# Slab Rolling on a Rolling Cylinder

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# 1 Problem

Discuss the motion of a "slab" that rolls without slipping on a "cylinder", when the latter rolls without slipping on a horizontal plane.[1]

*This problem was suggested by Alexandre Tort. For the related case of one cylinder on/inside another, see [1, 2]. For the case of a sphere on a fixed cylinder, see pp. 212-214 of [3].*[2]

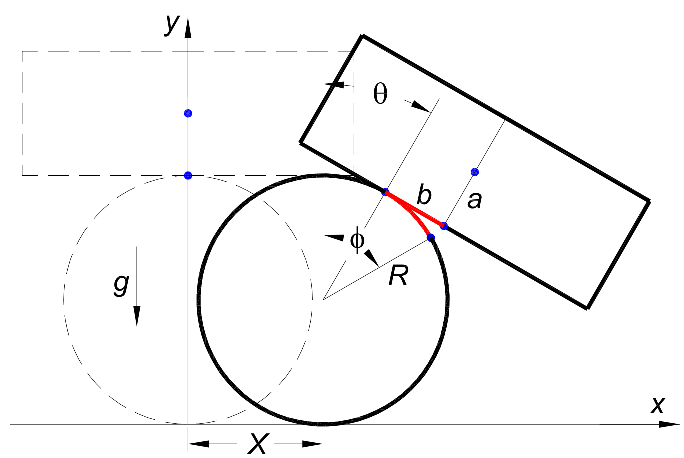
# 2 Solution

We will use a Lagrangian approach.

## 2.1 Coordinates and Constraints

When the slab, of thickness $2a$, mass $m$ and moment of inertia $kma^2$, is directly above the cylinder, of radius $R$, mass $M$ and moment of inertia $KMR^2$, and centered upon it, we define the line of contact of the cylinder with the horizontal plane to be the $z$-axis, at $x = y = 0$. Then, the condition of rolling without slipping for the cylinder is that when it has rolled (positive) distance $X$, the initial line of contact has rotated through angle $\phi = X/R$, clockwise with respect to the vertical, as shown in the figure below. This rolling constraint can be written as

$$X = R\phi. \tag{1}$$




[1]Either the "slab" or the "cylinder" (but not both) could have a very large moment of inertia if it is in the form of a "bobbin" with flanges that extend beyond the supporting surface.

[2]The author's interest in such problems was inspired in part by Bob Dylan: "It balances on your head just like a mattress balances on a bottle of wine." (Leopardskin Pill-Box Hat, 1966).

If the slab rolls without slipping such that a line (in the $x$-$y$ plane) from the center of the cylinder to the point of contact with the slab angle $\theta$ (positive clockwise) to the vertical, then the initial point of contact of the slab is at distance $b$ from the original point, and the initial point of contact of the cylinder has rotated by angle $\phi$. The second rolling constraint is that distance $b$ on the slab equals arc length $R(\phi - \theta)$ on the cylinder,

$$b = R(\phi - \theta)\,. \tag{2}$$

The vertical center of the cylinder is at $Y = R$, and the center of the slab is at

$$x = X + (a + R)\sin\theta + R(\phi - \theta)\cos\theta], \qquad y = R + (a + R)\cos\theta - R(\phi - \theta)\sin\theta. \tag{3}$$

Altogether there are 4 constraints on the 6 degree of freedom ($x$, $y$, $X$, $Y$, $\phi$, $\theta$), of the two-dimensional motion of the system, such that there are only two independent degrees of freedom, which we take to be the angles $\phi$ and $\theta$.

## 2.2 Energy

The total energy $E = T + V$ is conserved, where the potential energy $V$ (taken to be zero when $\phi = \phi_0$ and $\theta = \theta_0$),

$$V = mg(y - y_0) = mg\{(a + R)(\cos\theta - \cos\theta_0) - R[(\phi - \theta)\sin\theta - (\phi_0 - \theta_0)\sin\theta_0]\}, \tag{4}$$

depends on both coordinates $\phi$ and $\theta$.[3]

The kinetic energy of cylinder, whose axis is at $(X, Y)$, is

$$T_{\rm cyl} = \frac{M\dot{X}^2}{2} + \frac{I_{\rm cyl}\,\dot{\phi}}{2} = \frac{1 + K}{2} MR^2\dot{\phi}^2, \tag{5}$$

using the rolling constraint (1) and the expression $I_{\rm cyl} = KMR^2$ for the moment of inertia $I_{\rm cyl}$ in terms of parameter $K$.

