“Hidden” Momentum in an Oscillating Spring?

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1 Problem

[*This problem is too complicated to be a good illustration of “hidden” momentum is a mechanical system. Instead, see [1, 2]. It is the author’s view that an all-mechanical system (whose volume contains no macrscopic fields) contains no “hidden” momentum, see sec. 4 of [5]. The approximate stress tensor considered below seems to be insufficiently accurate for a meaningful calcultion.*]

The term “hidden” momentum was popularized by Shockley [3] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

Recently, a definition of “hidden” momentum has been proposed by Daniel Vanzella [4] (see also [5]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\,d{\rm Vol}, \quad (1)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,, \quad (2)$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

Does an isolated, oscillating spring contain “hidden” momentum according to the above definition? Consider also the subsystem of the spring to one size of its center.

2 Solution

We consider the spring to be a bar of rest mass m, rest length L, cross sectional area A and Young’s (elastic) modulus E. For simplicity, we assume that Poisson’s ratio is zero for the bar/spring.

We analyze the system in the lab frame, in which the center of the bar is at rest at the origin, and the oscillations are in the x-coordinate.

2.1 The Motion in a Low-Velocity Approximation

The low-velocity equation of motion for the displacement $s(x,t)$ of an element dx of the spring/bar, centered on x, is,[1]

$$F(x+dx) - F(x) = F'(x)\,dx = \frac{d}{dt}\left(\rho A\,dx\,\frac{ds}{dt}\right) \equiv \frac{d}{dt}\left(\rho A\,dx\,\dot{s}\right) \approx m\frac{dx}{L}\ddot{s}, \tag{3}$$

where $F(x)$ is the internal force across the y-z plane through point $(x,0,0)$, and ρ is the effective mass density of the moving element. That is,

$$\ddot{s} = \frac{L}{m}F'(x). \tag{4}$$

Due to the internal force F the element dx has stretched by amount,

$$\Delta x = s(x+dx) - s(x) = s'(x)dx. \tag{5}$$

We recall that stretching of an elastic medium can be related to its elastic modulus E as,[2]

$$\frac{F}{A} = E\frac{\Delta L}{L}. \tag{6}$$

Considering the entire bar/spring, the modulus E and the spring constant k are related by,

$$k = \frac{EA}{L}. \tag{7}$$

For the element dx, eq. (6) becomes,

$$F = EA\frac{\Delta x}{dx} = \frac{kL}{dx}\Delta x = kLs'. \tag{8}$$

Inserting this in the equation of motion (4) we find the wave equation,

$$\ddot{s} = \frac{kL^2}{m}s''. \tag{9}$$

We seek standing wave solutions of angular frequency ω,

$$s = g(x)\sin\omega t. \tag{10}$$

Inserting this in eq. (9) we find,

$$g'' = -\frac{m\,\omega^2}{kL^2}g, \tag{11}$$

which is solved by,

$$g = a\sin\left(\sqrt{\frac{m}{k}}\frac{\omega\,x}{L}\right), \tag{12}$$

[1] See sec. 2.4 for the next approximation.

[2] Strictly, eq. (6) holds only when the entire spring is at rest, so this analysis is not valid for $\dot{s}$ large compared to the speed of light.

where a is a constant, noting that the center of mass must remain fixed at $x = 0$, which requires the displacements to be antisymmetric in x.

The boundary conditions at the ends of the bar, $x = \pm L/2$, are that the stretch is zero there,

$$s'(\pm L/2, t) = 0, \tag{13}$$

which implies that,

$$\sqrt{\frac{m}{k}}\frac{\omega}{2} = \frac{(2n+1)\pi}{2}\,, \tag{14}$$

for n an integer. We will consider only the lowest mode of oscillation, $n = 0$, for which the angular frequency of oscillation is,

$$\omega = \pi\sqrt{\frac{k}{m}}\,, \tag{15}$$

and the standing waveform is,

$$s(x,t) = a \sin\frac{\pi x}{L} \sin\omega t. \tag{16}$$

The internal force F of eq. (8) is,

$$F(x,t) = kLs' = \pi a k \cos\frac{\pi x}{L} \sin\omega t, \tag{17}$$

where positive F implies that the element dx is under tension. The velocity $\dot{s}$ is,

$$\dot{s}(x,t) = a\,\omega \sin\frac{\pi x}{L} \cos\omega t. \tag{18}$$

2.2 Stress-Energy-Momentum Tensor

As discussed in sec. 2.3.1 of [1], keeping terms of order v^2/c^2 in the stress tensor of a mechanical system leads to inconsistencies, so we restrict the analysis here to order v/c, *i.e.*, order $\dot{s}/c$ and s'/c.

