

Leaky Syringe

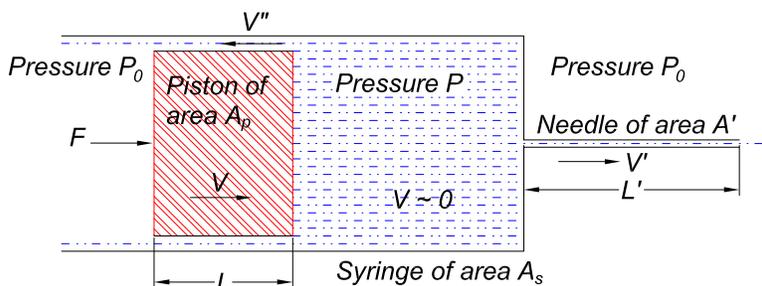
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1 Problem

Discuss the fluid flow in a syringe whose piston has area A_p smaller than the area A_s of the cylindrical body, when the piston is forced to move at constant velocity V .



This problem was suggested by Johann Otto.

2 Solution

When the area of a fluid flow changes abruptly, the fluid flow may pull away from the wall in the region of smaller area, which effect is called the *vena contracta*, as first noted by Torricelli in 1643 [1].

In the present problem, this could mean that the fluid flow inside the needle (where the velocity is V') and in the annulus between the piston and the outer cylinder (where the velocity is V'') has areas smaller than that of the needle and the annulus. If so, there would be five unknown quantities, the pressure P in the syringe (where the flow velocity is approximately zero), the flow velocities V' and V'' , and the two reduced areas A' and A'' associated with those flow velocities.

However, there would then be only three equations to determine these five unknowns: mass conservation (assuming an incompressible fluid), and energy and momentum conservation (in the sense that the energy and momentum gained by the fluid is supplied by the agent that exerts force F on the piston). Note that if the fluid has pulled away from the walls of the needle, the piston and the syringe, there is no fluid friction at these surfaces, and hence no additional equations involving the fluid viscosity.

To obtain a solution to the present problem, in which the fluid leaves the syringe via two flow paths, we suppose that the *vena contracta* does not occur, such that the fluid fills the needle and the annulus around the piston, and there is fluid friction at the walls of these regions.¹ Then, there are only three unknowns, P , V' and V'' , which can be determined

¹This contrasts with the related case of a shock absorber (hydraulic brake) [2], in which there is only a single flow path, and the *vena contracta* must be taken into account.

using mass conservation, and Poiseuille’s law for flow with friction in the needle and in the annulus.

The equation of mass conservation is

$$VA_s = V'A' + (V + V'')A'' \approx V'A' + V''A'', \quad (1)$$

where $A'' = A_s - A_p$ is the cross sectional area of the annulus, and $V + V''$ is the velocity of the fluid in the annulus relative to the moving piston, taking V'' to be the velocity of the fluid in the annulus relative to the syringe. For a narrow annulus, $V'' \gg V$, and we will neglect the small correction due to V in the following.

Poiseuille’s law [3, 4, 5, 6, 7, 8] for flow of a fluid with dynamic viscosity η in the needle of radius R' and length L' is

$$Q' = V'A' = \frac{\pi R'^4(P - P_0)}{8\eta L'}, \quad (2)$$

where P_0 is the atmospheric pressure at the outlet of the needle.

The flow in the annulus is a variant of so-called Taylor-Couette [9] flow, where the two cylindrical surfaces have a relative axial velocity, rather than being in relative rotation. For simplicity, we suppose that flow in the annulus is sufficiently well approximate by (plane Poiseuille) flow in a rectangular slot of height $R_s - R_p$ and width $2\pi R_s$, which approximately obeys (for $V'' \gg V$),

$$Q'' \approx V''A'' = \frac{\pi R_s^2(R_s - R_p)^2(P - P_0)}{6\eta L}. \quad (3)$$

Using eqs. (2)-(3) in eq. (1), the pressure P inside the syringe is estimated to be

$$P \approx P_0 + \frac{\eta VA_s}{\pi \left[\frac{R'^4}{8L'} + \frac{R_s^2(R_s - R_p)^2}{6L} \right]}. \quad (4)$$

The flow velocities are then

$$V' \approx V \frac{A_s}{A'} \frac{1}{1 + \frac{3LR'^4}{4L'R_s^2(R_s - R_p)^2}}, \quad V'' \approx V \frac{A_s}{A''} \frac{1}{1 + \frac{4L'R_s^2(R_s - R_p)^2}{3LR'^4}}, \quad (5)$$

which are both large compared to V .

2.1 $L, L' \rightarrow 0$

If the lengths L and L' of the piston/annulus and of the needle are very short, friction in the annulus and in the needle become negligible. In this case the fluid flow will exhibit the *vena contracta* [1] at both the annulus and the “needle” (which is now simply a circular aperture in the end wall of the syringe).

For a solution when viscosity is neglected, we estimate the areas of the flow in the annulus and in the “needle” to be 1/2 their geometrical areas, as holds approximately in many cases.

And, since the work done on the fluid is the kinetic energy of the fluid (when viscosity and gravity are neglected), we have that (for $V \gg V''$)

$$\begin{aligned} F_{\text{total}} &= FV - P_0 \frac{A'}{2} V' - P_0 \frac{A''}{2} V'' = PA_p V - P_0 \frac{V'A' + V''A''}{2} \\ &= \frac{d\text{KE}_{\text{fluid}}}{dt} \approx \frac{\rho V'A' V'^2}{2} + \frac{\rho V''A'' V''^2}{2} = \frac{\rho}{4} (V'^3 A' + V''^3 A''), \end{aligned} \quad (6)$$

noting that the mass of fluid accelerated per second from rest in the syringe to velocity V' in the “needle” is approximately $\rho V'A'/2$, and that accelerated each second to velocity V'' in the annulus is $\rho V''A''/2$, where A' and A'' are the geometric areas. Also, the time rate of change of momentum of the fluid is

$$\begin{aligned} \frac{dp_{\text{fluid}}}{dt} &\approx \frac{\rho V'A'}{2} V' - \frac{\rho V''A''}{2} V'' = F_{\text{total}} = F + P_0 A'' - P(A_s - A') - P_0 A' \\ &= (P - P_0)(A' - A''). \end{aligned} \quad (7)$$

Finally, mass conservation, eq. (1), implies that

$$VA_s \approx \frac{V'A' + V''A''}{2}. \quad (8)$$

We now have three equations for the three unknowns P , V' and V'' , so in principle these unknowns are determined. However, eq. (6) is a cubic equation so the solution is algebraically intricate, and we do not give it here.²

The special case when $A' = 0$ (no needle) is the same as the hydraulic brake considered in sec. 2.2 of [2]. The case when $A'' = 0$ corresponds to a syringe without a leak (and with a zero-length “needle”), for which both eqs. (6) and (7) give $P = P_0 + \rho V'^2/2$ (for $A_p = A_s = A$ and mass conservation $VA = V'A'/2$), as also follows from Bernoulli’s equation³ (which does apply to this simple case, unlike the cases where $A' = 0$ and the present example where both A' and A'' are nonzero).

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²Surprisingly, the solution for the case of nonzero lengths of the needle and piston, when effects of viscosity must be considered, is analytically simpler than the case of zero length of the needle and piston, for which viscosity can be ignored.

³Bernoulli’s equation applies in situations where the kinetic + potential energy of the fluid is conserved, and where the fluid flow is steady. See [10] for further discussion of this point.

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