The kinetic energy of the slab, whose axis is at $(x, y)$, is, using $I_{\rm slab} = kma^2$,

$$T_{\rm slab} = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + \frac{I_{\rm slab}\,\dot{\theta}^2}{2} = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + \frac{kma^2\,\dot{\theta}^2}{2}\,. \tag{6}$$

From eq. (3) we have,

$$\dot{x} = (a + R)\cos\theta\,\dot{\theta} - R(\phi - \theta)\sin\theta\,\dot{\theta} + R(\dot{\phi} - \dot{\theta})\cos\theta, \tag{7}$$

$$\dot{y} = -(a + R)\sin\theta\,\dot{\theta} - R(\phi - \theta)\cos\theta\,\dot{\theta} - R(\dot{\phi} - \dot{\theta})\sin\theta, \tag{8}$$

so the kinetic energy of the slab can be written as

$$\begin{aligned} T_{\rm slab} &= \frac{m}{2}\left[(a + R)^2\,\dot{\theta}^2 + R^2(\phi - \theta)^2\,\dot{\theta}^2 + R^2(\dot{\phi} - \dot{\theta})^2 + 2R(a + R)\,\dot{\theta}(\dot{\phi} - \dot{\theta})\right] + \frac{kma^2\,\dot{\theta}^2}{2} \\ &= \frac{m}{2}\left[(1 + k)a^2\dot{\theta}^2 + R^2(\phi - \theta)^2\,\dot{\theta}^2 + 2aR\,\dot{\theta}\,\dot{\phi} + R^2\dot{\phi}^2\right]. \end{aligned} \tag{9}$$

The total kinetic energy $T_{\rm cyl} + T_{\rm slab}$ is

$$T = \frac{[m + (1 + K)M]R^2}{2}\dot{\phi}^2 + maR\,\dot{\phi}\,\dot{\theta} + \frac{(1 + k)ma^2 + mR^2(\phi - \theta)^2}{2}\,\dot{\theta}^2. \tag{10}$$

[3] This contrasts with the case of a cylinder rolling on/inside another cylinder [1, 2], where the potential energy does not depend on $\phi$, such that there is a second conserved quantity for the system.

### 2.3 Equations of Motion

#### 2.3.1 $\phi$

The $\phi$-equation for the Lagrangian $\mathcal{L} = T - V$ is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = [m + (1 + K)M]R^2\,\ddot{\phi} + maR\,\ddot{\theta} = \frac{\partial \mathcal{L}}{\partial \phi} = mgR\sin\theta. \qquad (11)$$

#### 2.3.2 $\theta$

The $\theta$-equation can be written as

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= aR\,\ddot{\phi} + \left[(1+k)a^2 + R^2(\phi - \theta)^2\right]\,\ddot{\theta} + 2R^2(\phi - \theta)(\dot{\phi} - \dot{\theta})\,\dot{\theta} \\ &= \frac{1}{m}\frac{\partial \mathcal{L}}{\partial \theta} = g\left[(a + R)\sin\theta + R(\phi - \theta)\cos\theta - R\sin\theta\right] \\ &= g\left[a\sin\theta + R(\phi - \theta)\cos\theta\right]. \qquad (12)\end{aligned}$$

### 2.4 $\phi = \phi_0 =$ Constant

Before discussing the general case, we consider the special case that the cylinder is fixed, but the initial angle $\phi_0$ in not necessarily zero.

Then, the $\phi$-equation of motion (11) is to be ignored, and the $\theta$-equation (12) becomes

$$\left[(1+k)a^2 + R^2(\phi_0 - \theta)^2\right]\,\ddot{\theta} - 2R^2(\phi_0 - \theta)\,\dot{\theta}^2 = g\left[a\sin\theta + R(\phi_0 - \theta)\cos\theta\right]. \qquad (13)$$

The potential energy (4) becomes

$$\frac{V}{m} = g\{(a + R)(\cos\theta - \cos\theta_0) - R[(\phi_0 - \theta)\sin\theta - (\phi_0 - \theta_0)\sin\theta_0]\}, \qquad (14)$$

the kinetic energy (10) becomes

$$\frac{T}{m} = \frac{(1+k)a^2 + R^2(\phi_0 - \theta)^2}{2}\,\dot{\theta}^2, \qquad (15)$$

and the total energy becomes

$$\begin{aligned}\frac{E}{m} &= \frac{(1+k)a^2 + R^2(\phi_0 - \theta_0)^2}{2}\,\dot{\theta}_0^2 = \frac{(1+k)a^2 + R^2(\phi_0 - \theta)^2}{2}\,\dot{\theta}^2 \\ &\quad + g\{(a + R)(\cos\theta - \cos\theta_0) - R[(\phi_0 - \theta)\sin\theta - (\phi_0 - \theta_0)\sin\theta_0]\}. \qquad (16)\end{aligned}$$