In the rest frame (the * frame) of an element dx of the bar/spring the stress-energy-momentum tensor has the form,

$$T^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho^\star c^2 & & \mathbf{0} & \\ \hline & -F/A & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right), \tag{19}$$

where the effective mass density $\rho^\star$ is normalized to the unstretched volume $A\,dx$, and includes a contribution from the elastic energy of the stretched segment (whose spring constant is kL/dx, recalling eq. (8)),

$$\rho^\star = \frac{m}{AL} + \frac{1}{c^2}\frac{1}{A\,dx}\frac{1}{2}\frac{kL}{dx}\Delta x^2 = \frac{m}{AL} + \frac{kLs'^2}{2Ac^2}\,, \tag{20}$$

recalling eq. (5).

In the lab frame the segment has velocity $\dot{s}$ with Lorentz factor,

$$\gamma(x) = \frac{1}{\sqrt{1 - \dot{s}^2/c^2}}\,, \tag{21}$$

and the Lorentz transformation from the $\star$ frame of the segment to the lab frame is,

$$\mathsf{L}^{\mu\nu}(x) = \left(\begin{array}{c|ccc} \gamma & \gamma\dot{s}/c & 0 & 0 \\ \hline \gamma\dot{s}/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{22}$$

Hence, the stress-energy-momentum tensor in the lab frame is,

$$T^{\mu\nu} = (\mathsf{L}T^\star\mathsf{L})^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2(\rho^\star c^2 - \dot{s}F/Ac^2) & \gamma^2\dot{s}(\rho^\star c^2 - F/A)/c & 0 & 0 \\ \hline \gamma^2\dot{s}(\rho^\star - F/A)/c & \gamma^2(\dot{s}^2\rho^\star - F/A) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{23}$$

As a check, we evaulate the 4-force density (2) in the interior of the compresses rod, where it should be zero.

$$\begin{aligned} f^0 &= \partial_0 T^{00} + \partial_i T^{0i} = \frac{\partial T^{00}}{\partial ct} + \frac{\partial T^{0x}}{\partial x} = -\frac{\gamma^2\ddot{s}F}{Ac^3} + \gamma^2\dot{s}'\left(\rho^\star c - \frac{F}{Ac}\right) \\ &= \frac{\gamma^2 a\,\omega^2 F}{Ac^3}\sin\frac{\pi x}{L}\sin\omega t + \frac{\pi\gamma^2 a\,\omega}{L}\left(\rho^\star c - \frac{F}{Ac}\right)\cos\frac{\pi x}{L}\cos\omega t \end{aligned} \tag{24}$$

2.3 "Hidden" Momentum

2.3.1 Entire Spring

The momentum density $\mathbf{p}$ in the oscillating bar/spring is, recalling eqs. (10) and (20),

$$\begin{aligned} \mathbf{p} &= \frac{T^{0x}}{c}\,\hat{\mathbf{x}} = \gamma^2\dot{s}\left(\rho^\star - \frac{F}{Ac^2}\right)\hat{\mathbf{x}} \approx \left(1 + \frac{\dot{s}^2}{c^2}\right)\dot{s}\left(\frac{m}{AL} + \frac{kLs'^2}{2Ac^2} - \frac{kLs'}{Ac^2}\right)\hat{\mathbf{x}} \\ &\approx \frac{m}{AL}\dot{s}\left(1 - \frac{kL^2 s'}{mc^2}\right)\hat{\mathbf{x}}, \end{aligned} \tag{25}$$

where in this section we neglect terms higher than second order in s and its derivatives. The total momentum $\mathbf{P}$ is,

$$\mathbf{P} = A\int_{-L/2}^{L/2} \mathbf{p}\,dx = 0, \tag{26}$$

as $\dot{s}$ is antisymmetric in x while s' is symmetric, such that $\mathbf{p}$ is antisymmetric. Of course, the position $\mathbf{x}_{\rm cm}$ and the velocity $\mathbf{v}_{\rm cm}$ of the center of mass/energy are zero in the lab frame. If we take the boundary of the bar/spring to be its physical surface, then the velocity of the bounding surfaces corresponding to $x = \pm L/2$ is $\mathbf{v}_b = \dot{s}\,\hat{\mathbf{x}}$. Then, the boundary integral in the first form of "hidden" momentum in eq. (1) is,

$$\oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}$$
$$= \frac{AL}{2}[p_x(L/2) + p_x(-L/2) - \rho(L/2)\dot{s}(L/2) - \rho(-L/2)\dot{s}(-L/2)]\,\hat{\mathbf{x}} = 0, \qquad (27)$$

and hence the "hidden" momentum is zero,

$$\mathbf{P}_{\rm hidden} = \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = 0. \qquad (28)$$

2.3.2 The Spring for $x > 0$

Consider now the subsystem consisting of the bar/spring at $x > 0$.

