Conducting Spherical Shell with a Circular Orifice

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(September 21, 2002)

1 Problem

A conducting spherical shell with a circular orifice of half angle θ_0 is at electric potential V_0. Show that the difference between the charge densities on the inner and outer surfaces is independent of position, and estimate the ratio of the electric charge on the inner surface to that on the outer.

Correct results can be inferred from "elementary" arguments based on superposition, and more "exact" derivations can be based on Legendre polynomial expansions, Green's functions, or the method of inversion.

2 Solution

This mixed boundary value problem is taken from the classic essay by G. Green (1828) [1], where it was discussed using what are now called Green's functions. We first present an "elementary" solution, and then seek confirmation based on an expansion of the potential in a series of Legendre polynomials, and further confirmation via the methods of Green and Thomson.

2.1 Elementary Solution via Superposition

We wish to relate the problem of the shell with orifice (which we will also call a spherical bowl in case angle θ_0 is large) to the simpler problem of a complete shell that is at potential V_0 and hence has total charge $Q_0 = aV_0$ (in Gaussian units), uniform radial electric field $E_0 = Q_0/a^2 = V_0/a$ at its outer surface, zero electric field in the interior, and uniform surface charge density $\sigma_0 = Q_0/4\pi a^2 = V_0/4\pi a$.

Following a well-known argument, to a first approximation the configuration of the shell with orifice is equivalent to removing a spherical cap from the complete shell, keeping surface charge density σ_0 on that cap. The electric field above and below the spherical cap due to this charge density is $\pm 2\pi\sigma_0 = \pm E_0/2$. Subtracting this field from that of the initially complete sphere, we find the approximate field in the orifice to be $E_0/2$.

The shell with orifice has charge density σ_+ on its outer surface and σ_- on its inner surface, with total charges $Q_\pm$ on these surfaces. All of the field lines that emanate from the charge distribution σ_- on the inner surface pass through the orifice of area $A_{\rm orifice}$, so the average electric field at the orifice is the same as if charge Q_- were distributed across the orifice; namely $E_{\rm orifice} \approx 4\pi Q_-/A_{\rm orifice}$. Equating this to $E_0/2 = Q_0/2a^2$, we find,

$$Q_- \approx Q_0 \frac{A_{\rm orifice}}{8\pi a^2} \approx Q_0 \frac{\theta_0^2}{8} \approx aV_0 \frac{\theta_0^2}{8}\,, \tag{1}$$

where the latter forms hold for a circular orifice of half angle θ_0, whose area is $A_{\rm orifice} \approx \pi a^2 \theta_0^2$.

To a first approximation, the charge Q_+ on the outer surface of the shell with orifice remains Q_0. Hence, we estimate that,

$$\frac{Q_-}{Q_+} \approx \frac{A_{\rm orifice}}{8\pi a^2} \approx \frac{\theta_0^2}{8} \,. \tag{2}$$

We can write the charge density on the outer surface of the conducting shell with orifice as $\sigma_+ = \sigma_0 + \Delta\sigma_+$, where $\Delta\sigma_+ \ll \sigma_0$ (except for very small values of distance s from the edge of the orifice where we expect that σ_+ varies as $1/\sqrt{s}$. Similarly, the charge density σ_-, whose integral is $Q_- \ll Q_0$, obeys $\sigma_- \ll \sigma_0$ with the possible exception of very small values of distance s.

The electric fields due the densities σ_0, $\Delta\sigma_+$ and σ_- on the conducting shell with orifice must sum to zero inside the material of the shell. The electric field due to charge density σ_0 is zero in the interior a complete shell, and remains near zero when that density exists on the shell except in the orifice. Hence, the electric field inside the conducting material of the shell due to the charge densities $\Delta\sigma_+$ and σ_- must sum to zero, or very nearly so. The usual argument based on Gaussian pillboxes tells us that the electric field just inside charge density $\Delta\sigma_+$ is $-4\pi\Delta\sigma_+$, while that just outside charge density σ_- is $4\pi\sigma_-$. Thus, we conclude that $\Delta\sigma_+ \approx \sigma_-$ for all points not close to the edge of the orifice, and we might suppose this relation holds there as well. Expressing this conclusion as a relation between the densities σ_+ and σ_-, we have,

$$\sigma_+ - \sigma_- = \sigma_0 + \Delta\sigma_+ - \sigma_- \approx \sigma_0 = \frac{V_0}{4\pi a} \,. \tag{3}$$

Assuming eq. (3) to be correct, we can deduce the first correction to the total charge Q_+ on the outer surface of the shell with orifice. Namely,

$$Q_+ \approx Q_0 + Q_- - Q_{\rm cap} = Q_0 - Q_- = Q_0 \left(1 - \frac{\theta_0^2}{8}\right), \tag{4}$$

and hence,

$$Q_+ + Q_- \approx Q_0 = aV_0. \tag{5}$$

We infer that if a small hole could be created spontaneously in a conducting spherical shell, the charge originally at the place of the hole would redistribute itself half on the outer and half on the inner surface of the shell with orifice. The value of the electric potential of the shell would not change during this process.

Since the above model does not necessarily reproduce all details of the actual charge distribution on the shell with orifice, it is useful to confirm the results with alternative analyses.

2.2 Solution via Legendre Series

By expanding the potential in a Legendre series and applying boundary conditions at the surface of the shell, we find "exact" confirmation of the result (3) and approximate confirmation of eq. (2).

We work in a spherical coordinate system (r, θ, φ) in which the center of the sphere of radius a is at the origin, and the $+z$ axis passes through the center of the circular orifice. Once we have an expression for the azimuthally symmetric potential $V(r, \theta)$, we obtain the electric field as $\mathbf{E} = -\nabla V$ and the surface charge densities $\sigma_\pm$ as,

$$\sigma_\pm(\theta) = \pm \frac{E_r(r = a_\pm, \theta)}{4\pi} = \mp \frac{1}{4\pi} \frac{\partial V(r = a_\pm, \theta)}{\partial r}\,, \qquad (6)$$

in Gaussian units. The charge density on the outer surface is labeled σ_+.

As the potential $V(r, \theta)$ is finite at the origin and at large r, an appropriate Legendre-series expansion is,

$$V(r < a, \theta) = \sum_n A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta), \qquad (7)$$

$$V(r > a, \theta) = \sum_n A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta), \qquad (8)$$

which is continuous across the conducting spherical shell at $r = a$. Hence, we obtain the following series expansions for the charges densities (6),

$$\sigma_+(\theta) = \frac{1}{4\pi a} \sum_n (n+1) A_n P_n(\cos\theta), \qquad (9)$$

$$\sigma_-(\theta) = \frac{1}{4\pi a} \sum_n n A_n P_n(\cos\theta). \qquad (10)$$

The potential V_0 on the conductor, whose coordinates are $(r = a, \theta_0 \le \theta \le \pi)$, provides a partial boundary condition at $r = a$,

$$V_0 = \sum_n A_n P_n(\cos\theta < \cos\theta_0), \qquad (11)$$

With this condition we can confirm relation (3) between the charge densities (9) and (10),

$$\sigma_+ - \sigma_- = \frac{1}{4\pi a} \sum_n A_n P_n(\cos\theta < \cos\theta_0) = \frac{V_0}{4\pi a}\,. \qquad (12)$$

Indeed, whatever the shape of the orifice(s) in the conducting shell, condition (11) holds for the remaining conducting region, and hence relation (12) holds also [2].

