729 Variants of Poynting's Theorem

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1 Problem

Discuss how by rearranging terms in Poynting's theorem [1], one can display (at least) 729 variants thereof.[1]

2 Solution

2.1 The Standard Version of Poynting's Theorem

Poynting's theorem [1][2] expresses energy conservation in electromagnetic phenomena in the form,[3]

$$\frac{\partial \text{ energy density}}{\partial t} + \boldsymbol{\nabla} \cdot \textbf{energy current density} = \text{source power density}. \tag{1}$$

In the standard version, the sources on the right side are "nonelectromagnetic" in character, such as batteries or dynamos that convert chemical (*i.e.*, quantum electrodynamic) or "mechanical" (really another form of quantum field) energy into "electromagnetic" form as understood in the context of "classical" electrodynamics.

The author finds it instructive to characterize the nonelectromagnetic power source by a nonelectromagnetic field $\mathbf{E}'$ that acts on the "free" conduction current $\mathbf{J}_{\rm free}$ according to an extension of Ohm's law,

$$\mathbf{J}_{\rm free} = \sigma(\mathbf{E} + \mathbf{E}'), \tag{2}$$

where $\mathbf{E}$ is the usual electric field and σ is the conductivity of the medium that supports the conduction current. This permits us to relate the nonelectromagnetic field $\mathbf{E}'$ to electromagnetic quantities,

$$\mathbf{E}' = \frac{\mathbf{J}_{\rm free}}{\sigma} - \mathbf{E}. \tag{3}$$

The total density of power delivered by the nonelectromagnetic source to the electromagnetic system is then,

$$P_{\rm nonelectromagnetic,total} = \mathbf{J}_{\rm free} \cdot \mathbf{E}' = \frac{J_{\rm free}^2}{\sigma} - \mathbf{J}_{\rm free} \cdot \mathbf{E}. \tag{4}$$

[1] This problem is an extension of [2] in which the usual Poynting's theorem was recast as eq. (13) involving the magnetization density $\mathbf{M}$. For a review of a different class of Poynting-vector alternatives, see [3].

[2] Heaviside's independent discovery of this theorem can be traced in [4, 5, 6].

[3] Apparently, the form (1) was first considered by Umov [8] as an extrapolation to energy flow of Euler's continuity equation for mass flow [9].

The first term on the right side of eq. (4) is the density of Joule heating of the conductive medium, and we regard this power as "lost" with respect to the electromagnetic system. In contrast, the second term represents power that is transferred from the nonelectromagnetic system into energy stored, or flowing, within the electromagnetic system,

$$P_{\rm nonelectromagnetic,transferred} = -\mathbf{J}_{\rm free} \cdot \mathbf{E} = -\mathbf{E} \cdot \left(\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) = \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \boldsymbol{\nabla} \times \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}, \qquad (5)$$

where we have used the third and fourth macroscopic Maxwell equations and a vector-calculus identity. Of course, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, where $\mathbf{P}$ and $\mathbf{M}$ are the densities of electric and magnetic dipoles. We identify the Poynting vector $\mathbf{S}$,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \qquad (6)$$

as describing the flow (current density) of electromagnetic energy, and,

$$\frac{\partial u}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \qquad (7)$$

as the time rate of change of the electromagnetic energy density u.

While Poynting's theorem (5) clearly suggests that $\mathbf{S}$ describes the flow of electromagnetic energy, it does not in general identify the electromagnetic energy density u. This leaves open the possibility that some alternative form of eq. (5) might be preferable.

2.2 Poynting's Theorem for Linear Media

For so-called linear media in which $\mathbf{E}$ is proportional to $\mathbf{D}$ and $\mathbf{B}$ is proportional to $\mathbf{H}$, the expression (7) is a perfect differential and we can write the electromagnetic energy density u as,

$$u = \frac{\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}}{2}, \qquad (8)$$

and Poynting's theorem reads,[4]

$$\boldsymbol{\nabla} \cdot \mathbf{S} + \frac{\partial u}{\partial t} = -\mathbf{J}_{\rm free} \cdot \mathbf{E} = P_{\rm nonelectromagnetic,transferred}. \qquad (11)$$

[4]Poynting's original derivation [1] assumed linear media, and did not proceed as in our eq. (5). Rather, he began with the energy density (8) written as $u = \epsilon E^2/2 + \mu H^2/2$, and took its time derivative,

$$\frac{\partial u}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \boldsymbol{\nabla} \times \mathbf{E} = -\mathbf{E} \cdot \left(-\frac{\partial \mathbf{D}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{H} \right) - \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{J}_{\rm free} \cdot \mathbf{E} - \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}). \qquad (9)$$

The volume integral of this is,

$$\int \frac{\partial u}{\partial t}\, d{\rm Vol} = -\int \mathbf{J}_{\rm free} \cdot \mathbf{E}\, d{\rm Vol} - \int (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{Area}, \qquad (10)$$

2.3 729 Variants

In general, the (classical) macroscopic electromagnetic fields **E**, **D**, **B** and **H** are related by the forms,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{12}$$

where **P** and **M** are the densities of (quantum) electric and magnetic dipole moments. A detailed understanding of the densities **P** and **M** is beyond the scope of "classical" electrodynamics.