The total momentum $\mathbf{P}$ is,

$$\begin{aligned}
\mathbf{P}(x > 0) &= A\int_0^{L/2} dx\,\mathbf{p} \approx \frac{m}{L}\int_0^{L/2} dx\,\dot{s}\left(1 - \frac{kL^2 s'}{mc^2}\right)\hat{\mathbf{x}} \\
&= \frac{m}{L}\int_0^{L/2} dx\,\left(a\omega\cos\omega t\sin\frac{\pi x}{L} - \frac{\pi a^2\omega kL}{4mc^2}\sin 2\omega t\sin\frac{2\pi x}{L}\right)\hat{\mathbf{x}} \\
&= \frac{m\,a\,\omega}{\pi}\cos\omega t - \frac{a^2\,\omega kL}{8c^2}\sin 2\omega t. \qquad (29)
\end{aligned}$$

recalling eq. (25). The mass density ρ is, recalling eq. (20),

$$\rho = \frac{T^{00}}{c^2} \approx \gamma^2\rho^\star \approx \frac{m}{AL}\left(1 + \frac{\dot{s}^2}{c^2}\right) + \frac{kLs'^2}{2Ac^2} = \frac{m}{AL}\left(1 + \frac{\dot{s}^2}{c^2} + \frac{kL^2 s'^2}{2mc^2}\right), \qquad (30)$$

so,

$$\begin{aligned}
M &= A\int_0^{L/2} dx\,\rho \approx \frac{m}{L}\int_0^{L/2} dx\,\left(1 + \frac{\dot{s}^2}{c^2} + \frac{kL^2 s'^2}{2mc^2}\right) \\
&= \frac{m}{L}\int_0^{L/2} dx\,\left(1 + \frac{a^2\,\omega^2}{c^2}\cos^2\omega t\sin^2\frac{\pi x}{L} + \frac{\pi^2 a^2 k}{2mc^2}\sin^2\omega t\cos^2\frac{\pi x}{L}\right) \\
&= \frac{m}{2} + \frac{\pi^2 a^2 k}{4c^2}\cos^2\omega t + \frac{\pi^2 a^2 k}{8c^2}\sin^2\omega t = \frac{m}{2} + \frac{\pi^2 a^2 k}{8c^2} + \frac{\pi^2 a^2 k}{8c^2}\cos^2\omega t, \qquad (31)
\end{aligned}$$

and,

$$\dot{M} \approx -\frac{\pi^2 a^2 k\omega}{8c^2}\sin 2\omega t. \qquad (32)$$

We can also calculate,

$$
\begin{aligned}
Mx_{\rm cm} &= A\int_0^{L/2} dx\,(x+s)\rho \approx \frac{m}{L}\int_0^{L/2} dx\,\left[x\left(1+\frac{\dot{s}^2}{c^2}+\frac{kL^2 s'^2}{2mc^2}\right)-\dot{s}\right] \\
&= \frac{m}{L}\int_0^{L/2} dx\,\left[x\left(1+\frac{a^2\omega^2}{c^2}\cos^2\omega t\sin^2\frac{\pi x}{L}+\frac{\pi^2 a^2 k}{2mc^2}\sin^2\omega t\cos^2\frac{\pi x}{L}\right)-a\,\omega\cos\omega t\sin\frac{\pi x}{L}\right] \\
&= \frac{mL}{4}-\frac{a\,\omega\, m}{\pi}\cos\omega t \qquad (33)
\end{aligned}
$$

recalling eq. (18) for $\dot{s}$. Then,

$$
M\dot{x}_{\rm cm} = \frac{d(Mx_{\rm cm})}{dt} - \frac{\dot{M}}{M}Mx_{\rm cm}, \qquad (34)
$$

where from eq. (33),

$$
\frac{d(Mx_{\rm cm})}{dt} \approx \frac{m}{L}\int_0^{L/2} x\left(\frac{2\dot{s}\ddot{s}}{c^2}+\frac{kL^2 s'\dot{s}'}{mc^2}\right) - \frac{a\,\omega^2 m}{\pi}\cos\omega t. \qquad (35)
$$

