A Slingshot Orbit

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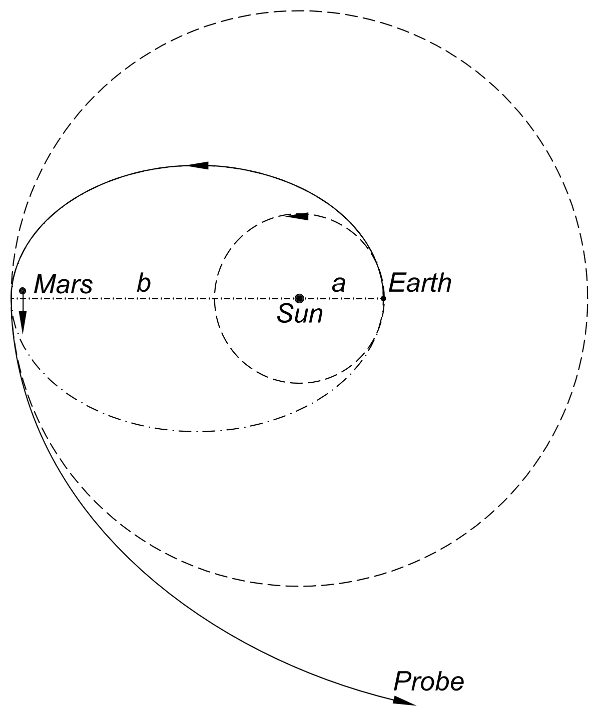
1 Problem

A space probe is launched from Earth into a transfer orbit whose maximum radius b is slightly larger than the distance from the Sun to Mars. The launch time is such that when the probe reaches radius b it has a near collision with Mars, which deflects the velocity of the probe by $\approx 180°$ with respect to Mars, which gives a forward boost to the velocity of the probe with respect to the Sun.



What is the largest distance from the Sun to which the probe can now travel?

As an intermediate step, calculate such parameters of the transfer orbit as its eccentricity ϵ, characteristic radius r_0, energy E, angular momentum L, and the maximum and minimum velocities v_a and v_b.

You may make the approximations that the orbits of Earth and Mars are circular with radii a and b, respectively, that the masses of Earth and Mars do not affect the transfer orbit between the two planets, that the mass of the Earth and Sun can be ignored during the near collision between the probe and Mars, and that the masses of Earth and Mars can again be ignored after the near collision.

You may also ignore the complication that the distance of closest approach needed for Mars to deflect the probe by 180° is less than its radius.

Calculate the velocity v_0 relative to the Earth needed to launch the probe into the transfer orbit, assuming $\mathbf{v}_0$ is parallel to $\mathbf{v}_E$, the velocity of the Earth with respect to the Sun.

Compare v_0 with the escape velocity v_e for the probe from both the Earth and Sun, assuming that $\mathbf{v}_e$ is also parallel to $\mathbf{v}_E$.

