Magnetic Field at the Origin of a Grounded, Conducting Sphere Circled by a Moving Charge Q

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1 Problem

An electric charge Q moves with constant angular velocity ω in a circle of radius a whose origin is at the center of a grounded, conducting sphere of radius $b < a$. What is the magnetic field at the center of the sphere according to an observer that moves with charge Q?

2 Solution

In the (rotating) frame of the observer no "ordinary" charge is in motion, so it might be that there is no magnetic field in this frame.

However, since the the origin of the rotating frame has no translational velocity with respect to the lab frame, we expect the magnetic field at the origin in the rotating frame to differ from that at the origin in the lab frame only by a rotation about the axis $\boldsymbol{\omega}$. In the present example, the magnetic field $\mathbf{B}_0$ at the origin is constant in time and parallel to angular velocity vector $\boldsymbol{\omega}$, so the magnetic field at the origin in the rotating frame is the same nonzero, constant vector $\mathbf{B}_0$.

As noted by Schiff [1], the effect of the "distant stars" is to make "fictitious" charges and currents appear in the rotating frame, such that the electric and magnetic fields in that frame appear to have sources other than "ordinary" charges and currents [2].

The charge Q induces a surface charge density σ on the grounded, conducting sphere given by eq. (2.5) of [3],

$$\sigma = -\frac{Q}{4\pi ab}\frac{1 - b^2/a^2}{[1 - 2(b/a)\cos\theta + b^2/a^2]^{3/2}}, \tag{1}$$

where θ is the polar angle between the radial vector to charge Q and that to an element of area on the surface of the sphere, supposing that the velocity $v_Q = a\omega$ is much less than the speed of light such that effects of retardation can be ignored. In this approximation, the surface charge density rotates with angular velocity ω, although the bulk of the sphere is at rest in the lab frame.

The electric and magnetic fields can now be calculated everywhere in the lab frame, although analytic expressions for these fields involve elliptic integrals. Here we only calculate the magnetic field at the origin in the limit that $a \gg b$, in which case the surface charge density is

$$\sigma \approx -\frac{Q}{4\pi ab}[1 + 3(b/a)\cos\theta]. \tag{2}$$

The magnitude B_Q of the magnetic field at the origin in the lab frame due to moving charge Q is

$$B_Q = \frac{\mu_0 Q v_Q}{4\pi a^2} = \frac{\mu_0 Q \omega}{4\pi a}\,, \tag{3}$$

where the sphere is assumed to have the same magnetic permeability μ_0 as the vacuum. The magnitude B_σ of the field at the origin due to the rotating surface charge distribution σ is

$$\begin{aligned} B_\sigma &= \frac{\mu_0}{4\pi b^2}\int d\text{Area}\,\sigma v_\sigma \cos\theta \\ &\approx \frac{\mu_0}{4\pi ab}\int_{-1}^{1}\int_0^{2\pi} b^2\, d\cos\theta\, d\phi\,\left(-\frac{Q}{4\pi b^2}\right)[1+3(b/a)\cos\theta](b\omega\cos\theta)\cos\theta \\ &= -\frac{\mu_0 Q\omega}{12\pi a}\,. \end{aligned} \tag{4}$$

The total magnetic field at the origin in the lab frame is

$$B_0 = B_Q + B_\sigma \approx \frac{\mu_0 Q\omega}{6\pi a}\,, \tag{5}$$

when $a \gg b$.

References

[1] L.I. Schiff, *A Question in General Relativity*, Proc. Nat. Acad. Sci. **25**, 391 (1939), http://physics.princeton.edu/~mcdonald/examples/EM/schiff_proc_nat_acad_sci_25_391_39.pdf

[2] K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug. 8, 2008), http://physics.princeton.edu/~mcdonald/examples/rotatingEM.pdf

[3] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).