“Hidden” Momentum in an Unbalanced Tire

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1 Problem

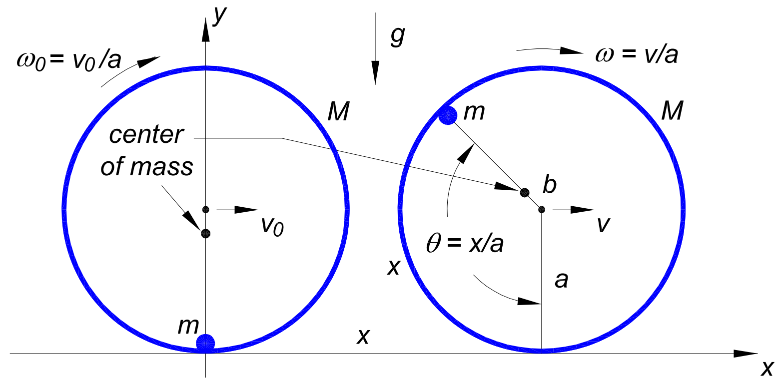
Discuss the motion of an unbalanced tire that rolls without slipping on a horizontal surface, subject to conservation of energy.

In what sense(s) can this system be said to contain “hidden momentum”?

2 Solution

2.1 The Motion

We consider the tire to be a hoop of radius a and mass M that is unbalanced due to an additional mass m at some fixed point on the rim, as shown in the figure below.



Taking the origin of an x-y coordinate system to be at a point of contact of the tire with the road when mass m is at its lowest position, when the point of contact is at x the geometric center of the tire has horizontal velocity v and the center of mass of the unbalanced tire has rotated through angle $\theta = x/a$. The angular velocity ω of the tire is

$$\omega = \dot\theta = \frac{v}{a}\,. \tag{1}$$

The mass m has coordinates

$$x_m = x - a\sin\theta, \qquad y_m = a - a\cos\theta, \tag{2}$$

and velocity

$$\dot x_m = v - a\,\omega\cos\theta = (1-\cos\theta)v, \qquad \dot y_m = a\,\omega\sin\theta = \sin\theta v, \qquad v_m^2 = 2(1-\cos\theta)v^2. \tag{3}$$

The conserved energy E of the system is

$$E = Mga + mga(1-\cos\theta) + Mv^2 + \frac{mv_m^2}{2} = Mga + mga(1-\cos\theta) + [M + (1-\cos\theta)m]v^2. \tag{4}$$

Setting this equal to the initial energy $E_0 = Mga + Mv_0^2$, the velocity of the geometric center of the tire is related by

$$v^2(\theta) = \frac{Mv_0^2 - mga(1 - \cos\theta)}{M + (1 - \cos\theta)m}\,. \tag{5}$$

The tire has continuous rolling motion only if $v_0^2 > 2mga/M$; otherwise the motion is oscillatory. In any case, the geometric center of the tire does not move at constant speed for nonzero m.

For small mass $m \ll M$ and rolling motion,

$$v^2 \approx v_0^2 \left[1 - \frac{m}{M}\left(1 + \frac{ga}{v_0^2}\right)(1 - \cos\theta)\right]. \tag{6}$$

2.2 Hidden Momentum

An observer of the motion of the unbalanced tire finds this to be somewhat odd compared to that of a balanced tire. The observer might say that there is some "hidden" aspect of the tire that leads to its peculiar behavior.

If the observer can't see the perturbing mass m, then it is "hidden" to him, and he might say that the momentum and energy associated with mass m are therefore "hidden". We consider such an observer to be rather unobservant, and we don't pursue his possible thought process any further.

A view expressed in [1] is that an observer who is aware of both masses M and m, and able to compute both momenta

$$\mathbf{P}_M = Mv\,\hat{\mathbf{x}}, \qquad \text{and} \qquad \mathbf{P}_m = m(\dot{x}_m\,\hat{\mathbf{x}} + \dot{y}_m\,\hat{\mathbf{y}}) = mv[(1 - \cos\theta)\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}], \tag{7}$$

might consider the quantity

$$\mathbf{P} - m_{\text{total}}\mathbf{v}_{\text{geometric center}} = \mathbf{P}_M + \mathbf{P}_m - (M + m)v\,\hat{\mathbf{x}} = mv(-\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}) \tag{8}$$

to be a kind of "hidden momentum". The author's view is that if one is aware of mass m and its effect on the motion of the system, one should not say that this effect is "hidden".

In 1967, Shockley [2] gave the term "hidden momentum" a particular meaning in electrodynamics. The issues here can be traced back to the debate between Ampère and Biot as to the force law between two current elements $d\mathbf{I}_1 = I_1 d\mathbf{l}_1$ and $d\mathbf{I}_2 = I_2 d\mathbf{l}_2$. Ampère [3, 4, 5] argued that the force on element 1 due to element 2 is (in Gaussian units)

$$d\mathbf{F}_{\text{on }1} = \frac{3(d\mathbf{I}_1 \cdot \hat{\mathbf{r}})(d\mathbf{I}_2 \cdot \hat{\mathbf{r}}) - 2(d\mathbf{I}_1 \cdot d\mathbf{I}_2)}{c^2 r^2}\,\hat{\mathbf{r}} \qquad \text{(Ampere)}, \tag{9}$$

while Biot and Savart [6] claimed that[1]

$$d\mathbf{F}_{\text{on }1} = \frac{d\mathbf{I}_1 \times \mathbf{B}_2}{c}, \qquad \text{where} \quad \mathbf{B}_2 = \frac{d\mathbf{I}_2 \times \hat{\mathbf{r}}}{cr^2} \qquad \text{(Biot-Savart)}. \tag{10}$$

[1]Biot and Savart were not very explicit as to the forms (10), which are stated more clearly in [5, 7].