The boundary condition for the rest of the spherical shell is that the radial component $E_r = -\partial V(r = a, \theta < \theta_0)/\partial r$ of the electric field is continuous there, which leads to,

$$0 = \sum_n (2n+1) A_n P_n(\cos\theta > \cos\theta_0). \qquad (13)$$

We can combine the partial conditions (11) and (13) into a condition over the whole range of θ,

$$\sum_n A_n P_n(\cos\theta) = \begin{cases} -2 \sum_n n A_n P_n(\cos\theta) & (\theta < \theta_0), \\ V_0 & (\theta > \theta_0). \end{cases} \qquad (14)$$

For an approximate solution, we keep terms only up to the largest order n for which $P_n(\cos\theta_0) \approx 1$, in which case we can obtain simple analytic results via the usual method of evaluation of coefficient A_m by multiplying the boundary condition by P_m and integrating over $\cos\theta$. Since the Legendre polynomial P_n has $n-1$ zeroes over the interval $0 < \theta < \pi$, we must have $\theta_0 \lesssim \pi/n$, or $n \lesssim \pi/\theta_0$. Thus, we approximate the condition (14) as,

$$\sum_{n=0}^{\pi/\theta_0} A_n P_n(\cos\theta) \approx \begin{cases} -2\sum_{n=0}^{\pi/\theta_0} nA_n & (\theta < \theta_0), \\ V_0 P_0 & (\theta > \theta_0), \end{cases} \tag{15}$$

using $P_0 = 1$. We set $A_n = 0$ for $n > \pi/\theta_0$.

To isolate A_m for $1 < m < \pi/\theta_0$ we multiply eq. (15) by P_m and integrate from -1 to 1 with respect to $d\cos\theta$ to find,

$$\begin{aligned} A_m &= \frac{2m+1}{2}\left(-2\sum_{n=0}^{\pi/\theta_0} nA_n \int_{\cos\theta_0}^{1} P_m \, d\cos\theta + V_0 \int_{-1}^{\cos\theta_0} P_0 P_m \, d\cos\theta\right) \\ &\approx \frac{2m+1}{2}\left(-2\sum_{n=0}^{\pi/\theta_0} nA_n \int_{\cos\theta_0}^{1} d\cos\theta + V_0 \int_{-1}^{1} P_0 P_m \, d\cos\theta - V_0 \int_{\cos\theta_0}^{1} d\cos\theta\right) \\ &= V_0\delta_{0m} - \frac{2m+1}{2}\left(2\sum_{n=0}^{\pi/\theta_0} nA_n + V_0\right)(1-\cos\theta_0) \\ &\approx V_0\delta_{0m} - \frac{(2m+1)\theta_0^2}{4}\left(2\sum_{n=0}^{\pi/\theta_0} nA_n + V_0\right), \end{aligned} \tag{16}$$

recalling that $P_m \approx 1$ for $\cos\theta > \cos\theta_0$. To complete the evaluation of A_m we need the sum $\sum_{n=0}^{\pi/\theta_0} nA_n$, which we find from eq. (16) to obey,

$$\begin{aligned} \sum_{m=0}^{\pi/\theta_0} mA_m &= -\sum_{m=0}^{\pi/\theta_0} m(2m+1)\frac{\theta_0^2}{4}\left(2\sum_{n=0}^{\pi/\theta_0} nA_n + V_0\right) \\ &\approx -\frac{\pi^3}{6\theta_0}\left(2\sum_{n=0}^{\pi/\theta_0} nA_n + V_0\right), \end{aligned} \tag{17}$$

since,

$$\sum_{m=0}^{\pi/\theta_0} m(2m+1) \approx \int_0^{\pi/\theta_0} x^2 \, dx = \frac{2}{3}(\pi/\theta_0)^3. \tag{18}$$

Solving eq. (17) we have,

$$\sum_{n=0}^{\pi/\theta_0} nA_n \approx -V_0 \frac{\pi^3/6\theta_0}{1+\pi^3/3\theta_0}, \tag{19}$$

and,

$$2\sum_{n=0}^{\pi/\theta_0} nA_n + V_0 \approx V_0 \left(1 - \frac{\pi^3/3\theta_0}{1+\pi^3/3\theta_0}\right) = V_0 \frac{1}{1+\pi^3/3\theta_0} \approx \frac{3\theta_0}{\pi^3} V_0. \tag{20}$$

In sum, we approximate the Fourier coefficients as,

$$A_n \approx \begin{cases} V_0\delta_{0n} - \frac{(2n+1)3\theta_0^3}{4\pi^3} V_0 & (n < \pi/\theta_0), \\ 0 & (n > \pi/\theta_0). \end{cases} \tag{21}$$

The potential on the axis $\theta = 0$ inside the spherical shell is,

$$V(r < a, 0) = \sum_{n=0}^{\pi/\theta_0} A_n \left(\frac{r}{a}\right)^n P_n(1) \approx V_0 \left(1 - \sum_0^{\pi/\theta_0} \frac{(2n+1)3\theta_0^3}{4\pi^3} \left(\frac{r}{a}\right)^n\right), \tag{22}$$

which drops from V_0 at the center of the sphere to,

$$V(a, 0) \approx V_0 \left(1 - \sum_0^{\pi/\theta_0} \frac{(2n+1)3\theta_0^3}{4\pi^3}\right) \approx V_0 \left(1 - \frac{3\theta_0}{4\pi}\right) \tag{23}$$

at the center of the circular orifice. The equipotential surfaces, which would be spherical in the absence of the orifice, are deflected towards the orifice, and in some cases the deflection forms a "bubble" that passes through the orifice into the interior of the spherical shell. Very similar behavior occurs for the case of a circular hole in a conducting plane, as illustrated in Fig. 3.14 of [3]. Associated with this behavior is the presence of some charge on the interior surface of the spherical shell.

