explain how the dragging of the ether by the Earth renders a wave theory of light compatible with the observation of stellar aberration. See, for example, [3].

The phase velocity $\mathbf{u}'_p$ of the wave in the moving frame is given by

$$\mathbf{u}'_p = \frac{\omega'}{k'^2}\mathbf{k}' = \frac{\omega'}{k^2}\mathbf{k} = \frac{\omega - \mathbf{k} \cdot \mathbf{v}}{k^2}\mathbf{k} = \mathbf{u}_p - (\mathbf{v} \cdot \hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p = \mathbf{u}_p - \mathbf{v}_\parallel = \mathbf{u}_p - \mathbf{v} + \mathbf{v}_\perp \tag{8}$$

if $\mathbf{k} \cdot \mathbf{v} < \omega$, noting that the components of velocity $\mathbf{v}$ that are parallel and perpendicular to velocity $\mathbf{u}_p$ are $\mathbf{v}_\parallel = (\mathbf{v} \cdot \hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p$ and $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel$, respectively. When $\mathbf{k} \cdot \mathbf{v} > \omega$, the phase velocity in the moving frame is

$$\mathbf{u}'_p = -\frac{\omega'}{k^2}\mathbf{k} = -\frac{\mathbf{k} \cdot \mathbf{v} - \omega}{k^2}\mathbf{k} = \mathbf{u}_p - (\mathbf{v} \cdot \hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p, \tag{9}$$

so the form of the phase velocity transformation is independent of the magnitude of $\mathbf{k} \cdot \mathbf{v}$. However, the transformation of the wave velocity is NOT the same as the transformation of velocity of a particle ($\mathbf{u}' = \mathbf{u} - \mathbf{v}$) if the direction $\mathbf{k}$ of the wave is different from that of the boost $\mathbf{v}$.

Example:

$$\mathbf{k} = \frac{(1, -1, 0)}{\sqrt{2}} \qquad \omega = 1, \qquad \text{and} \qquad \mathbf{v} = (v, 0, 0). \tag{10}$$

Then $k = 1$ and the phase velocity in the original frame is

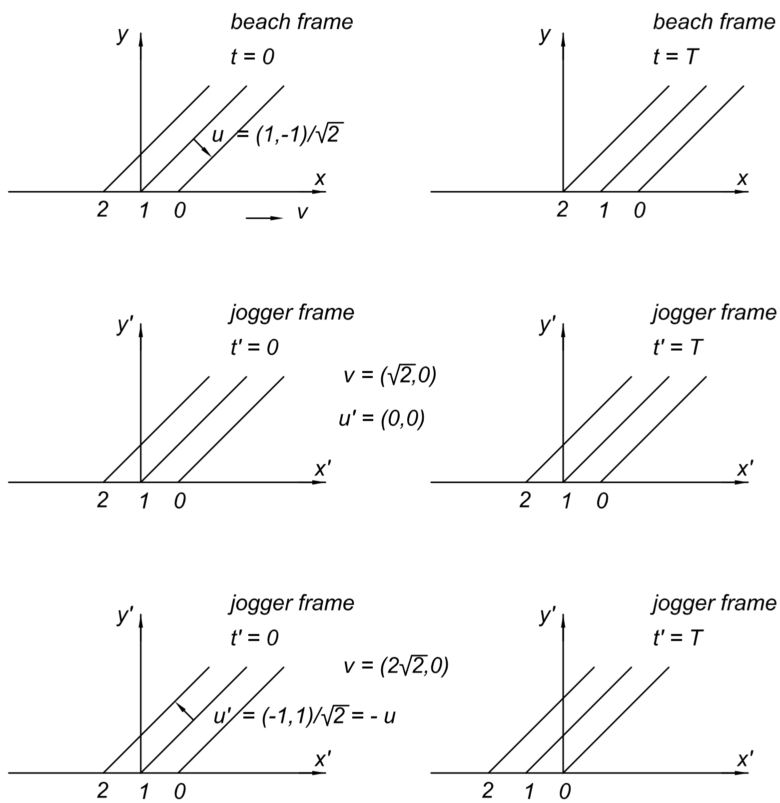
$$\mathbf{u}_p = \frac{\omega}{k^2}\mathbf{k} = \frac{(1, -1, 0)}{\sqrt{2}}\,. \tag{11}$$