The boundary integral is,

$$
\begin{aligned}
&\oint_{\rm boundary} (\mathbf{x}-\mathbf{x}_{\rm cm})\,(\mathbf{p}-\rho\mathbf{v}_b)\cdot d\mathbf{Area} \\
&= \frac{AL}{2}\left[p_x(L/2)-\rho(L/2)\dot{s}(L/2)-p_x(0)-\rho(0)\dot{s}(0)\right]\,\hat{\mathbf{x}} \\
&\approx \frac{AL}{2}\left[\frac{m}{AL}\dot{s}\left(1-\frac{kL^2 s'}{mc^2}\right)-\frac{m}{AL}\dot{s}\right]_{L/2}\hat{\mathbf{x}} \\
&= -\left.\frac{kL^2\dot{s}s'}{2c^2}\right|_{L/2}\hat{\mathbf{x}} = 0. \qquad (36)
\end{aligned}
$$

Combining these results, the x-component of the "hidden" momentum in the bar/spring for $x>0$ is,

$$
\begin{aligned}
P_{x,\rm hidden}(x>0) &= P_x - M\dot{x}_{\rm cm} - \oint_{\rm boundary}(x-x_{\rm cm})\,(\mathbf{p}-\rho\mathbf{v}_b)\cdot d\mathbf{Area} \qquad (37) \\
&\approx \frac{m}{L}\int_0^{L/2} -\dot{s}\frac{KL^2 s'}{mc^2}\,dx - \frac{m}{L}\int_0^{L/2}(x+s)\left(\frac{2\dot{s}\ddot{s}}{c^2}+\frac{KL^2 s'\dot{s}'}{mc^2}\right)+\frac{\dot{M}}{M}Mx_{\rm cm}.
\end{aligned}
$$

The first integral is zero since $\dot{s}s' \propto \sin 2\pi x/L$. In the second integral,

$$
\frac{2\dot{s}\ddot{s}}{c^2}+\frac{kL^2 s'\dot{s}'}{mc^2} = \frac{a^2\omega^3\sin 2\omega t}{2c^2}\left(3\sin^2\frac{\pi x}{L}-1\right), \qquad (38)
$$

so, noting that,

$$
\int_0^{L/2} x\left(3\sin^2\frac{\pi x}{L}-1\right)\,dx = \frac{L^2}{16}\,, \qquad \int_0^{L/2}\sin\frac{\pi x}{L}\left(3\sin^2\frac{\pi x}{L}-1\right)\,dx = \frac{L}{\pi}\,, \qquad (39)
$$

we have,

$$P_{x,\text{hidden}}(x>0) \approx -\frac{ma^2\omega^3 \sin 2\omega t}{2c^2}\left(\frac{L}{16}+\frac{a\cos\omega t}{\pi}\right)+\frac{\dot M}{M} M x_{\text{cm}}. \qquad (40)$$

Possibly the "hidden" momentum would be zero at order $1/c^2$ if we used the relativistic form of the oscillation.

2.4 Relativistic Oscillation at Order $1/c^2$

"Hidden" momentum in electromechanical examples is always of order $1/c^2$, so we consider the oscillations of the bar/spring at this order.

The relativistic version of the equation of motion (9) is,

$$\gamma K L^2 s'' = \frac{d}{dt}\gamma m^\star \dot s = \dot\gamma m^\star \dot s + \gamma m^\star \ddot s = \gamma m^\star \ddot s\left(1+\frac{2\gamma^2\dot s^2}{c^2}\right) \approx m^\star \ddot s\left(1+\frac{5\dot s^2}{2c^2}\right), \qquad (41)$$

where,

$$\gamma = \frac{1}{\sqrt{1-\dot s^2/c^2}}, \qquad (42)$$

and the approximation holds at order $1/c^2$. We seek a perturbative solution of the form,

$$s = s_0 + s_1 + \ldots, \qquad (43)$$

where s_0 satisfies the nonrelativistic equation of motion (9) and is given by eq. (16). Then, s_1 is of order $1/c^2$, obeys the boundary condition that $s_1'(\pm L/2) = 0$, and vanishes at $x = 0$ (being a term in the lowest mode of oscillation). Using eq. (43) in eq. (37) for the "hidden" momentum at $x > 0$, the first integral is again zero, while the terms in the second integral due to s_1 are of order $1/c^4$.

Thus, the nonzero "hidden" momentum (40) remains valid at order $1/c^2$, and provides an example of "hidden" momentum in a subsystem of an all-mechanical system.

References

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