The charge density σ_- on the inner surface of the spherical shell follows from eq. (10),

$$\sigma_- \approx \frac{1}{4\pi a} \sum_{n=0}^{\pi/\theta_0} nA_n P_n(\cos\theta) \approx \frac{V_0}{4\pi a} \sum_{n=0}^{\pi/\theta_0} n \frac{(2n+1)3\theta_0^3}{4\pi^3} P_n(\cos\theta). \tag{24}$$

Although there is no physical charge in the region $\theta < \theta_0$, eq. (24) is formally defined there, and $\int_{-1}^{1} \sigma_-\, d\cos\theta = 0$. This permits the charge Q_- on the inner surface of the spherical shell with a circular orifice of area $A_{\rm orifice} \approx \pi a^2 \theta_0^2$ to be calculated as,

$$\begin{aligned} Q_- &= \int_{-1}^{\cos\theta_0} 2\pi a^2 \sigma_-\, d\cos\theta = 2\pi a^2 \int_{-1}^{1} \sigma_-\, d\cos\theta - 2\pi a^2 \int_{\cos\theta_0}^{1} \sigma_-\, d\cos\theta \\ &\approx \frac{V_0}{4\pi a} \sum_{n=0}^{\pi/\theta_0} n \frac{(2n+1)3\theta_0^3}{4\pi^3} A_{\rm orifice} \approx \frac{V_0}{8\pi a} A_{\rm orifice} \approx aV_0 \frac{\theta_0^2}{8}. \end{aligned} \tag{25}$$

We see that the result $Q_- \approx V_0 A_{\rm orifice}/8\pi a$ holds for a small orifice of any shape [2].

The charge density σ_+ on the outer surface to that on the inner surface by eq. (12), so the charge Q_+ on the outer surface is given by,

$$\begin{aligned} Q_+ &= Q_- + \int_{-1}^{\cos\theta_0} 2\pi a^2 \frac{V_0}{4\pi a}\, d\cos\theta \\ &\approx aV_0 \left(\frac{\theta_0^2}{8} + \frac{1+\cos\theta_0}{2}\right) \approx aV_0 \left(\frac{\theta_0^2}{8} + 1 - \frac{\theta_0^2}{4}\right) \approx aV_0 \left(1 - \frac{\theta_0^2}{8}\right). \end{aligned} \tag{26}$$

The total charge on the spherical shell is,

$$Q_+ + Q_- \approx aV_0, \tag{27}$$

which is the same as the charge on a complete conducting sphere of radius a at potential V_0.

The ratio of the charge on the inner surface to that on the outer surface is,

$$\frac{Q_-}{Q_+} \approx \frac{A_{\rm orifice}}{2A_{\rm sphere}} \approx \frac{\theta_0^2}{8}. \tag{28}$$

The approximate result (28) confirms eq. (2), but is based on the truncated set of Fourier coefficients (21). Hence, additional confirmation is still desirable.

2.3 Solution via Green's Functions

Green [1] introduced what are now called Green's functions to provide a (then) new derivation of Poisson's integral for a sphere of radius a, namely that the potential at an interior point (b, θ, ϕ) is given in terms of the potential on the surface S as,

$$V(r, \theta, \phi) = \frac{a^2 - b^2}{4\pi a} \int \frac{V(r' = a, \theta', \phi')}{R^3} dS', \tag{29}$$

where R is the distance from the interior point to area element on the surface [4]. He then considered the case of a shell at potential V_0 with a circular orifice of half angle $\theta_0 \ll 1$, and imagined the effect of the orifice is the same as the superposition of the spherical cap of half angle θ_0 on a complete shell and a conducting disk of radius $a\theta_0$ of charge opposite to that of the cap. After some very clever analysis he found the charge densities σ_+ and σ_- to be related by eq. (3), with σ_- given by,

$$\sigma_-(\theta) \approx \frac{V_0}{4\pi^2 a} \left(\frac{\sqrt{2}\theta_0/2}{\sqrt{1 - \cos\theta}} - \tan^{-1} \frac{\sqrt{2}\theta_0/2}{\sqrt{1 - \cos\theta}} \right). \tag{30}$$

The total charge Q_- on the inner surface is therefore,

$$\begin{aligned}
Q_- &= \int_{-1}^{\cos\theta_0} 2\pi a^2 \sigma_- \, d\cos\theta \\
&= \frac{aV_0}{2\pi} \int_{-1}^{\cos\theta_0} \left(\frac{\sqrt{2}\theta_0/2}{\sqrt{1 - \cos\theta}} - \tan^{-1} \frac{\sqrt{2}\theta_0/2}{\sqrt{1 - \cos\theta}} \right) d\cos\theta \\
&= \frac{aV_0}{2\pi} \int_{\theta_0/2}^{1} \left(x - \tan^{-1} x \right) \frac{\theta_0^2}{x^3} \, dx \\
&= \frac{aV_0\theta_0^2}{2\pi} \left[-\frac{1}{x} + \frac{1}{2}\left(1 + \frac{1}{x^2}\right) \tan^{-1} x + \frac{1}{2x} \right]_{\theta_0/2}^{1} \\
&\approx \frac{aV_0\theta_0^2}{2\pi} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{aV_0\theta_0^2}{8} \left(1 - \frac{2}{\pi} \right) \approx \frac{aV_0\theta_0^2}{22}.
\end{aligned} \tag{31}$$

As we will see in the following section, the "exact" form for σ_-, which in effect includes terms to all orders in θ_0, changes the factor $\sqrt{1-\cos\theta}$ to $\sqrt{\cos\theta_0 - \cos\theta}$. This has the effect of shifting the upper limit of the x integration from 1 to ∞, which in turn changes the factor $\pi/4 - 1/2$ to $\pi/4$. It is surprising that higher-order terms could change the result by a factor of 3, which shows the delicacy of Green's approximations.

For completeness, we review Green's derivation of eq. (30).

We write the total charge density σ at radius $r = a$ as,

$$\sigma(\theta) = \sigma_+ + \sigma_- = \sigma_0 + \Delta\sigma, \tag{32}$$

where $\sigma_0 = V_0/4\pi a$ relates the uniform charge distribution σ_0 on a complete shell that is at potential V_0, which is also the potential of the conducting shell with orifice. In that orifice, the total charge distribution vanishes, so $\Delta\sigma(\theta < \theta_0) = -\sigma_0$.

The potential $V(r,\theta,\varphi)$ in the interior of the shell due to charge distribution (32) can be written,

$$V(r,\theta,\varphi) = V_0 + \Delta V, \tag{33}$$

Thus, the potential ΔV is due to the charge distribution $\Delta\sigma$. Since the potential of the shell with orifice is V_0, we obtain the partial boundary condition that $\Delta V(r = a, \theta > \theta_0) = 0$.

To complete a solution, we need to determine the charge distribution $\Delta\sigma$ that is induced on a grounded, conducting shell with orifice of half angle θ_0 due to a uniform charge distribution $-\sigma_0$ on the spherical cap of the orifice.