The phase velocity in the moving frame is

$$\mathbf{u}'_p = \mathbf{u}_p - (\mathbf{v}\cdot\hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p = \frac{(1,-1,0)}{\sqrt{2}} - \frac{v}{\sqrt{2}}\frac{(1,-1,0)}{\sqrt{2}} = \frac{(1-v/\sqrt{2}, v/\sqrt{2}-1,0)}{\sqrt{2}}\,. \qquad (12)$$

The tranverse velocity $u'_{p,\perp} = u'_{p,y}$ is never the same as $u_{\perp,p} = u_y$ for nonzero v, and if $v > 1/k_\perp = \sqrt{2}$ then the sign the transverse component $u'_{p,\perp}$ of the phase velocity is opposite in the moving frame to that of $u_{p,\perp}$ in the original frame.

We illustrate this behavior by considering a train of water waves with velocity $\mathbf{u}_p = (1,-1)/\sqrt{2}$ impinging on a beach that lies along the x axis, as shown in the figure below.



The intercept of a wavefront with the beach has velocity $\mathbf{u}_{\rm intercept} = (\sqrt{2}, 0)$. If a jogger runs along the beach with velocity $\mathbf{v} = \mathbf{u}_{\rm intercept}$, then the wavetrain appears to be at rest with respect to the jogger, as confirmed by eq. (12).

If the jogger runs with velocity $\mathbf{v} = 2\mathbf{u}_{\rm intercept}$, then the phase velocity of the wavetrain according to the jogger is $\mathbf{u}'_p = -\mathbf{u}_p$, *i.e.*, the wavetrain appears to recede from the beach opposite to the direction of motion of the waves according to an observer at rest.

2.2 Galilean Transformation of Group Velocity

In a medium that is at rest on average and whose oscillations are characterized by the dispersion relation $\omega = \omega(k)$, a wave packet with frequency components near a central value $\omega_0 = \omega(k_0)$ has group velocity $\mathbf{u}_g$ given by

$$\mathbf{u}_g = \boldsymbol{\nabla}_{\mathbf{k}}\,\omega(k_0) = \frac{\partial\omega(k_0)}{\partial\mathbf{k}} = \frac{d\omega(k_0)}{dk}\,\hat{\mathbf{k}}. \qquad (13)$$

In a moving frame related to the rest frame of the medium by the Galilean transformation (4) we have seen that $\mathbf{k}' = \mathbf{k}$ and $\omega' = \omega - \mathbf{k}\cdot\mathbf{v}$, so the group velocity in that frame is given by[1]

$$\mathbf{u}'_g = \frac{\partial\omega'(k'_0)}{\partial\mathbf{k}'} = \frac{\partial\omega'(k'_0)}{\partial\mathbf{k}} = \frac{d\omega(k_0)}{dk}\,\hat{\mathbf{k}} - \mathbf{v} = \mathbf{u}_g - \mathbf{v}. \tag{14}$$

That is, the Galilean transformation of group velocity has the same form as that of particle velocity, eq. (1), in contrast with the transformation (8) for phase velocity.[2]

This result is in agreement with the quantum view that waves have quanta that behave like particles, where the effective particle velocity of a wave packet of such quanta is the group velocity, not the phase velocity. For discussion of different aspects of quantum theory and Galilean transformations, see [1, 4].

Group velocity is the same as the velocity of energy flow in most examples of wave phenomenon [5]. A pictorial derivation of the Galilean transformation of energy flow velocity has been given in sec. 11.2 of [2] (without comment that energy flow velocity is the same as group velocity, which is the same as phase velocity for light waves with the dispersion relation $\omega = kc$, and hence the pictorial derivation is contradictory for light waves).