#### 2.4.1 Small Oscillations

We first seek an oscillatory solution, of the form

$$\theta = \theta_0 + \alpha\, e^{i\omega t}, \qquad \dot{\theta} = i\alpha\omega\, e^{i\omega t}, \qquad \ddot{\theta} = -\alpha\omega^2\, e^{i\omega t}, \qquad (17)$$

where $\alpha$ is small. Using this trial solution in eq. (13), and keeping terms only to order $\alpha$, we have

$$\sin\theta \approx \sin\theta_0 + \alpha\, e^{i\omega t}\, \cos\theta_0, \qquad \cos\theta \approx \cos\theta_0 - \alpha\, e^{i\omega t}\, \sin\theta_0, \tag{18}$$

$$\begin{aligned} & -\alpha\omega^2 \left[(1+k)a^2 + R^2(\phi_0-\theta_0)^2\right]\, e^{i\omega t} \\ \approx\ & g\left[a\left(\sin\theta_0 + \alpha\, e^{i\omega t}\, \cos\theta_0\right) + R(\phi_0 - \theta_0 - \alpha\, e^{i\omega t})\left(\cos\theta_0 - \alpha\, e^{i\omega t}\, \sin\theta_0\right)\right] \\ \approx\ & g\left[a\sin\theta_0 + R(\phi_0-\theta_0)\cos\theta_0\right] + \alpha g\left[a\cos\theta_0 - R(\phi_0-\theta_0)\sin\theta_0 - R\cos\theta_0\right]\, e^{i\omega t}. \end{aligned} \tag{19}$$

The constant term must be zero, which tells us that

$$\tan\theta_0 = \frac{R}{a}(\theta_0 - \phi_0) = -\frac{b_0}{a}\,, \tag{20}$$

recalling eq. (2).

The terms in $\alpha\, e^{i\omega t}$ must be the same on both sides of eq. (19), which tells us that the angular frequency $\omega$ of small oscillations is related by

$$\omega = \sqrt{\frac{g\{R[\cos\theta_0 - (\theta_0-\phi_0)\sin\theta_0] - a\cos\theta_0\}}{(1+k)a^2 + R^2(\phi_0-\theta_0)^2}} = \sqrt{\frac{g(R\cos\theta_0 - a/\cos\theta_0)}{a^2(1+k+\tan^2\theta_0)}}\,. \tag{21}$$

Oscillatory solutions exist only for

$$R > \frac{a}{\cos^2\theta_0}\,, \qquad \cos\theta_0 < \sqrt{\frac{a}{R}}\,, \tag{22}$$

which always requires that $R > a$.

In particular, $\omega = \sqrt{g(R-a)/a^2(1+k)}$ for $\phi_0 = 0 = \theta_0$. For this case, the constant energy is

$$\frac{E}{m} = \frac{(1+k)a^2}{2}\,\dot\theta_0^2 = \frac{(1+k)a^2}{2}\,\dot\theta^2 - g(a+R)(1-\cos\theta), \tag{23}$$

and we record the full equation of motion,

$$\left[(1+k)a^2 + R^2\,\theta^2\right]\ddot\theta + 2R^2\,\theta)\,\dot\theta^2 = g\left[a\sin\theta - R\,\theta\cos\theta\right]. \tag{24}$$

For reference, we also record that in this case,

$$\dot\theta^2 \;=\; \dot\theta_0^2 + \frac{2g(a+R)(1-\cos\theta)}{(1+k)a^2}\,, \tag{25}$$

$$\left[(1+k)a^2 + R^2\,\theta^2\right]\ddot\theta \;=\; g(a\sin\theta - R\,\theta\cos\theta) - 2R^2\,\theta\left(\dot\theta_0^2 + \frac{2g(a+R)(1-\cos\theta)}{(1+k)a^2}\right). \tag{26}$$

For a solid slab of half width $c$, $k = (1 + c^2/a^2)/3$, so for $\phi_0 = 0 = \theta_0$,

$$\omega = \sqrt{\frac{3g(R-a)}{4a^2 + c^2}} \ \underset{\text{cube}}{\overset{\text{solid}}{\longrightarrow}}\ \sqrt{\frac{3g(R-a)}{5a^2}}\,. \tag{27}$$