Ampére showed that the two forms for $d\mathbf{F}_{\text{on }1}$ are equivalent when computing the force of one closed current loop on another, and argued that his form was superior in that it satisfied Newton's 3rd law. However, Ampère's form (9) does not factorize, in contrast to eq. (10) of Biot and Savart, such that the latter is compatible with a field theory of magnetism whereas the former is less obviously so. This led to awkwardness throughout most of the 1800's in which advocates of electromagnetic field theory avoided discussion of current elements (moving electric charges) except in closed circuits.

A consistent vision of electrodynamics of current elements/moving charges requires understanding of some kind of "hidden momentum" that restores compatibility with Newton's laws of motion. This vision was supplied by Poynting [8] and Poincaré [9] who argued that the electromagnetic field supports both flow of energy and storage of momentum, where (in vacuum) the density $\mathbf{p}_{\text{EM}}$ of momentum stored in the electromagnetic field is

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \tag{11}$$

and $\mathbf{S}$ is the Poynting vector that describes the flow of energy. The electromagnetic field momentum,

$$\mathbf{P}_{\text{EM}} = \int \mathbf{p}_{\text{EM}}\, d\text{Vol}, \tag{12}$$

is the "hidden momentum" that makes the Biot-Savart force law consistent with Newton's 3rd law.[2]

Shockley's contribution [2] was to note that there exist nominally static situations in which the electromagnetic momentum (12) is nonzero while there is no obvious equal and opposite mechanical momentum, as seems to be required by the center-of-energy theorem.[3] For Shockley, the electromagnetic momentum was "obvious", and he characterized the equal and opposite mechanical momentum as "hidden". This specialized usage of the term "hidden momentum" is somewhat unfortunate, and has proven to be controversial for many readers. *An unbalanced tire, considered as a purely mechanical system, cannot have "hidden momentum" in the sense of Shockley, as for him there must first be electromagnetic momentum before there can be "hidden momentum".*

It is possible to define the term "hidden momentum" without mention of electromagnetism in such a way that the more general definition includes the "hidden momentum" of Shockley as a special case. Following a suggestion by D. Vanzella [12], we define [13]

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\, (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\text{cm}})\, d\text{Vol}, \tag{13}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}, \tag{14}$$

[2] For an explicit verification of this, see [10].

[3] See, for example, [11].

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.[4,5]

According to the last form of eq. (13), "hidden momentum" can only exist in a subsystem that has a volume interaction with another subsystem, which implies that it exists only in field theories. In this sense, definition (13) participates in the spirit of the debate of Ampére and Biot-Savart (while the definition (8) does not).

In closing, we apply definition (13) to the present example, taking the subsystem to be only the rolling, unbalanced tire. The tire makes contact with the road (another subsystem), and the velocity of the tire is zero at the point of contact (for rolling without slipping), so that the boundary integral in eq. (13) is zero. Then,

$$\mathbf{P}_{\rm hidden} = \mathbf{P} - M_{\rm total}\mathbf{v}_{\rm cm}. \tag{15}$$

The center of mass of the unbalanced tire is at distance b from the geometric center of the tire, where

$$\frac{b}{a} = \frac{m}{M+m}\,, \tag{16}$$

and has coordinates

$$x_{\rm cm} = x - b\sin\theta, \qquad y_{\rm cm} = a - b\cos\theta, \tag{17}$$

and velocity

$$\dot{x}_{\rm cm} = v - b\,\omega\cos\theta = \left(1 - \frac{b}{a}\cos\theta\right)v = \left(1 - \frac{m}{M+m}\cos\theta\right)v, \tag{18}$$

$$\dot{y}_{\rm cm} = b\,\omega\sin\theta = \frac{b}{a}\sin\theta\,v = \frac{m}{M+m}\sin\theta\,v. \tag{19}$$

The "hidden momentum" of the unbalanced tire according to eq. (15) is

$$\begin{aligned} P_{{\rm hidden},x} &= P_{M,x} + P_{m,x} - (M+m)\dot{x}_{\rm cm} \\ &= Mv + m(1-\cos\theta)v - (M+m)\left(1 - \frac{m}{M+m}\cos\theta\right)v = 0, \end{aligned} \tag{20}$$

$$P_{{\rm hidden},y} = P_{M,y} + P_{m,y} - (M+m)\dot{y}_{\rm cm} = mv\sin\theta - (M+m)\frac{m}{M+m}v\sin\theta = 0. \tag{21}$$

That is, there is no "hidden momentum" in an unbalanced tire in the sense of the general definition (13), which follows from the use of this term by Shockley.

[4] With this definition of "hidden momentum", both the electromagnetic field momentum and the equal and opposite mechanical momentum in Shockley's example are described as "hidden".

[5] As discussed in sec. 3 of [13], we do not advocate replacing $\mathbf{v}_{\rm cm}$ by $\mathbf{v}_{\rm geometric\ center}$ in definition (13).

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