2.3 Galilean Transformation of the Wave Equation

The wave equation is

$$\nabla^2 f - \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{15}$$

in the inertial frame with coordinates $(\mathbf{x}, t)$ where the medium is at rest and the wave velocity is u (at angular frequency ω). The Galilean coordinate transformation to an inertial frame that moves with velocity $\mathbf{v}$ with respect to the rest frame of the medium is given by eq. (4). The transformations of derivatives with respect to the coordinates are

$$\frac{\partial}{\partial x} = \frac{\partial\mathbf{x}'}{\partial x}\cdot\boldsymbol{\nabla}' + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial\mathbf{x}'}{\partial t}\cdot\boldsymbol{\nabla}' + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{\partial}{\partial t'} - \mathbf{v}\cdot\boldsymbol{\nabla}', \tag{16}$$

so the wave equation (15) transforms to

$$\nabla'^2 f' - \frac{1}{u^2}\frac{\partial^2 f'}{\partial t'^2} + \frac{2}{u^2}\frac{\partial}{\partial t'}\mathbf{v}\cdot\boldsymbol{\nabla}' f' - \frac{(\mathbf{v}\cdot\boldsymbol{\nabla}')^2}{u^2} f' = 0 \tag{17}$$

[1] If $\omega < \mathbf{k}\cdot\mathbf{v}$ then $\mathbf{k}' = -\mathbf{k}$ and $\omega' = -\omega + \mathbf{k}\cdot\mathbf{v}$, so that the result of eq. (14) still obtains.

[2] A counterintuitive result of the transformation (14) is that while the phase and group velocities might have the same direction in the rest frame of a medium, their directions will be different in frames where the medium has a velocity whose direction is different from the common direction of the wave velocities in the rest frame. For example, consider a medium with the dispersion relation (in the rest frame of that medium) $\omega = uk$ for constant u. Then, for a packet of waves all propagating in the $\hat{\mathbf{x}}$-direction, $\mathbf{u}_p = \mathbf{u}_g = u\,\hat{\mathbf{x}}$, which is not of the form $\omega' = u'k'$; a medium that is dispersion free in its rest frame has dispersion in frames where the medium is moving (according to Galilean transformations). In the $'$ frame where the medium has velocity $\mathbf{v}$, the dispersion relation transforms to $\omega' = uk' - \hat{\mathbf{k}}'\cdot\mathbf{v}$ according to eqs. (6) and (7), while the waves of the packet still have phase velocity in the $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$ direction. However, the group velocity in the $'$ frame is $\mathbf{u}'_g = \boldsymbol{\nabla}_{\mathbf{k}'}\omega' = u\,\hat{\mathbf{k}}' - \mathbf{v} = \mathbf{u}_g - \mathbf{v}$. Unless velocity $\mathbf{v}$ is along the x axis, the group velocity in the $'$ frame has a component perpendicular to the phase velocity. There is no formal contradiction here, but we see that care is required to identify the group velocity of waves in a moving medium. In particular, we should not write $\omega' = (u - v\cos\theta)k'$, where the angle θ between $\mathbf{v}$ and $\mathbf{k} = \mathbf{k}'$ is constant for waves in the packet, and thereby incorrectly conclude that $\mathbf{u}'_g = (u - v\cos\theta)\hat{\mathbf{k}}'$.

in the moving frame.

The wave function (5) is readily verified to be a solution to equation (17) for $u = \omega/k$.

The noninvariance of the wave equation (15) under Galilean transformations led W. Voigt in 1887 to deduce a form of the coordinate transformation for which the wave equation is invariant [6], namely what is now called the Lorentz transformation if $u = c$. He considered a (conformal) generalization of eq. (4),[3]

$$\mathbf{x}' = s[a(\mathbf{x}_\parallel - \mathbf{v}t) + \mathbf{x}_\perp] = s[\mathbf{x} + (a-1)(\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} - a\mathbf{v}t], \qquad t' = bs\left(t - \frac{\mathbf{v}}{u^2}\cdot\mathbf{x}\right), \tag{18}$$

where a, b and s are positive constants, and

$$\mathbf{x}_\parallel = (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}, \qquad \mathbf{x}_\perp = \mathbf{x} - (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}. \tag{19}$$