This solution could be readily implemented if we knew the charge distribution induced on a grounded, conducting shell with orifice by a unit charge at an arbitrary point on the spherical cap of the orifice. While this approach is now associated with the name of Green, he did not in fact use this approach in 1828, and it was W. Thomson who first used this method to provide a solution valid for any angle θ_0, as described in to following section.

Rather, the approach of Green in his Essay [1] was to consider that part of the potential ΔV due to the uniform charge distribution $\Delta\sigma = -\sigma_0$ on the spherical cap $(r = a, \theta, \theta_0)$ to be equivalent to that of a uniform flat disk of radius $a\theta_0 \ll a$ which carries charge density $-\sigma_0$. By the usual argument using a Gaussian pillbox, the charge density on that disk is related to the electric field and the potential by,

$$-\frac{\partial \Delta V(r = a_-)}{\partial r} = \Delta E_r(r = a^-) = 2\pi\sigma_0. \tag{34}$$

Thus we have a mixed boundary value problem, with knowledge of the potential over part of the boundary and knowledge of the normal derivative of the potential.

We can apply Poisson's integral (29) to the potential ΔV, in which case we learn that

$$\Delta V(r,\theta,\phi) = \frac{a^2 - b^2}{4\pi a} \int_{\rm cap} \frac{\Delta V(r' = a, \theta', \phi')}{R^3} dS', \tag{35}$$

since $\Delta V = 0$ on the conducting shell.

(More to come...)

2.4 Solution by Inversion for a Spherical Bowl

Apparently, W. Thomson (Lord Kelvin) first deduced the surface charge distribution on a conducting spherical shell with a circular orifice of any size (henceforth called a spherical bowl) in 1847, using his method of inversion [5] starting from the charge density (38) on the surface of a thin conducting disk, but he published the result only in 1872 [6]. In his discussion of Thomson's calculation [7], Maxwell seemed unaware of Green's prior work. Thomson's solution by inversion for the spherical bowl is also discussed by Jeans [8].

In the notation of the present paper, Thomson found the charge density σ_- on the inner surface of the spherical bowl to be,

$$\sigma_-(\theta) = \frac{V_0}{4\pi^2 a} \left(\sqrt{\frac{1-\cos\theta_0}{\cos\theta_0 - \cos\theta}} - \tan^{-1} \sqrt{\frac{1-\cos\theta_0}{\cos\theta_0 - \cos\theta}} \right). \tag{36}$$

We see that Green's result (30) is the small-angle limit of eq. (36).

The total charge Q_- on the inner surface follows from eq. (36) as,

$$\begin{aligned} Q_- &= \int_{-1}^{\cos\theta_0} 2\pi a^2 \sigma_- \, d\cos\theta \\ &= \frac{aV_0}{2\pi} \int_{-1}^{\cos\theta_0} \left(\sqrt{\frac{1-\cos\theta_0}{\cos\theta_0 - \cos\theta}} - \tan^{-1} \sqrt{\frac{1-\cos\theta_0}{\cos\theta_0 - \cos\theta}} \right) d\cos\theta \\ &= \frac{aV_0}{2\pi} \int_{\sqrt{1-\cos\theta_0}/\sqrt{1+\cos\theta_0}}^{\infty} \left(x - \tan^{-1} x \right) \frac{4\sin^2(\theta_0/2)}{x^3} \, dx \\ &= \frac{2aV_0 \sin^2(\theta_0/2)}{\pi} \left[-\frac{1}{x} + \frac{1}{2}\left(1 + \frac{1}{x^2}\right) \tan^{-1} x + \frac{1}{2x} \right]_{\sqrt{1-\cos\theta_0}/\sqrt{1+\cos\theta_0}}^{\infty} \\ &= \frac{aV_0}{2\pi} \left[\pi \sin^2(\theta_0/2) + \sin\theta_0 - \theta_0 \right] \\ &\approx \frac{aV_0\theta_0^2}{8}, \end{aligned} \tag{37}$$

where the approximation holds for small θ_0. This derivation is the firmest evidence we offer in support of the "elementary" result (2).

As angle θ_0 approaches π, the spherical bowl approaches a thin conducting disk of radius $b = a(\pi - \theta_0) \ll a$. In this limit the charge density (36) becomes,

$$\sigma_-(r) = \frac{V_0}{2\pi^2 \sqrt{b^2 - r^2}}, \qquad \text{and} \qquad Q_- = \frac{bV_0}{2\pi}, \tag{38}$$

which is the well-known result for the charge density on one side of a conducting disk, r being the distance from the center of the disk (see the Appendix for a highly geometric derivation of eq. (38)). Also in this limit, the charge distribution $\sigma_0 = V_0/4\pi a$ is small compared to σ_-. Hence, the charge distribution on the other side of the thin conducting disk is $\sigma_+ = \sigma_0 + \sigma_- = \sigma_-$. As expected, the charge distribution is the same on both sides of a conducting disk.

We now present details of a derivation leading to eq. (36), following Thomson [6]. The starting point is based on the discussion of eqs. (32)-(33), that the difference between the charge distribution on a conducting spherical bowl and the uniform charge distribution σ_0 on a complete sphere at the same potential is the same as that induced on a grounded spherical bowl by charge distribution $-\sigma_0$ on the spherical cap that completes the spherical bowl.

Thomson solved this problem by first finding the charge distribution induced on a grounded spherical bowl by a unit charge at an arbitrary point on the spherical cap, using his method of inversion.

We begin with a conducting disk of radius b at potential V_0. Then, from eq. (38) and the geometry shown in Fig. 1, the charge density $\sigma_{\rm disk}$ on each side of the disk can be written,

$$\sigma_{\rm disk} = \frac{V_0}{2\pi^2\sqrt{(b+r)(b-r)}} = \frac{V_0}{2\pi^2\sqrt{EP \cdot PD}} = \frac{V_0}{2\pi^2\sqrt{AP \cdot PB}}\,, \tag{39}$$

where APB is any chord that passes through point P and ECD is the diameter that contains P.

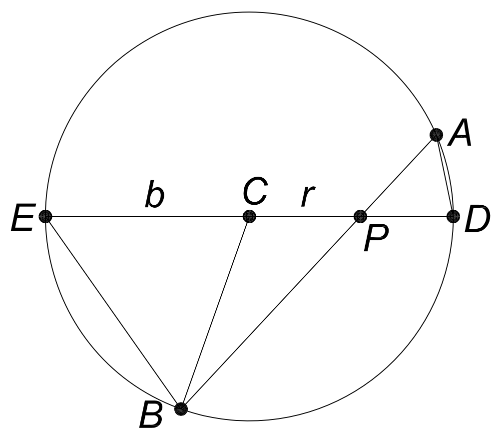

Figure 1: Point P on diameter ECD is at distance r from the center of a circle of radius b. APB is any chord that contains P. Triangles ADP and BEP are similar since $\angle DAP = \angle DAB = (\angle BCD)/2 = \angle BED = \angle BEP$. Hence $EP/PB = AP/PD$, and $AP \cdot PB = EP \cdot PD = (b+r)(b-r) = b^2 - r^2$.

