Static Magnetic Field as Determined by Its Value on a Surface

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1 Problem

Deduce an expression for a static magnetic field **B**, and a corresponding vector potential **A**, based on knowledge of the field on a closed surface surrounding the observation point.

2 Solution

This problem is a variation on so-called vector diffraction theory. For discussion of determination of the magnetic field based only on its value along an axis, see [1, 2].

We recall the formalism of Kottler [3, 4] for fields with time dependence $e^{-i\omega t}$ in vacuum,

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \int_V \left(\frac{ik}{c}\mathbf{J}(\mathbf{x}')\frac{e^{ikr}}{r} + \rho(\mathbf{x}')\boldsymbol{\nabla}'\frac{e^{ikr}}{r} \right) d\mathrm{Vol}' + \frac{i}{\omega}\oint_S (\mathbf{J}\cdot\hat{\mathbf{n}}')\boldsymbol{\nabla}'\frac{e^{ikr}}{r}\, d\mathrm{Area}' \\ &\quad -\frac{1}{4\pi}\boldsymbol{\nabla}\times\oint_S \left\{ [\hat{\mathbf{n}}'\times\mathbf{E}(\mathbf{x}')]\frac{e^{ikr}}{r} + \frac{i}{k}\boldsymbol{\nabla}\times[\hat{\mathbf{n}}'\times\mathbf{B}(\mathbf{x}')]\frac{e^{ikr}}{r} \right\} d\mathrm{Area}', \qquad (1) \end{aligned}$$

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{1}{c}\int_V \mathbf{J}(\mathbf{x}')\times\boldsymbol{\nabla}'\frac{e^{ikr}}{r}\, d\mathrm{Vol}' \\ &\quad -\frac{1}{4\pi}\boldsymbol{\nabla}\times\oint_S \left\{ [\hat{\mathbf{n}}'\times\mathbf{B}(\mathbf{x}')]\frac{e^{ikr}}{r} - \frac{i}{k}\boldsymbol{\nabla}\times[\hat{\mathbf{n}}'\times\mathbf{E}(\mathbf{x}')]\frac{e^{ikr}}{r} \right\} d\mathrm{Area}', \qquad (2) \end{aligned}$$

where $\hat{\mathbf{n}}'$ is the outward unit vector normal to surface S, $r = |\mathbf{x}-\mathbf{x}'|$, c is the speed of light in vacuum, $k = \omega/c$, and Gaussian units are employed. See the Appendix of [5] for derivations and discussion of these forms.

For a region with no currents the magnetic field can be related to a vector potential that follows from eq.(2 as

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\oint_S \left\{ \mathbf{B}(\mathbf{x}')\times\hat{\mathbf{n}}'\frac{e^{ikr}}{r} - \frac{i}{k}\boldsymbol{\nabla}\times[\mathbf{E}(\mathbf{x}')\times\hat{\mathbf{n}}']\frac{e^{ikr}}{r} \right\} d\mathrm{Area}', \qquad (3)$$

assuming that we can take the curl after performing the integrations. If **E** and **B** are zero everywhere on the surface of a region then **A** is zero in its interior, according to eq. (3). The prescription of eq. (3) cannot be extended to all of space since there must be currents somewhere if **B** is nonzero.

If the bounding surface is a perfect conductor, then $\mathbf{E}(\mathbf{x}')\times\hat{\mathbf{n}}' = 0$, *and*

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\oint_S \mathbf{B}(\mathbf{x}')\times\hat{\mathbf{n}}'\frac{e^{ikr}}{r}\, d\mathrm{Area}' = \frac{1}{c}\oint_S \frac{\mathbf{K}(\mathbf{x}')\, e^{ikr}}{r}\, d\mathrm{Area}', \qquad (4)$$

where $\mathbf{K}$ *is the surface current density. We recognize this as the (Lorenz-gauge) retarded vector potential, assuming that all currents lie on the bounding surface.*

In the static limit, $\omega = 0 = k$, the electric field does not depend the current density $\mathbf{J}$ or the magnetic field, and the magnetic field does not depend on the electric field. Noting that $\boldsymbol{\nabla}'(1/r) = \hat{\mathbf{r}}/r^2 = -\boldsymbol{\nabla}(1/r)$, we obtain

$$\mathbf{E}(\mathbf{x}) = \int_V \rho(\mathbf{x}') \frac{\hat{\mathbf{r}}}{r^2}\, d\mathrm{Vol}' + \frac{1}{4\pi} \oint_S \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')]}{r^2}\, d\mathrm{Area}', \tag{5}$$