The transformations of derivatives are now

$$\frac{\partial}{\partial x} = \frac{\partial \mathbf{x}'}{\partial x}\cdot\boldsymbol{\nabla}' + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = s\left[\frac{\partial}{\partial x'} + (a-1)\hat{v}_x(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}') - b\frac{v_x}{u^2}\frac{\partial}{\partial t'}\right], \tag{20}$$

$$\frac{\partial}{\partial t} = \frac{\partial \mathbf{x}'}{\partial t}\cdot\boldsymbol{\nabla}' + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = s\left[b\frac{\partial}{\partial t'} - a\mathbf{v}\cdot\boldsymbol{\nabla}'\right], \tag{21}$$

so that

$$\nabla^2 = s^2\left[\nabla'^2 + \frac{b^2 v^2}{u^4}\frac{\partial^2}{\partial t'^2} - \frac{2ab}{u^2}\frac{\partial}{\partial t'}\mathbf{v}\cdot\boldsymbol{\nabla}' + (a^2-1)(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}')^2\right], \tag{22}$$

and the wave equation (15) with $u = c$ transforms to

$$\nabla'^2 f' - \frac{b^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 f'}{\partial t'^2} + \left[a^2\left(1 - \frac{v^2}{c^2}\right) - 1\right](\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}')^2 f' = 0 \tag{23}$$

in the moving frame for any nonzero value of s. If

$$a = b = \frac{1}{\sqrt{1 - v^2/u^2}}, \tag{24}$$

then the wave equation in the moving frame reduces to

$$\nabla'^2 f' - \frac{1}{u^2}\frac{\partial^2 f'}{\partial t'^2} = 0, \tag{25}$$

i.e., its form is invariant under the transformation (18) with a and b given by eq. (24) for any nonzero value of s.

A consequence is that the wave velocity is u in both the rest frame of the medium and in the moving frame. This would make sense only if the wave velocity had a more universal character than understood in the year 1887. Voigt noted that his transformation predicted aberration of stellar lightwaves observed on the moving Earth. He also noted that

[3] Voigt actually used transformation (18) subject to the condition that $d = 1/a$ which makes $x'_\parallel = x_\parallel - vt$ as in the classic Galilean transformation. Transformation (18) with arbitrary d was first considered by Poincaré in 1905 [7].

his transformation assumed that the wave velocity was independent of position, which is not typically the case for hydrodynamic waves, but is true for light waves (in vacuum). However, he did not further develop the notion that the speed of light c is a universal constant.

Voigt assumed the existence of a luminiferous ether, and did not notice that if the wave equation has the same form in all inertial frames, then the ether has no special rest frame (that can be determined from observations of lightwaves) and hence may not exist.[4]

The development of the theory of relativity by Lorentz, Poincaré and Einstein appears not to have been influenced by the 1887 paper of Voigt prior to 1908, when Lorentz belatedly became aware of it.

2.4 Lorentz Transformation of Phase Velocity

For earlier discussions of this topic see [8]-[11]. (Oct. 19, 2019: see also [12]).

We have seen in sec. 2.1 that in a Galilean transformation of phase velocity $\mathbf{u}_p$ by a velocity $\mathbf{v}$, the component of phase velocity transverse to $\mathbf{v}$ is not invariant. This alerts us that phase velocity will not be part of a Lorentz 4-velocity vector. However, we can still display an explicit form for the Lorentz transformation of phase velocity.

We write a 4-vector as $a_\mu = (a_0, \mathbf{a})$ and the square of its invariant length as $a_\mu a^\mu = a_0^2 - \mathbf{a}^2$. In particular, the position 4-vector is $x_\mu = (ct, \mathbf{x})$.