We take the attitude that any electric dipoles formed by free electric charges and currents do not contribute to the densities **P** and **M**, but are represented in Maxwell's equations in the quantities $\rho_{\rm free}$ and $\mathbf{J}_{\rm free}$.

Using the relations (12) to we can replace any/all of the 6 field vectors on the right side of Poynting's theorem (5) with either of two vectors in the relations (12), and move the "extra" terms from the right side to the left side, to obtain $3^6 = 729$ variants of the form,

$$P_{{\rm source},n} = \boldsymbol{\nabla} \cdot \mathbf{S}_n + \frac{\partial u_n}{\partial t}\, . \tag{13}$$

We display only a few of these below:

$$P_{\rm source,0} = -\mathbf{J}_{\rm free} \cdot \mathbf{E}, \qquad \mathbf{S}_0 = \mathbf{E} \times \mathbf{H}, \qquad \frac{\partial u_0}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\, . \tag{14}$$

Variant 0 is, of course, the standard version of Poynting's theorem.

$$P_{\rm source,1} = -\mathbf{E} \cdot \left(\mathbf{J}_{\rm free} + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M}\right) = -\mathbf{E} \cdot \mathbf{J}_{\rm total},$$
$$\mathbf{S}_1 = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}, \qquad u_1 = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\, . \tag{15}$$

Variant 1 looks a lot like the microscopic version of Poynting's theorem, but here the **E** and **B** are macroscopic averages over their microscopic counterparts. This might be called the

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which Poynting interpreted as indicating that the time rate of change of the electromagnetic energy inside a volume equals the rate $-\mathbf{J}_{\rm free} \cdot \mathbf{E}$ of work done by the fields within that volume, minus the flux $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ of energy that leaves the volume across its surface.

A notable feature of this derivation is that it deduces $-\mathbf{J}_{\rm free} \cdot \mathbf{E}$ to be the power delivered by the fields to the free current density $\mathbf{J}_{\rm free}$ without any model of those currents, other than that they obey Maxwell's equations.

While some derivations of the magnetic field energy density, $u_{\rm mag} = \mathbf{B} \cdot \mathbf{H}/2$, *start from a statement that the power delivered by the fields to the free current density is* $-\mathbf{J}_{\rm free} \cdot \mathbf{E}$, *Maxwell's first argument for* $u_{\rm mag}$, *given (1856) on p. 63 of [10], was a generalization of the energy density in the case of permanent magnetism. See also Arts. 632-636 of [11], and sec. A.28.1.6 of [12].*

In 1861, Maxwell used his famous "idler-wheel" model of electric currents, to deduce the magnetic energy density $u_{\rm mag}$ *on pp. 286-288 of [13], and then that* $-\mathbf{J}_{\rm free} \cdot \mathbf{E}$ *is the power density delivered to the currents, pp. 288-289. See also sec. A.28.2.3 of [12]. Maxwell considered his deduction of* $-\mathbf{J}_{\rm free} \cdot \mathbf{E}$ *to be model dependent, and did not include it in his great paper of 1864 [14], nor in his* Treatise *[11].*

"pure" electromagnetic field version of Poynting's theorem in that for all other variants the "material" fields $\mathbf{P}$ or $\mathbf{M}$ appear somewhere in $\mathbf{S}$ or u.

$$P_{\rm source,2} = -\mathbf{E}\cdot\left(\mathbf{J}_{\rm free} + \frac{\partial \mathbf{P}}{\partial t}\right) + \mathbf{M}\cdot\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{S}_2 = \mathbf{E}\times\mathbf{H}, \qquad u_2 = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}, \tag{16}$$

$$P_{\rm source,3} = -\mathbf{E}\cdot\left(\mathbf{J}_{\rm free} + \frac{\partial \mathbf{P}}{\partial t}\right) - \mu_0\mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial t} \qquad \mathbf{S}_3 = \mathbf{E}\times\mathbf{H}, \qquad u_3 = \frac{\epsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2}, \tag{17}$$