Next, we invert the disk with respect to a sphere of radius s whose center O is not in the plane of the disk, as shown in Fig. 2. The plane that contains the disk inverts into a sphere that passes through the center of inversion O, and the disk inverts into a spherical bowl the occupies part of that sphere. The distance OP from the center of inversion to a point P on the bowl is related to the distance OP' of the inverse point on the disk by

$$OP \cdot OP' = s^2. \tag{40}$$

The principle of the method of inversion is that if we relate the charge $dq = \sigma_{\rm bowl} dS$ in area element dS about point P on the bowl to charge $dq' = \sigma_{\rm disk}(V_0) dS'$ in element dS' about point P' on the conducting disk whose potential is V_0 according to $dq = -dq'(OP/s)$,

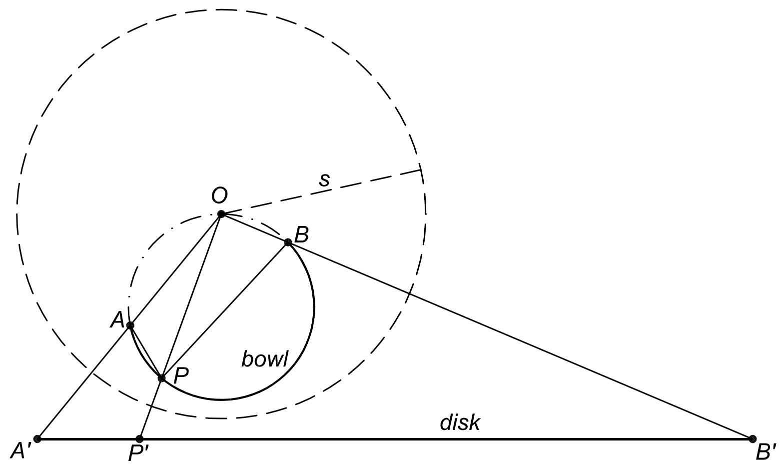

Figure 2: The inverse of a disk with respect to a sphere of radius s centered at point O is a spherical bowl that lies on a sphere that contains point O. The plane $OA'B'$ is not necessarily perpendicular to the plane of the disk, and in general $A'B'$ is not a diameter of the disk, but only a chord.

then the charge distribution on the spherical bowl is that for the case that the bowl is a grounded conductor in the presence of charge sV_0 at point O.

Since the conducting disk has the same charge distribution $\sigma_{\rm disk}$ on both of its sides, the method of inversion tells us that both the inner and outer surfaces of a grounded, conducting spherical bowl have the same charge distributions induced by a charge placed anywhere on the spherical cap that completes the bowl. As the case of a conducting spherical bowl at potential V_0 is the superposition of a complete shell of charge density $\sigma_0 = V_0/4\pi a$ and a grounded conducting bowl when charge density $-\sigma_0$ covers the spherical cap, we have another confirmation of relation (3) that $\sigma_+ - \sigma_- = \sigma_0$.

Area element dS about point P on the bowl is the inverse of element dS' about P' on the disk. Hence,

$$\frac{dS}{dS'} = \left(\frac{OP}{OP'}\right)^2 = \frac{(OP)^4}{s^4}\,, \tag{41}$$

using eq. (39).

Following the spirit of Green, we desire the charge distribution at point P on each side of a grounded conducting bowl induced by unit charge at point O, which we obtain from eqs. (40)-(41) as,

$$\begin{aligned}\sigma_{\rm bowl}(P, V=0, q_O=1) &= \frac{dq}{dS} = -\frac{1}{sV_0}dq'\frac{OP}{s}\frac{s^4}{dS'\cdot(OP)^4} = -\frac{\sigma_{\rm disk}(V_0)}{V_0}\frac{s^2}{(OP)^3} \\ &= -\frac{s^2}{2\pi^2(OP)^3\sqrt{A'P'\cdot P'B'}}\,. \end{aligned} \tag{42}$$

Expression (42) will be more useful if we can replace distances $A'P'$ and $P'B'$ measured on the disk by quantities related to the spherical bowl. Referring to Fig. 2, we see that triangles APO and $A'P'O$ are similar, so that,

$$\frac{A'P'}{AP} = \frac{OA'}{OP} = \frac{s^2}{OA \cdot OP}\,, \tag{43}$$

since A and A' are inverse points with respect to the sphere of radius s about O. Likewise, similar triangles BPO and $B'P'O$ lead to ,

$$\frac{P'B'}{PB} = \frac{OB'}{OP} = \frac{s^2}{OB \cdot OP}\,. \tag{44}$$

Thus,

$$A'P' \cdot P'B' = \frac{OB'}{OP} = \frac{s^4}{(OP)^2} \frac{AP \cdot PB}{OA \cdot OB}\,. \tag{45}$$

If we keep points O and P' fixed then point P' is fixed also, but we can vary the chord $A'P'B'$ and consequently the location of points A and B as well. Under such variation the ratio $s^4/(OP)^2$ remains constant, and the product $A'P' \cdot P'B'$ also remains constant according to the logic of eq. (39) and Fig. 1. Hence, we obtain the peculiar theorem that the ratio $(AP{\cdot}PB)/(OA{\cdot}OB)$ is also constant during such variation, which result Thomson attributes to Liouville.

In any case, we see that eq. (42) for the charge density at point P can also be written,

$$\sigma_{\rm bowl}(P, V = 0, q_O = 1) = -\frac{1}{2\pi^2 (OP)^2} \frac{\sqrt{OA \cdot OB}}{\sqrt{AP \cdot PB}}\,. \tag{46}$$

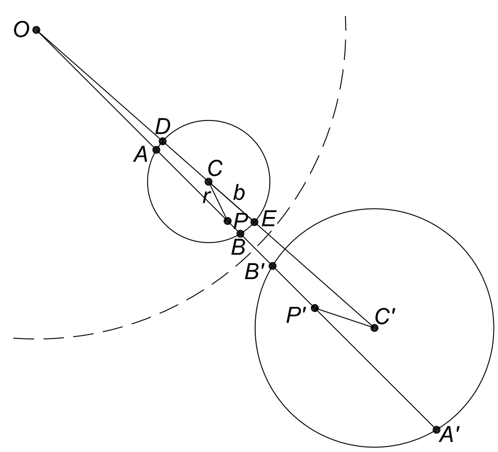

Figure 3: When the center of inversion, O, lies in the plane of the original disk whose center is C', then the inverse of that disk is another disk with center at C in the same plane. While point C' is not the inverse of point C [9], all other primed points are the inverses of their unprimed partner. When the chord $A'P'B'$ lies along the line OP', triangles OBD and OAE are similar, and hence $OA \cdot OB = OD \cdot OE = (a - b)(a + b) = a^2 - b^2$.