The Lorentz transformation of a 4-vector a_μ from one inertial frame to another with velocity $\mathbf{v}$ with respect to the first can be written

$$a'_0 = \gamma(a_0 - a_\parallel v/c), \tag{26}$$

$$\mathbf{a}'_\parallel = \gamma(\mathbf{a}_\parallel - a_0\mathbf{v}/c), \tag{27}$$

$$\mathbf{a}'_\perp = \mathbf{a}_\perp, \tag{28}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp, \qquad \mathbf{a}_\parallel = (\mathbf{a}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}, \tag{29}$$

and the 4-vector has components $(a_0, \mathbf{a})$ and $(a'_0, \mathbf{a}')$ in the first and second frames, respectively.

We recall that the 3-velocity $\mathbf{u}$ of a particle obeys $u < c$ and can be embedded in the 4-velocity

$$u_\mu = \gamma_u(c, \mathbf{u}), \qquad \text{where} \qquad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}. \tag{30}$$

The Lorentz transformation (26)-(28) of the particle 4-velocity is

$$\gamma_{u'} = \gamma\gamma_u(1 - \mathbf{u}\cdot\mathbf{v}/c^2), \tag{31}$$

$$\mathbf{u}'_\parallel = \frac{\gamma\gamma_u}{\gamma_{u'}}(\mathbf{u}_\parallel - \mathbf{v}) = \frac{\mathbf{u}_\parallel - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2} = \frac{(\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}, \tag{32}$$

$$\mathbf{u}'_\perp = \mathbf{u}_\perp = \mathbf{u} - (\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}. \tag{33}$$

[4]The convective derivative $D/Dt = \partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$ for a particle/observer with velocity $\mathbf{u}$ transforms to $\partial/\partial t' + (\mathbf{u} - \mathbf{v})\cdot\boldsymbol{\nabla}' = D/Dt'$ under a Galilean transformation, so the form of the Navier-Stokes equation is invariant under this transformation. However, one would not thereby infer that fluids have no rest frame.

Equations (32) and (33) can be combined to give

$$\mathbf{u}' = \frac{\mathbf{u} - \mathbf{v} - \mathbf{u}_\perp(\mathbf{u}\cdot\mathbf{v})/c^2}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}\,, \tag{34}$$

which reverts to the Galilean transformation (1) of particle velocity in the low-velocity limit.

Returning to consideration of waves, we can associate the plane wave (2), $\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)$, with a wave 4-vector $k_\mu = (\omega/c, \mathbf{k})$ such that the phase ϕ of the wave is a Lorentz scalar,

$$\phi = \mathbf{k}\cdot\mathbf{x} - \omega t = -k_\mu x^\mu. \tag{35}$$

Surfaces of constant phase propagate with phase velocity,

$$\mathbf{u}_p = \frac{\omega}{k^2}\mathbf{k} = \mathbf{u}_{p\|} + \mathbf{u}_{p\perp} = u_{p\|}\,\hat{\mathbf{v}} + u_{p\perp}\,\hat{\mathbf{k}}_\perp, \tag{36}$$

where $\mathbf{u}_{p\|} = \mathbf{u}\cdot\hat{\mathbf{v}}$, and $\hat{\mathbf{k}}_\perp$ is a unit vector in the direction of $\mathbf{u}_{p\perp} = \mathbf{u}_p - \mathbf{u}_{p\|}$ (and not the component perpendicular to $\mathbf{v}$ of the unit vector $\hat{\mathbf{k}} = \hat{\mathbf{u}}_p$). If we write,

$$\mathbf{k} = k_\|\,\hat{\mathbf{v}} + k_\perp\,\hat{\mathbf{k}}_\|, \tag{37}$$

and note that $\omega = k\,u_p$, we also have,

$$k_\| = \frac{\omega\,u_{p\|}}{u_p^2}\,, \qquad k_\perp = \frac{\omega\,u_{p\perp}}{u_p^2}\,. \tag{38}$$

In a frame that moves with velocity $\mathbf{v}$ with respect to the original frame the phase velocity is,

$$\mathbf{u}'_p = \frac{\omega'}{k'^2}\mathbf{k}' = \frac{\omega'}{k'}\hat{\mathbf{u}}'_p = \mathbf{u}'_{p\|} + \mathbf{u}'_{p\perp}. \tag{39}$$

