Can Salmon Swim up a Waterfall after Leaping into It?

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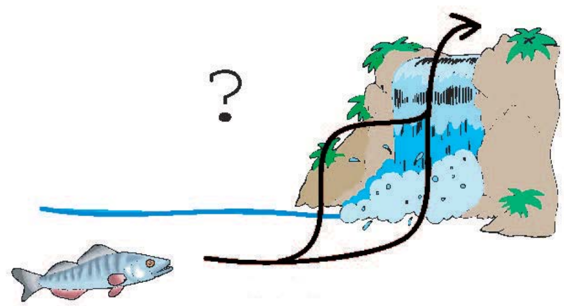
1 Problem

Spawning salmon can leap over waterfalls, but if they leap into the waterfall they may be washed back down, as seen in the video, http://www.youtube.com/watch?v=iM0mn5unvoM



Discuss whether a salmon can leap with initial vertical speed v_0 from the base of a waterfall of height H, entering the water at distance h below the top of the fall, and swim upwards to the top, assuming that the salmon swims with acceleration a relative to the downward water flow (and has maximum speed v_0 relative to the water).

Discuss also what is the maximum height H of a waterfall that the salmon could swim up without leaping, remaining in water at all times.

This problem is adapted from the Princeton University Physics 101 Learning Guide 1.

2 Solution

2.1 Leaping without Swimming

If a salmon can leap upwards with speed v_0 it can reach the top of a waterfall of height H_0 directly, without swimming in the falling water, where,

$$v_0^2 = 2gH_0. \tag{1}$$

2.2 Swimming without Leaping

If a salmon with maximum water speed v_0 tries to swim up a waterfall of height H, remaining in water at all times, it cannot make upward progress unless the velocity of the water at the base of the fall is less than v_0. The vertical speed of the water at the base of the waterfall is given by,

$$v_{\text{base}}^2 = 2gH, \tag{2}$$

supposing that the water speed at the top of the fall is near zero. The requirement that $v_{\text{base}} < v_0$ for the salmon to swim upwards from the base of the fall leads to,

$$H < \frac{v_0^2}{2g}, \tag{3}$$

where g is the acceleration due to gravity.

As the salmon swims upwards the speed of the water relative to the ground decreases, so if the salmon can maintain constant speed v_0 relative to the water even when swimming upwards, it continues to rise and eventually reaches the top of the waterfall.

But, as the salmon rises, it must expend energy against the force of gravity, and it might be that the salmon never reaches the top.

We suppose that the salmon can exert maximum power equal to that needed to swim with speed horizontally v_0 against the friction of the water. The force of friction can be modeled as [1],

$$F_{\text{drag}}(v) = \frac{1}{2}\rho C A v^2, \tag{4}$$

where ρ is the density of the water, $C \approx 0.01$ is the (small) drag coefficient, A is the surface area of the salmon and v is its speed relative to the water. We approximate the volume V of the salmon as $AT/2$ where $T \approx 0.05$ m is its average thickness. Then,

$$F_{\text{drag}}(v) = \frac{MCv^2}{T}. \tag{5}$$

As the salmon swims it pushes on the water, which pushes back on the salmon with force F_s, which equals F_{drag} when swimming horizontally at constant speed. Hence, the maximum power the salmon can exert is,

$$P_0 = F_{\text{drag}}(v_0)v_0 = \frac{MCv_0^3}{T}. \tag{6}$$

When the salmon swims vertically its equation of motion is,

$$F_s - F_{\text{drag}} - Mg = Ma_{\text{ground}}. \tag{7}$$

where the vertical acceleration upwards relative to the ground is related to the acceleration a of the salmon relative to the falling water by $a_{\text{ground}} = a - g$. Thus we can write,

$$F_s = F_{\text{drag}} + M(g + a_{\text{ground}}) = F_{\text{drag}} + Ma, \tag{8}$$

which is the equation of motion of the salmon in the accelerated frame of the falling water. On multiplying eq. (8) by the velocity v of the salmon relative to the falling water, we obtain the power relation,

$$P_s = P_{\text{drag}} + Mav, \tag{9}$$

where the power P_s exerted by the salmon is bounded by P_0 of eq. (6).