This result is remarkable in that it does not depend on the radius s of the sphere of inversion nor (directly) on the radius of the spherical bowl. In particular, we can take point O to lie in the plane of the original disk and outside its bounding circle, in which case the spherical bowl degenerates into another disk (which lies between point O and the original disk). Hence, charge distribution (46) also holds for the case of a grounded, conducting disk in the presence of unit charge at point O. We write the distance from point O to the center of the grounded disk as a, the radius of the disk as b, the distance from the center of the disk to point P as r, and the distance OP as R. We have seen that the ratio $(OA \cdot OB)/(AP \cdot PB)$ is independent of the choice of chord $A'P'B'$ for fixed points O, P and P'. We can conveniently evaluate this ratio of the case that the chord $A'P'B$ lies along the line OP', as shown in Fig 3. By the argument in the caption of Fig. 1, $AP \cdot PB = b^2 - r^2$, and by a very similar argument $OA \cdot OB = a^2 - b^2$. Thus, eq. (46) tells us that the charge distribution induced on each side of a grounded, conducting circular disk of radius b by unit charge in the plane of the disk at distance $a > b$ is,

$$\sigma_{\rm disk}(r) = -\frac{1}{2\pi^2 R^2} \frac{\sqrt{a^2 - b^2}}{\sqrt{b^2 - r^2}}\,, \qquad (47)$$

where R is the distance between the exterior unit charge and the point of interest on the disk. This result was first given by Green, p. 181 of [1].

We can now complete the calculation of the charge distribution induced on a grounded, conducting spherical bowl of radius a by uniform charge distribution $-\sigma_0$ on the spherical cap that completes the bowl. We will calculate the distribution $\sigma_-(\theta)$ on the inner surface of the bowl, and obtain the distribution σ_+ on the outer surface via relation (3). Integrating eq.(46) over points $O = (a, \theta' < \theta_0, \varphi')$ on the spherical cap, for point $P = (a, \theta > \theta_0, 0)$ we have

$$\sigma_-(\theta > \theta_0, V = 0, \sigma_{\rm cap} = -\sigma_0) = \int_{\cos\theta_0}^{1} a^2 \; d\cos\theta' \int_0^{2\pi} d\varphi' \frac{\sigma_0}{2\pi^2 (OP)^2} \frac{\sqrt{OA \cdot OB}}{\sqrt{AP \cdot PB}}\,. \qquad (48)$$

Points A, B, O and P in the integrand of eq. (48) all lie in the same plane, but according to the theorem of Liouville proved above, we are free to chose for this any plane that contains points O and P. Points A and B are then the intersection of this plane with the rim of the spherical bowl. Thomson suggests that we always choose the plane $ABOP$ to contain the "south pole" S of the bowl, *i.e.*, the intersection of the axis of the bowl with its surface, as shown in Fig. 4.

We introduce angles α and β as shown in Fig. 4 so that the lengths of lines AS, PS, AP and PB are related by,

$$\begin{aligned} AS &= 2d \sin\alpha, & (49) \\ PS &= 2d \sin\beta, & (50) \\ AP &= 2d \sin(\alpha - \beta), & (51) \\ PB &= 2d \sin(\alpha + \beta). & (52) \end{aligned}$$

Using the identity $\sin(\alpha - \beta) \sin)\alpha + \beta) = \sin^2\alpha - \sin^2\beta$, we find that,

$$AP \cdot PB = (AS)^2 - (PS)^2 = 2a^2(\cos\theta_0 - \cos\theta), \qquad (53)$$

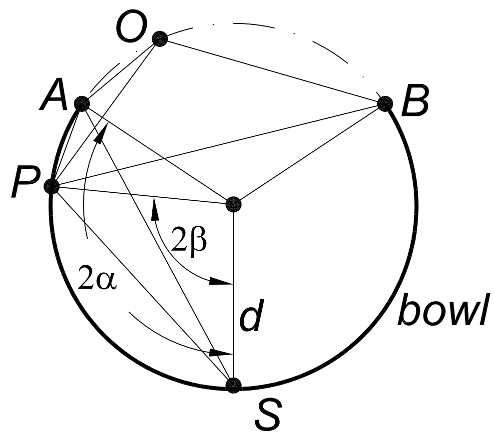

Figure 4: The plane $ABOP$ may be chosen to contain the "south pole" S of the spherical bowl without changing the result of eq. (46). This plane does not, in general, contain the center of the spherical bowl, but its intersection with the bowl is an arc $APSB$ of a circle with radius $d \leq a$. The arc AS subtends angles 2α with respect to the center of arc $APSB$, arc PS subtends angle 2β. Then, arc AP subtends angle $2(\alpha - \beta)$, and arc PSB subtends angle $2(\alpha + \beta)$.

noting also that $(AS)^2 = 2a^2(1 + \cos\theta_0)$, *etc.* By a similar construction that emphasizes point O rather than point P, we also have that,

$$OA \cdot OB = (OS)^2 - (AS)^2 = 2a^2(\cos\theta' - \cos\theta_0). \qquad (54)$$

Forms (53) and (54) are convenient in that their lefthand sides appear to depend on the azimuthal coordinates of points A, B, O and P, while the righthand sides depend only on the polar coordinates.

Thus, the only remaining azimuthal dependence of the integrand of eq. (48) is that due to length OP, which can be expressed as,

$$(OP)^2 = 2a^2(1 - \cos\gamma) = 2a^2(1 - \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\varphi'), \qquad (55)$$

where γ is the angle subtended by arc OP with respect to the center of the spherical bowl. Combining eqs. (48) and (53)-(55), we have,

$$\begin{aligned}
\sigma_-(\theta) &= \frac{V_0}{16\pi^3 a}\int_{\cos\theta_0}^{1} d\cos\theta' \sqrt{\frac{\cos\theta' - \cos\theta_0}{\cos\theta_0 - \cos\theta}} \int_0^{2\pi} d\varphi' \frac{1}{1 - \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\varphi'} \\
&= \frac{V_0}{16\pi^3 a}\int_{\cos\theta_0}^{1} d\cos\theta' \sqrt{\frac{\cos\theta' - \cos\theta_0}{\cos\theta_0 - \cos\theta}} \frac{2\pi}{\sqrt{(1 - \cos\theta\cos\theta')^2 - \sin^2\theta\sin^2\theta'}} \\
&= \frac{V_0}{8\pi^2 a\sqrt{\cos\theta_0 - \cos\theta}} \int_{\cos\theta_0}^{1} d\cos\theta' \frac{\sqrt{\cos\theta' - \cos\theta_0}}{\cos\theta' - \cos\theta} \\
&= \frac{V_0}{8\pi^2 a\sqrt{\cos\theta_0 - \cos\theta}} \int_0^{\sqrt{1-\cos\theta_0}} \frac{2x^2\ dx}{x^2 + \cos\theta_0 - \cos\theta}
\end{aligned}$$