The Lorentz transformation (26)-(28) of the wave 4-vector k_μ is,

$$\omega' = \gamma(\omega - \mathbf{k}\cdot\mathbf{v}) = \gamma(\omega - k_\|\,v) = \frac{\gamma\,\omega}{u_p^2}(u_p^2 - u_{p\|}\,v), \tag{40}$$

$$\mathbf{k}'_\| = \gamma(\mathbf{k}_\| - \omega\mathbf{v}/c^2) = \gamma\,\hat{\mathbf{v}}(k_\| - \omega v/c^2) = \frac{\gamma\,\omega}{c^2\,u_p^2}\,\hat{\mathbf{v}}\left(c^2\,u_{p\|} - u_p^2\,v\right), \tag{41}$$

$$\mathbf{k}'_\perp = \mathbf{k}_\perp = k_\perp\,\hat{\mathbf{k}}_\perp = \frac{\omega\,u_{p\perp}}{u_p^2}\,\hat{\mathbf{k}}_\perp. \tag{42}$$

Combining eqs. (41) and (42) we find,

$$k'^2 = \frac{\omega^2}{c^4\,u_p^4}[c^4\,u_{p\perp}^2 + \gamma^2(c^2\,u_{p\|} - u_p^2\,v)^2], \tag{43}$$

and hence the Lorentz transformation of phase velocity is,[5]

$$u'_p = \frac{\omega'}{k'} = \frac{\gamma\,c^2(u_p^2 - u_{p\|}\,v)}{\sqrt{c^4\,u_{p\perp}^2 + \gamma^2(c^2\,u_{p\|} - u_p^2\,v)^2}} = \frac{u_p - v\,u_{p\|}/u_p}{\sqrt{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\|}/u_p - u_p\,v/c^2)^2}}\,, \tag{44}$$

[5]Equation (44) can be used to define an index of refraction $n' = c/u'_p$ in a moving medium [13].

and the direction of the wavefront is,

$$\hat{\mathbf{u}}'_p = \hat{\mathbf{k}}' = \frac{\mathbf{k}'_\parallel + \mathbf{k}'_\perp}{k'} = \frac{\gamma(c^2\, u_{p\parallel} - u_p^2\, v)\,\hat{\mathbf{v}} + c^2\, u_{p\perp}\,\hat{\mathbf{k}}_\perp}{\sqrt{c^4\, u_{p\perp}^2 + \gamma^2 (c^2\, u_{p\parallel} - u_p^2\, v)^2}} = \frac{(u_{p\parallel}/u_p - u_p\, v/c^2)\,\hat{\mathbf{v}} + (u_{p\perp}/\gamma u_p)\,\hat{\mathbf{k}}_\perp}{\sqrt{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p\, v/c^2)^2}}\,. \tag{45}$$

The compenents of the transformed phase velocity are,

$$\mathbf{u}'_{p\parallel} = u'_p \frac{\mathbf{k}'_\parallel}{k'} = \frac{\gamma^2\, c^2 (u_p^2 - u_{p\parallel}\, v)(c^2\, u_{p\parallel} - u_p^2\, v)}{c^4\, u_{p\perp}^2 + \gamma^2 (c^2\, u_{p\parallel} - u_p^2\, v)^2}\,\hat{\mathbf{v}} = \frac{(u_p - v\, u_{p\parallel}/u_p)(u_{p\parallel}/u_p - u_p\, v/c^2)}{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p\, v/c^2)^2}\,\hat{\mathbf{v}}, \tag{46}$$

$$\mathbf{u}'_{p\perp} = u'_p \frac{\mathbf{k}'_\perp}{k'} = \frac{\gamma\, c^4 (u_p^2 - u_{p\parallel}\, v)\, u_{p\perp}}{c^4\, u_{p\perp}^2 + \gamma^2 (c^2\, u_{p\parallel} - u_p^2\, v)^2}\,\hat{\mathbf{k}}_\perp = \frac{(u_p - v\, u_{p\parallel}/u_p)\, u_{p\perp}/\gamma u_p}{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p\, v/c^2)^2}\,\hat{\mathbf{k}}_\perp. \tag{47}$$

The cumbersome expression (44) reduces to the Galilean result (8) in the low-velocity limit.