Hence, if a salmon with speed $v = v_0$ relative to the water enters the base of a waterfall where the water speed is $v_{\rm water} = \sqrt{2gH} = v_0$, it cannot accelerate upwards with respect to the downward water flow, and remains at the base of the fall, expending large amounts of energy just to stay in place with respect to the ground.

If the waterfall has height $H < H_0 = v_0^2/2g$ and the salmon enters its base with vertical speed v_0 relative to the downward water flow, then it swims upwards even when its acceleration a with respect to the water is zero, and will eventually reach the top of the waterfall. This activity, however, requires significant expenditure of energy, whereas if the salmon leaps out of the water with vertical speed v_0 it reaches the top with no additional energy cost. Hence, salmon "instinctively" choose to leap, rather than swim, up waterfalls of height $H < H_0$,

Can a salmon reach the top of a waterfall of height $H > H_0 g$ by first leaping and then swimming?

2.3 Swimming after Leaping

If instead the salmon leaps out of the water at the base of the fall with upward speed v_0, and enters the waterfall, whose height above the base is H, at height h below its top, the ground speed $v_{\rm salmon,ground,\mathit{h}}$of the salmon as it enters the water is given by,

$$v_{\rm salmon,ground,\mathit{h}} = \sqrt{v_0^2 - 2g(H-h)}, \tag{10}$$

provided that $H - h < H_0 = v_0^2/2g$. The initial ground speed $v_{\rm water,\mathit{h}}$ of the water that the salmon enters is given by,

$$v_{\rm water,\mathit{h}} = -\sqrt{2gh}, \tag{11}$$

so, the initial speed of the salmon with respect to the falling water is,

$$v_{\rm salmon,water,\mathit{h}} = v_{\rm salmon,ground,\mathit{h}} - v_{\rm water,\mathit{h}}. \tag{12}$$

If $v_{\rm salmon,water,\mathit{h}} > v_0$ the salmon cannot maintain the waterspeed v_i and slows down to (at most) waterspeed v_0 rather quickly.

If $v_{\rm salmon,water,\mathit{h}} < v_0$ the salmon can maintain this speed with respect to the water, and even increase the waterspeed $v_{\rm salmon,water}$ up to v_0. Hence, a critical case is when $v_{\rm salmon,water,\mathit{h}} = v_0$.

However, the salmon has upward motion with respect to the ground only if,

$$v_{\rm salmon,ground,\mathit{h}} = v_{\rm salmon,water,\mathit{h}} + v_{\rm water,\mathit{h}} = v_{\rm salmon,water,\mathit{h}} - \sqrt{2gh} > 0, \tag{13}$$

so the critical case is that $v_0 = v_{\rm salmon,water,\mathit{h}} = \sqrt{2gh}$ and that,

$$v_{\rm salmon,ground,\mathit{h}} = 0 = \sqrt{v_0^2 - 2g(H-h)} = \sqrt{2v_0^2 - 2gH}, \tag{14}$$

for which,

$$H = \frac{v_0^2}{g} = 2H_0. \tag{15}$$

Hence, in principle salmon can combine leaping and swimming to ascend waterfalls higher than they can leap directly. It is unclear whether this happens in practice.

The above argument can be summarized by noting that if a salmon leaps by height H_0 *into a waterfall of height* $2H_0$, *it enters the water at rest with respect to the ground but with speed* $v_0 = \sqrt{2gH_0}$ *with respect to the water, which waterspeed it can maintain. For* H *slightly less than* $2H_0$, *the salmon can enter the waterfall at height slightly less than* H_0 *with slightly positive ground speed, and maintain positive ground speed until it eventually reaches the top of the waterfall.*

References

[1] M.F.M. Osborne, *The Hydrodynamic Performance of Migratory Salmon*, J. Exp. Biol. **38**, 361 (1961), http://physics.princeton.edu/~mcdonald/examples/fluids/osborne_jeb_38_365_61.pdf