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{x}') \times \hat{\mathbf{r}}}{r^2}\, d\mathrm{Vol}' + \frac{1}{4\pi} \oint_S \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')]}{r^2}\, d\mathrm{Area}', \tag{6}$$

If there are no currents within the volume of integration, the static magnetic field there can be deduced from the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \oint_S \frac{\mathbf{B}(\mathbf{x}') \times \hat{\mathbf{n}}'}{r}\, d\mathrm{Area}' \qquad \text{(static limit)}, \tag{7}$$

recalling eq. (2). The example of a static, toroidal magnetic field (for which $\mathbf{B} = 0$ outside the torus but $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area}$ is nonzero for loops that link the torus) suggests that eqs. (3) and (7) are restricted to simply connected regions.

2.1 Uniform Axial Field

As an example, consider a uniform axial field, $\mathbf{B} = B_0\, \hat{\mathbf{z}}$ that is generated by azimuthal currents about the z-axis. The associated vector potential has only the azimuthal component

$$A_\phi = \frac{\rho B_0}{2}\,. \tag{8}$$

in a cylindrical coordinate system (ρ, ϕ, z).

We take the point of observation to be $(\rho, 0, 0)$. As the surface of integration for eq. (7) we consider a cylinder of radius $a > \rho$ with faces at $-z_1$ and z_2. Then, $\mathbf{B} \times \hat{\mathbf{n}}' = B_0\, \hat{\boldsymbol{\phi}}$ and

$$\begin{aligned}
A_\phi &= A_y = \frac{1}{4\pi} \int_0^{2\pi} a\, d\phi \int_{-z_1}^{z_2} dz\, \frac{B_0 \cos\phi}{\sqrt{z^2 + a^2 + \rho^2 - 2a\rho\cos\phi}} \\
&= \frac{aB_0}{4\pi} \int_0^{2\pi} \cos\phi\, d\phi\, \ln \frac{z_2 + \sqrt{z_2^2 + a^2 + \rho^2 - 2a\rho\cos\phi}}{-z_1 + \sqrt{z_1^2 + a^2 + \rho^2 - 2a\rho\cos\phi}} \\
&= \frac{aB_0}{4\pi} \int_0^{2\pi} \cos\phi\, d\phi \left[\ln\left(z_2 + \sqrt{z_2^2 + a^2 + \rho^2 - 2a\rho\cos\phi} \right) \right. \\
&\qquad \left. + \ln\left(z_1 + \sqrt{z_1^2 + a^2 + \rho^2 - 2a\rho\cos\phi} \right) - \ln\left(a^2 + \rho^2 - 2a\rho\cos\phi \right) \right] \\
&= -\frac{aB_0}{4\pi} \int_0^{2\pi} \cos\phi\, d\phi\, \ln\left(1 + \frac{\rho^2}{a^2} - 2\frac{\rho}{a}\cos\phi \right) = \frac{\rho B_0}{2}\,,
\end{aligned} \tag{9}$$

using 4.397.6 of [6]. A delicacy is our assumption that

$$\int_0^{2\pi} \cos\phi\, d\phi\, \ln\left(z + \sqrt{z^2 + a^2 + \rho^2 - 2a\rho\cos\phi} \right) = 0, \tag{10}$$

for nonzero values of z. This integral clearly goes to zero for large z, and the calculation (9) of A_ϕ must be independent of the values of z_1 and z_2.

2.2 Other Formulations

Section 14.3-4 of [7] gives a formalism by which $\mathbf{B}$ can be computed from knowledge of its normal component, $\mathbf{B} \cdot \hat{\mathbf{n}}$, on elliptical cylindrical surfaces, and sec. 18.2 describes the use of the tangential component $\mathbf{B} \times \hat{\mathbf{n}}$ on circular cylinders.[1]

References

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[7] A.J. Dragt, *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics* (Feb. 27, 2011), http://www.physics.umd.edu/dsat/

[1] The function $a_0(z) = a_0^{(0)}(z)$ used in [2] is the same as $C_0^{[1]}(z)$ in [7].