If the boost velocity $\mathbf{v}$ is parallel to the phase velocity $\mathbf{u}_p$, then from eqs. (40) and (41), or from eq. (44), we find that

$$\mathbf{u}'_p = \frac{\mathbf{u}_p - \mathbf{v}}{1 - \mathbf{u}_p \cdot \mathbf{v}/c^2} \qquad (\mathbf{u}_p \parallel \mathbf{v}), \tag{48}$$

so that the Lorentz transformation of phase velocity is the same as that for particle velocity, eq. (34), in this special case.

However, since the phase speed ω/k can have any value less than, equal to, or even greater than c,[6] the wave 4-vector k_μ can be spacelike, lightlike or timelike, respectively. This permits possibly surprising scenarios such as $\mathbf{u}_p = 2c\,\hat{\mathbf{x}}$, $\mathbf{v} = c\,\hat{\mathbf{x}}/2$, for which the transformed wave vector $\mathbf{k}'$ vanishes, the transformed waveform is the standing wave $\cos(\sqrt{3/4}\omega t')$, and the transformed phase velocity is formally infinite.

For the special case that $u_p = c$, eq. (40) becomes $\omega' = \gamma\,\omega(1 - u_{p\parallel} v/c^2)$, and eq. (43) becomes (after some algebra) $k'^2 = \gamma^2\omega^2(1 - u_{p\parallel} v/c^2)^2/c^2 = \omega'^2/c^2$. That is, if the phase velocity is c in one inertial frame it is also c in any other inertial frame.

2.5 Lorentz Transformation of Group Velocity

This section follows [14].

The result of sec. 2.3 that group velocity behaves like a particle velocity under Galilean transformations presages a similar behavior under Lorentz transformations.

A suitable 4-vector generalization of the 3-dimensional group velocity (13), $\mathbf{u}_g = \partial\omega/\partial\mathbf{k}$, is based on rewriting the dispersion relation $\omega = \omega(\mathbf{k})$ as

$$F(k_\mu) = F(\omega/c, \mathbf{k}) = 0, \tag{49}$$

and then taking the 4-vector gradient,

$$\left.\frac{\partial F}{\partial k_\mu}\right|_{F=0} = \left(c\frac{\partial F}{\partial \omega}, \frac{\partial F}{\partial \mathbf{k}}\right)_{F=0} = \left.\frac{\partial F}{\partial \omega}\right|_{F=0} \left(c, \frac{\partial \omega}{\partial \mathbf{k}}\right) = \left.\frac{\partial F}{\partial \omega}\right|_{F=0} (c, \mathbf{u}_g). \tag{50}$$

[6]A well-known case with phase speed greater than c is a waveguide. See, for example, sec. 8.3 of [2].

The invariant length of this gradient is

$$\sqrt{\frac{\partial F}{\partial k_\mu}\frac{\partial F}{\partial k^\mu}}\Bigg|_{F=0} = c\,\frac{\partial F}{\partial \omega}\bigg|_{F=0}\sqrt{1-u_g^2/c^2} = \frac{c}{\gamma_{u_g}}\,\frac{\partial F}{\partial \omega}\bigg|_{F=0}\,. \tag{51}$$

Dividing the 4-vector (50) by the Lorentz scalar (51) and multiplying by c we obtain the group-velocity 4-vector,

$$u_{g,\mu} = \gamma_{u_g}(c, \mathbf{u}_g). \tag{52}$$

This 4-vector is formally identical to that of the 4-velocity (30) of a particle, so the Lorentz transformation of the group velocity $\mathbf{u}_g$ has the form of eq. (34),

$$\mathbf{u}'_g = \frac{\mathbf{u}_g - \mathbf{v} - \mathbf{u}_{g\perp}(\mathbf{u}_g\cdot\mathbf{v})/c^2}{1-\mathbf{u}_g\cdot\mathbf{v}/c^2}\,. \tag{53}$$

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