$$P_{\rm source,4} = -\mathbf{E}\cdot\mathbf{J}_{\rm free} + \frac{\mathbf{P}}{\epsilon_0}\cdot\frac{\partial \mathbf{D}}{\partial t} - \mu_0\mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial t} \qquad \mathbf{S}_4 = \mathbf{E}\times\mathbf{H}, \qquad u_4 = \frac{D^2}{2\epsilon_0} + \frac{\mu_0 H^2}{2}, \tag{18}$$

$$P_{\rm source,5} = -\mathbf{E}\cdot\mathbf{J}_{\rm free} + \frac{\mathbf{P}}{\epsilon_0}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{M}\cdot\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{S}_5 = \mathbf{E}\times\mathbf{H}, \qquad u_5 = \frac{D^2}{2\epsilon_0} + \frac{B^2}{2\mu_0}, \tag{19}$$

Variants 2-5 keep the standard Poynting vector $\mathbf{E}\times\mathbf{H}$ and use 4 different energy densities based on $\mathbf{E}$ or $\mathbf{D}$ and $\mathbf{B}$ or $\mathbf{H}$, but the price is that the power-source terms include at least one of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$ or $\mathbf{H}$, which implies that the fields are sources for themselves. This is not logically excluded, but differs from the spirit of the standard Poynting theorem in which the source term is "nonelectromagnetic" (although written as $-\mathbf{J}_{\rm free}\cdot\mathbf{E}$ in eq. (5)). All of the 723 remaining variants share this feature that the fields are in some way sources of themselves.

$$P_{\rm source,6} = -\frac{\mathbf{D}}{\epsilon_0}\cdot(\mathbf{J}_{\rm free} + \boldsymbol{\nabla}\times\mathbf{M}) + c^2\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{P},$$
$$\mathbf{S}_6 = c^2\,\mathbf{D}\times\mathbf{B}, \qquad u_6 = \frac{D^2}{2\epsilon_0} + \frac{B^2}{2\mu_0}, \tag{20}$$

$$P_{\rm source,7} = -\frac{\mathbf{D}}{\epsilon_0}\cdot\mathbf{J}_{\rm free} + \frac{\mathbf{H}}{\epsilon_0}\cdot\boldsymbol{\nabla}\times\mathbf{P} - \mu_0\mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial t}$$
$$\mathbf{S}_7 = \frac{\mathbf{D}\times\mathbf{H}}{\epsilon_0}, \qquad u_7 = \frac{D^2}{2\epsilon_0} + \frac{\mu_0 H^2}{2}, \tag{21}$$

Variants 6 and 7 use "Poynting vectors" $\propto \mathbf{D}\times\mathbf{B}$ and $\mathbf{D}\times\mathbf{H}$, together with "matching" energy densities.

$$P_{\rm source,8} = -\mathbf{E}\cdot\left(\mathbf{J}_{\rm free} + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M} + \frac{\partial \mathbf{D}}{\partial t}\right)$$
$$\mathbf{S}_8 = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}, \qquad u_8 = \frac{\epsilon_0 E^2}{2} - \frac{\mathbf{E}\cdot\mathbf{D}}{2} + \frac{B^2}{2\mu_0}. \tag{22}$$

Variant 8 supposes that Maxwell's displacement current, $\partial\mathbf{D}/\partial t$, is included in the source currents, but this leads to a negative "matching" energy density for dielectric materials.

$$P_{\rm source,9} = -\frac{\partial(\epsilon_0 E^2/2 + \mu_0 H^2/2)}{\partial t} - \frac{\mathbf{D}}{\epsilon_0}\cdot\frac{\partial \mathbf{P}}{\partial t} - \frac{\mathbf{B}}{\mu_0}\cdot\frac{\partial \mathbf{M}}{\partial t} - \boldsymbol{\nabla}\cdot\frac{\mathbf{E}\times\mathbf{B}}{\mu_0} + \boldsymbol{\nabla}\cdot\frac{\mathbf{D}\times\mathbf{M}}{\epsilon_0},$$
$$\mathbf{S}_9 = c^2\,\mathbf{P}\times\mathbf{M}, \qquad u_9 = -\frac{P^2}{2\epsilon_0} - \frac{\mu_0 M^2}{2}. \tag{23}$$

Variant 9 is an example for how far one can go from the original statement of Poynting's theorem by using a "Poynting vector" $\mathbf{P} \times \mathbf{M}$, for which the "matching" energy density is negative, and the source terms are peculiar.

In the author's view, only variants 0 and 1 will find much "practical" use in physics, while the others are just formally correct rearrangements of terms with little useful physical meaning thereto.[5]

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