$$= \frac{V_0}{4\pi^2 a\sqrt{\cos\theta_0 - \cos\theta}}\left[x - \sqrt{\cos\theta_0 - \cos\theta}\tan^{-1}\frac{x}{\sqrt{\cos\theta_0 - \cos\theta}}\right]_0^{\sqrt{1-\cos\theta_0}}$$

$$= \frac{V_0}{4\pi^2 a}\left(\frac{\sqrt{1-\cos\theta_0}}{\sqrt{\cos\theta_0 - \cos\theta}} - \tan^{-1}\sqrt{\frac{1-\cos\theta_0}{\cos\theta_0 - \cos\theta}}\right), \tag{56}$$

using Dwight 858.536 to go from the first line to the second, and Dwight 122.1 to go from the $4^{\rm th}$ line to the $5^{\rm th}$.

Thomson [6] also gave an extension of eq. (46) in which the unit charge is not necessarily on the cap of the spherical bowl.

2.5 Solution in Toroidal Coordinates

The problem of a charged, conducting spherical bowl can also be solved in toroidal coordinates [10].

A Appendix: Charge Distribution on a Conducting Ellipsoid and on a Conducting Circular Disk

The charge distribution (38) on a thin, conducting disk can be deduced in a variety of ways. We record here a highly geometric derivation following Thomson (pp. 7 and 178-179 of [6]).

The starting point is the "elementary" result that the electric field is zero in the interior of a spherical shell of any thickness that has a uniform volume charge density between the inner and outer surfaces of the shell. A well-known geometric argument (due to Newton, p. 218 of [11]) for this is illustrated in Fig. 5.

The electric field at point $\mathbf{r}$ in the interior of the shell due to a lamina of thickness δ and area A_1 centered on point $\mathbf{r}_1$ that lies within a narrow cone whose vertex is point 0 is given by,

$$\mathbf{E}_1 = \frac{\rho d\mathrm{Vol}_1}{R_{01}^2}\hat{\mathbf{R}}_{01}, \tag{57}$$

where ρ is the volume charge density, $d\mathrm{Vol}_1 = A_1\delta$, $\mathbf{R}_{01} = \mathbf{r}_0 - \mathbf{r}_1$, and the center of the sphere is taken to be at the origin. Likewise, the electric field from a lamina of area A_2 centered on point $\mathbf{r}_2$ defined by the intercept with the shell of the same narrow cone extended in the opposite direction (forming a bicone) is given by,

$$\mathbf{E}_2 = \frac{\rho d\mathrm{Vol}_2}{R_{02}^2}\hat{\mathbf{R}}_{02}, \tag{58}$$

In the limit of bicones with small half angle, the two parts of the bicone as truncated by the shell are similar, so that,

$$\frac{A_1}{R_{01}^2} = \frac{A_2}{R_{02}^2}, \qquad \frac{d\mathrm{Vol}_1}{R_{01}^2} = \frac{d\mathrm{Vol}_2}{R_{02}^2}, \tag{59}$$

and, of course, $\hat{\mathbf{R}}_{02} = -\hat{\mathbf{R}}_{01}$. Hence, $\mathbf{E}_1 + \mathbf{E}_2 = 0$. Since this construction can be applied to all points in the material of the spherical shell, and for all pairs of surface elements subtended by (narrow) bicones, the total electric field in the interior of the shell is zero.

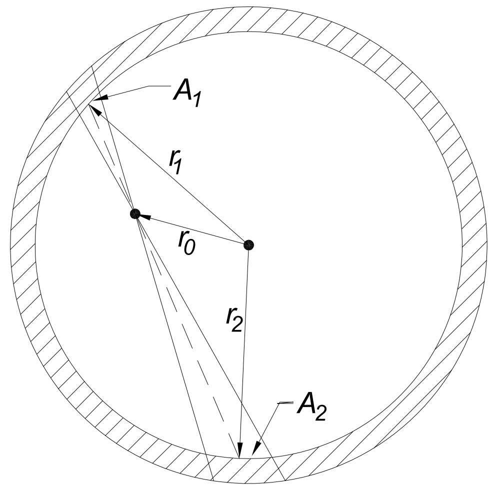

Figure 5: For any point $\mathbf{r}_0$ in the interior of a uniformly charged shell of charge, the axis of a narrow bicone intercepts the inner surface of the shell at points $\mathbf{r}_1$ and $\mathbf{r}_2$. The corresponding areas on the inner surface of the shell intercepted by the bicone are A_1 and A_2. In the limit of small areas, $A_1/R_{01}^2 = A_2/R_{02}^2$.

We now reconsider the above argument after arbitrary scale transformations have been applied to the rectangular coordinate axes,

$$x \to k_1 x, \qquad y \to k_2 y, \qquad z \to k_3 z. \tag{60}$$

A spherical shell of radius s is thereby transformed into an ellipsoid,

$$\frac{x^2}{s^2} + \frac{y^2}{s^2} + \frac{z^2}{s^2} = 1 \qquad \to \qquad \frac{x^2}{s^2/k_1^2} + \frac{y^2}{s^2/k_2^2} + \frac{z^2}{s^2/k_3^2} = 1. \tag{61}$$

As parameter s is varied, one obtains a set of similar ellipsoids, centered on the origin.

A small volume element obeys the transformation,

$$d\mathrm{Vol} = dxdydz \to k_1 k_2 k_3 dxdydz = k_1 k_2 k_3 d\mathrm{Vol}. \tag{62}$$

The three points 0, 1, and 2 in Fig. 5 lie along a line, so that,

$$\mathbf{R}_{01} = \mathbf{r}_0 - \mathbf{r}_1 = C\mathbf{R}_{02} = C(\mathbf{r}_0 - \mathbf{r}_2), \tag{63}$$

where C is a (negative) constant. This relation is invariant under the scale transformation (60), so that, together with eq. (62), the relation,

$$\frac{d\mathrm{Vol}_1}{R_{01}^2} = \frac{d\mathrm{Vol}_2}{R_{02}^2}\,, \tag{64}$$

is also invariant. Hence, if the ellipsoidal shell, which is the transform of the spherical shell of Fig. 5, contains a uniform volume charge density, the relation $\mathbf{E}_1 + \mathbf{E}_2 = 0$ remains true at the vertex of any bicone in the interior of the shell, which implies that the total electric field is zero there.