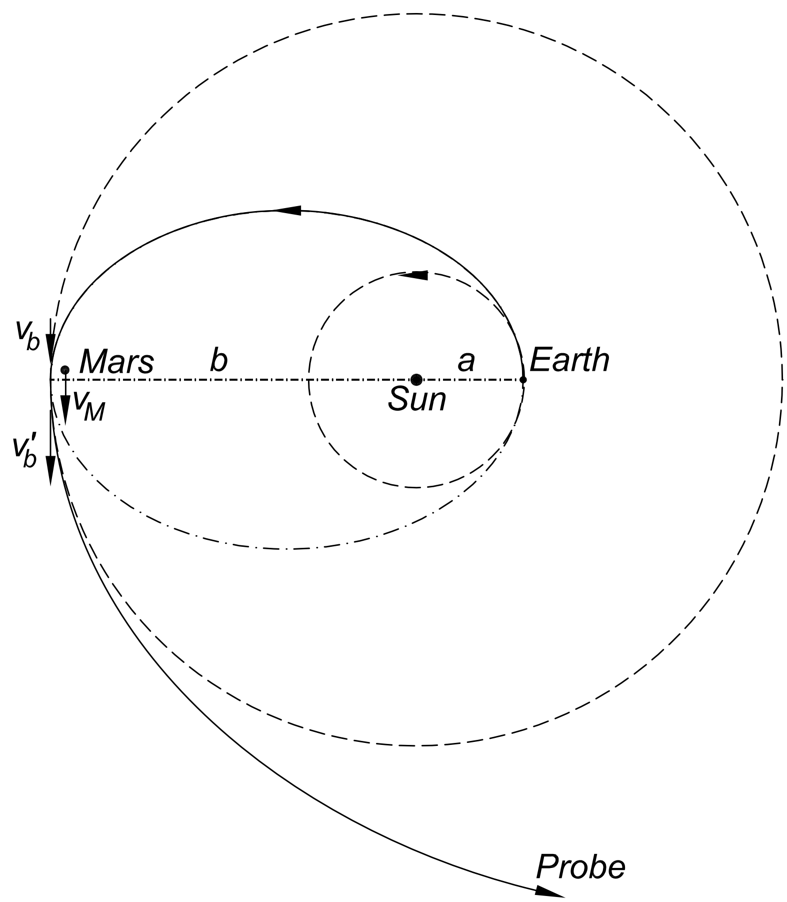
2 Solution

This problem is an example of a 4-body gravitational interaction. An Amusing web site on the n-body problem is http://www.soe.ucsc.edu/~charlie/3body/. See also [1].[1]

2.1 Solution Assuming that Mars is a Point Mass

Once we know the (purely azimuthal) velocity v_b' of the probe after its near collision with Mars at radius b, we can calculate the maximum distance $r_{\rm max}$ between the probe and the Sun according to the energy relation,

$$-\frac{GM_\odot}{r_{\rm max}} = -\frac{GM_\odot}{b} + \frac{v_b'^2}{2}\,. \tag{1}$$



[1]Oct. 17, 2019: Other discussions of the slingshot effect in unpowered spaceflight include [2]-[6]. This is distinct from the Oberth effect [7]-[9] in which a rocket pulse is more effective if it occurs when the rocket is in a region of low gravitational potential.

The velocity v_b' is the result of the elastic collision with Mars in which the initial velocity of the probe is v_b and the velocity v_M of Mars is given by,

$$v_M^2 = \frac{GM_\odot}{b}, \tag{2}$$

as follows from $F = ma$ assuming the orbit of Mars is circular. As will be verified below in detail, the velocity v_b of the probe at radius b is less than the orbital velocity v_M of Mars.

We analyze the collision in the center of mass frame, which is essentially the rest frame of Mars. In this frame, the initial (and final) speed of the probe is $v_M - v_b$, since the initial velocities are parallel. Assuming that the final velocity of the probe is also in the direction of Mars' velocity, we have at once that,

$$v_b' = v_b + 2v_M. \tag{3}$$

To determine v_b we consider the properties of the elliptical transfer orbit, whose general form can be written,

$$\frac{1}{r} = \frac{1 + \epsilon\cos\theta}{r_0}, \tag{4}$$

where the azimuthal angle θ is zero when the probe is launched from the Earth. Thus, the minimum and maximum radii a and b are related to r_0 and ϵ by,

$$\frac{1}{a} = \frac{1+\epsilon}{r_0}, \qquad \frac{1}{b} = \frac{1-\epsilon}{r_0}, \tag{5}$$

which tells us that,

$$r_0 = \frac{2ab}{a+b}, \qquad \text{and} \qquad \epsilon = \frac{b-a}{a+b}. \tag{6}$$