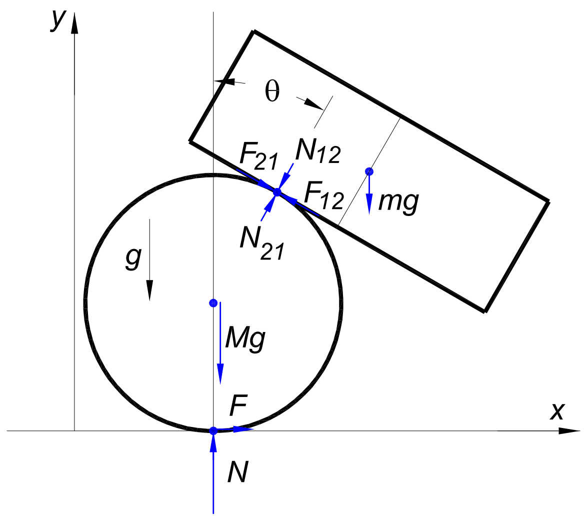
Note that it is possible to have small oscillations about a nonzero value of $\theta_0$, if the slab is appropriately off center with respect to the initial point of contact with the cylinder. For example, a cube of half length $a = R/2$ would oscillate about an initial configuration with $b_0 = -\sqrt{3}R/6$, $\theta_0 = 30^\circ$ and $\phi_0 = 15^\circ$ at angular frequency $\omega = \sqrt{2g/\sqrt{3}R} = 1.075\sqrt{g/R}$. This is about 2% less than the angular frequency $\omega = \sqrt{6g/5R} = 1.095\sqrt{g/R}$ for $\theta_0 = 0$.

### 2.4.2 Angle $\theta_s$ at which the Slab Falls off the Cylinder

As the slab rotates it may lose contact with (separate from) the cylinder, say at angle $\theta_s$.

This happens when the normal force $N_{12}$ of the cylinder on the slab vanishes, which occurs when the component of the gravitational force $mg$ equals the component of $m\mathbf{a}$ along an axis that makes angle $\theta_s$ to the vertical,

$$mg\cos\theta_s = m(-\ddot{x}_s\sin\theta_s - \ddot{y}_s\cos\theta_s). \tag{28}$$




From eqs. (7)-(8), we find

$$\begin{aligned}
\ddot{x} &= (a+R)(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) - R[(\phi-\theta)(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2) + \sin\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})] \\
&\quad + R[\cos\theta(\ddot{\phi}-\ddot{\theta}) - \sin\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})], \qquad (29)\\
\ddot{y} &= -(a+R)(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2) - R[(\phi-\theta)(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) + \cos\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})] \\
&\quad - R[\sin\theta(\ddot{\phi}-\ddot{\theta}) + \cos\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})], \qquad (30)
\end{aligned}$$

$$\begin{aligned}
\ddot{x}\sin\theta + \ddot{y}\cos\theta &= -(a+R)\,\dot{\theta}^2 - R[(\phi-\theta)\,\ddot{\theta} + \dot{\theta}(\dot{\phi}-\dot{\theta})] - R\,\dot{\theta}(\dot{\phi}-\dot{\theta}) \\
&= -R(\phi-\theta)\,\ddot{\theta} + (R-a)\,\dot{\theta}^2 - 2R\,\dot{\phi}\,\dot{\theta}. \qquad (31)
\end{aligned}$$

For constant angle $\phi_0$, the relation (28) becomes,

$$g\cos\theta_s = R(\phi_0 - \theta_s)\,\ddot{\theta}_s - (R-a)\,\dot{\theta}_s^2. \tag{32}$$

Equations (13) and (16) can be used in eq. (32) to determine $\ddot{\theta}_s$ and $\dot{\theta}_s$, but the resulting expression is lengthy. Even for the particular case that $\phi_0 = 0 = \theta_0$, the resulting expression for $\theta_s$ is complicated,

$$\begin{aligned}
g\cos\theta_s &= \frac{R\,\theta_s}{(1+k)a^2 + R^2\,\theta_s^2}\left[2R^2\,\theta_s\left(\dot{\theta}_0^2 + \frac{2g(a+R)(1-\cos\theta_s)}{(1+k)a^2} + g(R\,\theta_s\cos\theta_s - a\sin\theta_s)\right)\right] \\
&\quad - (R-a)\left(\dot{\theta}_0^2 + \frac{2g(a+R)(1-\cos\theta_s)}{(1+k)a^2}\right). \qquad (33)
\end{aligned}$$

One conclusion that can be drawn is that for $R \gg a$ the (thin) slab will not fall off the (large) cylinder (which is perhaps obvious without the preceding analysis).

### 2.5 Both $\phi$ and $\theta$ Vary

#### 2.5.1 Coupled Oscillations

We first consider the possibility of small, coupled oscillations in both $\phi$ and $\theta$, with equilibrium angles $\phi_0$ and $\theta_0$.