This proof is based on the premise that the ellipsoidal shell is bounded by two similar ellipsoids, and that the volume charge density in the shell is uniform.

If we let the outer ellipsoid of the shell approach the inner one, always remaining similar to the latter, we reach a configuration that is equivalent to a thin, conducting ellipsoid, since in both cases the electric field is zero in the interior. Hence, the surface charge distribution on a thin, conducting ellipsoid must the same as the projection onto its surface of a uniform charge distribution between that surface and a similar, but slightly larger ellipsoidal surface.

The charge σ per unit area on the surface of a thin, conducting ellipsoid is therefore proportional to the thickness, which we write as δd, of the ellipsoidal shell formed by that surface and a similar, but slightly larger ellipsoid,

$$\sigma = \rho \delta d, \tag{65}$$

where constant ρ is to be determined from a knowledge of the total charge Q on the conducting ellipsoid.

The thickness δd of a thin ellipsoidal shell at some point on its inner surface is the distance between the plane that is tangent to the inner surface at the specified point, and the plane that is tangent to the outer surface at the point similar to the specified point. These planes are parallel since the ellipsoids are parallel. In particular, if the semimajor axes of the inner ellipsoid are called a, b, and c, then those of the outer ellipsoid can be written $a + \delta a$, $b + \delta b$ and $c + \delta c$. Let the (perpendicular) distance from the plane tangent to the specified point on the inner ellipsoid to its center be called d, and the corresponding distance from the outer tangent plane be $d + \delta d$, so that δd is the desired thickness of the shell at the specified point. Then, the condition of similarity is that,

$$\frac{\delta a}{a} = \frac{\delta b}{b} = \frac{\delta c}{c} = \frac{\delta d}{d}\,. \tag{66}$$

Since the volume of an ellipsoid with semimajor axes a, b, and c is $4\pi abc/3$, the volume of the ellipsoidal shell is $4\pi(a+\delta a)(b+\delta b)(c+\delta c)/3 - 4\pi abc/3 = 4\pi abc(\delta d/d)$, using eq. (66). As the constant ρ has an interpretation as the uniform charge density within the material of the ellipsoidal shell, we find that the total charge Q on the conducting ellipsoid is related by,

$$Q = \rho \mathrm{Vol}_{\mathrm{shell}} = \frac{4\pi abc}{d} \rho \delta d, \tag{67}$$

and hence,

$$\sigma = \rho \delta d = \frac{Qd}{4\pi abc}\,. \tag{68}$$

It remains to find an expression for the distance d to the tangent plane. If we write the equation for the ellipsoid in the form,

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \tag{69}$$

then the gradient of f is perpendicular to the tangent plane. Thus, the vector $\mathbf{d}$ from the center of the ellipsoid to the tangent plane is proportional to ∇f. That is,

$$\mathbf{d} \propto \nabla f = 2\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right). \tag{70}$$

The unit vector $\hat{\mathbf{d}}$ is therefore,

$$\hat{\mathbf{d}} = \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}. \tag{71}$$

The magnitude d of the vector $\mathbf{d}$ is related to the vector $\mathbf{r} = (x, y, z)$ of the specified point on the ellipse by,

$$d = \mathbf{r} \cdot \hat{\mathbf{d}} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}. \tag{72}$$

At length, we have found the charge density on the surface of a conducting ellipsoid to be,

$$\sigma_{\text{ellipsoid}} = \frac{Q}{4\pi abc\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}, \tag{73}$$

where Q is the total charge.

The case of a thin, conducting elliptical disk in the x-y plane can be obtained from eq. (73) by letting c go to zero. For this, we note that eq. (69) for a general ellipsoid permits us to write,

$$c\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} = \sqrt{c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) + 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \to \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \tag{74}$$

The charge density on each side of a conducting elliptical disk is therefore,

$$\sigma_{\text{elliptical disk}} = \frac{Q}{4\pi ab\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}}}. \tag{75}$$

The charge density on each side of a conducting circular disk of radius b follows immediately as,

$$\sigma_{\text{circular disk}} = \frac{Q}{4\pi b\sqrt{b^2 - r^2}}, \tag{76}$$

where $r^2 = x^2 + y^2$. Such a disk has potential V_0, which can be found by calculating the potential at the center of the disk according to,

$$V_0 = V(r = 0, z = 0) = \int_0^b \frac{2\sigma(r)}{r} 2\pi r\, dr = \frac{Q}{b}\int_0^b \frac{dr}{\sqrt{b^2 - r^2}} = \frac{\pi Q}{2b}. \tag{77}$$

Hence, a conducting disk of radius b at potential V_0 has charge density,

$$\sigma_{\text{circular disk}} = \frac{V_0}{2\pi^2\sqrt{b^2 - r^2}} \tag{78}$$

on each side, which is the result quoted in eq. (38).

References

[1] G. Green, *Mathematical Papers*, (Macmillan, 1871), pp. 50-55, 57-61,
http://physics.princeton.edu/~mcdonald/examples/EM/green_papers.pdf

[2] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968), problems 46 and 47, p. 228, http://physics.princeton.edu/~mcdonald/examples/EM/smythe_50.pdf

[3] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999),
http://physics.princeton.edu/~mcdonald/examples/EM/jackson_ce3_99.pdf

[4] Green's derivation of eq. (29) is considered in prob. 5 of Ph501 Set 2,
http://physics.princeton.edu/~mcdonald/examples/ph501set2.pdf

[5] See, for example, secs. 5.09-5.102 of [2].

[6] W. Thomson (Lord Kelvin), *Determination of the Distribution of Electricity on a Circular Segment of Plane or Spherical Conducting Surface, Under Any Given Influence*, in *Papers on Electricity and Magnetism*, 2nd ed. (Macmillan, 1884), pp. 178-191, dated 1869, published in the 1st ed. of 1872. See also pp. 153-154 for the original statement of the result in a letter (in French) to Liouville.
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_electrostatics_magnetism.pdf

[7] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed. (Macmillan, 1904), Vol. 1, pp. 276-280, http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_treatise_v1_04.pdf

[8] J. Jeans, *The Mathematical Theory of Electricity and Magnetism*, 5th ed. (Cambridge U. Press, 1951), pp. 250-251, http://physics.princeton.edu/~mcdonald/examples/EM/jeans_em_11.pdf

[9] See, for example, sec. 4.21 of [2].

[10] N.M. Lebedev, I.P. Skalskaya and Y.S. Ufyland, *Worked Problems in Applied Mathematics* (Dover, 1979), prob. 501, p. 239,
http://physics.princeton.edu/~mcdonald/examples/EM/lebedev_65.pdf

[11] I. Newton, *Philosophiæ Naturalis Principia Mathematica*, 3rd ed. (1726),
http://physics.princeton.edu/~mcdonald/examples/mechanics/newton_principia.pdf