The characteristic radius r_0 has the useful property that $\ddot{r} = 0$ there, so $F_r = ma_r = m(\ddot{r} + r\dot{\theta}^2)$ tells us that,

$$\frac{GM_\odot m}{r_0} = mr_0\dot{\theta}^2 = \frac{L^2}{mr_0^3}, \tag{7}$$

where $L = mr^2\dot{\theta}$ is the angular momentum. Thus, the angular momentum of the transfer orbit is given by,

$$(L/m)^2 = GM_\odot r_0. \tag{8}$$

The maximum and minimum velocities now follow from the angular momentum as,

$$v_a^2 = \left(\frac{L/m}{a}\right)^2 = \frac{GM_\odot r_0}{a^2}, \qquad \text{and} \qquad v_b^2 = \frac{GM_\odot r_0}{b^2}. \tag{9}$$

The energy of the transfer orbit can be calculated at point a,

$$E = \frac{m}{2}v_a^2 - \frac{GM_\odot m}{a} = -\frac{GM_\odot m}{a+b}. \tag{10}$$

A more standard deduction of the relation between the velocities v_a and v_b and the angular momentum is based on an expression for the energy E of the transfer orbit in which the kinetic energy term $mr^2\dot{\theta}^2/2$ is replaced by $L^2/2mr^2$,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GM_\odot m}{r}\,. \tag{11}$$

Since $\dot{r} = 0$ when $r = a$ or b, we have,

$$\frac{L^2}{2ma^2} - \frac{GM_\odot m}{a} = \frac{L^2}{2mb^2} - \frac{GM_\odot m}{b}\,. \tag{12}$$

which quickly leads to eq. (8), and thence to eq. (10).

We can now find $r_{\rm max}$ using els. (1) (2), (3) and, (9),

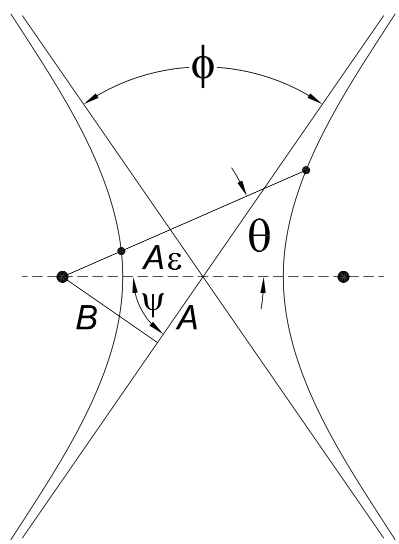
$$\frac{1}{r_{\rm max}} = \frac{1}{b} - \frac{(v_b + 2v_M)^2}{2GM_\odot} = \frac{1}{b} - \frac{r_0}{2b^2} - \frac{2}{b}\sqrt{\frac{r_0}{b}} - \frac{2}{b} = -\frac{1}{b} - \frac{r_0}{2b^2} - \frac{2}{b}\sqrt{\frac{r_0}{b}}\,. \tag{13}$$

Since $r_{\rm max}$ is negative, the probe can actually go arbitrarily far from the Sun. It gained enough energy in its collision with Mars to escape the Sun's gravity!

2.2 A More Realistic Deflection Angle

We have assumed that Mars deflects the probe by 180° in the Mars frame. This would require the probe to pass arbitrarily close to the center of Mars, which is unrealistic.

Here we calculate the deflection angle supposing the distance of closest approach is the radius of Mars.



In the Mars frame, where we ignore the gravity of the Earth and Sun, the trajectory of the probe is a hyperbola. We use the notation shown in the figure above, where the impact parameter is labeled B, and the distance from the intersection of the asymptotes to the focus is $A\epsilon$. The eccentricity ϵ is, related by

$$\epsilon^2 = 1 + \frac{B^2}{A^2}\,, \tag{14}$$

and the equation of the hyperbola is,

$$\frac{1}{r} = \frac{1 + \epsilon \cos\theta}{A(\epsilon^2 - 1)}\,, \tag{15}$$

taking the origin at the (left) focus and measuring azimuth θ from the line between the focus and the intersection of the asymptotes.