We seek an oscillatory solution of the form (17)-(18) for $\theta$, and

$$\phi = \phi_0 + \beta\, e^{i\omega t}, \tag{34}$$

where $\beta$ is small, for $\phi$. To use these trial solutions in eqs. (11)-(12), we have in addition to eq. (18) that

$$\sin\phi \approx \sin\phi_0 + \beta\, e^{i\omega t}\, \cos\phi_0, \qquad \cos\phi \approx \cos\phi_0 - \beta\, e^{i\omega t}\, \sin\phi_0. \tag{35}$$

Then, keeping terms only that are constant or proportional to $e^{i\omega t}$,[4] the $\theta$-equation of motion (12) becomes

$$\begin{aligned} &-\omega^2\beta a R\, e^{i\omega t} - \omega^2\alpha\, e^{i\omega t}\left[(1+k)a^2 + R^2(\phi_0-\theta_0)^2\right] \\ \approx\ & g\left\{a(\sin\theta_0 + \alpha\, e^{i\omega t}\cos\theta_0) + R[\phi_0 - \theta_0 + (\beta-\alpha)\, e^{i\omega t}](\cos\theta_0 - \alpha\, e^{i\omega t}\sin\theta_0)\right\}. \end{aligned} \tag{36}$$

The constant term in eq. (36) must be zero, which again tells us that

$$\tan\theta_0 = \frac{R}{a}(\theta_0 - \phi_0) = -\frac{b_0}{a}\,. \tag{37}$$

The terms in $e^{i\omega t}$ must be the same on both sides of eq. (36), which tells us that

$$\begin{aligned} &\omega^2\left[\beta a R + \alpha(1+k)a^2 + \alpha R^2(\phi_0-\theta_0)^2\right] \\ =\ & g\left\{R[(\alpha-\beta)\cos\theta_0 - \alpha(\theta_0-\phi_0)\sin\theta_0] - \alpha a\cos\theta_0\right\} \\ =\ & g\left[R(\alpha-\beta)\cos\theta_0 - \alpha a/\cos\theta_0\right]. \end{aligned} \tag{38}$$

To go further, we now consider the $\phi$-equation (11),

$$-\,\omega^2\, e^{i\omega t}\left\{\beta[m + (1+K)M]R^2 + \alpha m a R\right\} \approx mgR(\sin\theta_0 + \alpha\, e^{i\omega t}\cos\theta_0). \tag{39}$$

The constant term in eq. (39) must vanish, which implies that coupled oscillations are only possible for $\theta_0 = 0$. Then, from eq. (37) we have that $\phi_0 = b_0$ also, and eq. (38) becomes

$$\omega^2\left[\beta a R + \alpha(1+k)a^2\right] = g\left[R(\alpha-\beta) - \alpha a\right]. \tag{40}$$

In addition, the terms in $e^{i\omega t}$ on the left and right sides of eq. (39) must be the same, which implies

$$-\,\omega^2\left\{\beta[m + (1+K)M]R^2 + \alpha m a R\right\} = \alpha m g R. \tag{41}$$

This condition cannot be satisfied, so **there is no coupled oscillatory motion when the cylinder is free to roll**; it will always roll out from under the slab, which rotates until it falls off the cylinder at some angle $\theta_s$ of separation. An analysis of angle $\theta_s$ could be given via an extension of the discussion in sec. 2.4.2, but we will not pursue this here.

[4] We will not consider terms in $e^{2i\omega t}$ since the approximations (18) and (35) have omitted terms of this type.

### 2.5.2 Small Oscillations of the Slab

We next consider the possibility that as the cylinder rolls the slab executes small oscillatory motion in $\theta$, with an angular frequency that varies "slowly" with time. In the "instantaneous" approximation, the angular frequency $\omega(t)$ is just that associated with that found in sec. 2.4 for $\phi_0 = \phi(t)$.

It does not appear that analytic techniques are especially helpful in the next approximation, such that it is best to use numerical integration of the equations of motion (11)-(12) to carry the discussion further.

# References


[1] K.T. McDonald, *Cylinder Rolling on a Rolling Cylinder*, (Oct. 2, 2014),
http://physics.princeton.edu/~mcdonald/examples/2cylinders.pdf

[2] K.T. McDonald, *Cylinder Rolling inside Another Rolling Cylinder*, (Oct. 21, 2014),
http://physics.princeton.edu/~mcdonald/examples/2cylinders_in.pdf

[3] K.T. McDonald, *Motion of a Rigid Body Which is Rolling without Slipping*, Princeton U. Ph205 Lecture 20 (1980),
http://physics.princeton.edu/~mcdonald/examples/Ph205/ph205l20.pdf