The length A is related to the energy in the Mars frame via,

$$A = \frac{GM_M m}{2E} = \frac{GM_M}{(v_M - v_b)^2} = \frac{M_M}{M_\odot}\frac{b(a+b)}{3a + b - 2\sqrt{2a(a+b)}} = 4,500 \text{ km}, \tag{16}$$

since in the Mars frame the initial velocity of the probe is $v_M - v_b$.

We see that the angle between the sides $A\epsilon$ and B of the right triangle A-B-$A\epsilon$ is $\phi/2$, where ϕ is the deflection angle of the probe in the Mars frame. Hence,

$$\frac{1}{\epsilon} = \sin\phi/2. \tag{17}$$

From eq. (15), the distance d of closest approach is,

$$d = A(\epsilon - 1) = A\left(\frac{1}{\sin\phi/2} - 1\right), \tag{18}$$

so the angle of deflection in the Mars frame is related by,

$$\sin\phi/2 = \frac{A}{A+d}\,. \tag{19}$$

If we take the distance of closest approach to be the radius of Mars, $d = r_M = 3,435$ km, then the deflection angle in the Mars frame would be $\phi = 70^\circ$.

The deflection angle ψ in the Sun's frame is given by,

$$\tan\phi = \frac{(v_M - v_b)\sin\phi}{(v_M - v_b)\cos\phi - v_M} = -0.097, \tag{20}$$

noting that $v_b = v_M\sqrt{r_0/B} \approx 0.9 v_m$. Hence, the deflection angle $\psi \approx 174.4^\circ$ if the probe skims over the surface of Mars.

I believe that in this case the final velocity of the probe still exceeds the escape velocity of the solar system.

2.3 Launch Velocity

We first calculate the escape velocity v_e for the probe from both the Earth and Sun, assuming that $\mathbf{v}_E$ is parallel to $\mathbf{v}_E$, which is the most favorable direction for the launch. The velocity of the Earth with respect to the Sun is given by $F = ma$ as,

$$v_E^2 = \frac{GM_\odot}{a}\,, \tag{21}$$

assuming the orbit of the Earth is circular. For the probe to escape from the Earth-Sun system, its launch velocity $\mathbf{v}_e$ must be such that the total energy of the probe is zero. We calculate in the frame of the Sun, where the velocity of the probe (just after rocket burnout) is $\mathbf{v}_e + \mathbf{v}_E$, whose magnitude is $v_e + v_E$ since the two velocities are parallel. The energy relation for escape is,

$$E = 0 = \frac{m}{2}(v_e + v_E)^2 - \frac{GM_E m}{r_e} - \frac{GM_\odot m}{a} \,. \tag{22}$$

We note that the escape velocity from only the Earth is given by,

$$v_{e,E}^2 = \frac{2GM_E m}{r_e} \,, \tag{23}$$

so with aid of this and eq. (19), eq. (22) can be written as,

$$(v_e + v_E)^2 = v_{e,E}^2 + 2v_E^2. \tag{24}$$

Thus, the escape velocity from the Earth-Sun system is,

$$v_e = \sqrt{v_{e,E}^2 + 2v_E^2} - v_E. \tag{25}$$

Numerically, $v_{e,E} \approx 11,000$ m/s and $v_E \approx 30,000$ m/s, so $v_e \approx 13,800$ m/s.

To launch into the transfer orbit via velocity $\mathbf{v}_0 = v_0 \hat{\mathbf{v}}_E$, the magnitude of the probe velocity relative to the Sun just after rocket burnout is $v_0 + v_E$, so the energy relation is,

$$E = \frac{m}{2}(v_0 + v_E)^2 - \frac{GM_E m}{r_e} - \frac{GM_\odot m}{a} = \frac{m}{2} v_a^2 - \frac{GM_\odot m}{a} \,, \tag{26}$$

using eq. (10). Again using eq. (23), we find,

$$v_0 = \sqrt{v_{e,E}^2 + v_a^2} - v_E. \tag{27}$$

Since $v_a = v_E\sqrt{2b/(a+b)} \approx 33,000$ m/s, $v_0 \approx 4,800$ m/s.

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