Voltage Across the Terminals of a Receiving Antenna

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(June 25, 2007; updated February 26, 2018)

1 Problem

Deduce the no-load (open-circuit) voltage $V_{\rm oc}$ across the terminals of a short, center-fed linear dipole antenna of half height h when excited by a plane wave of wavelength $\lambda \gg h$ whose electric field vector $\mathbf{E}_{\rm in}$ is parallel to the dipole antenna.

The ratio $H_{\rm eff} = |V_{\rm oc}/E_{\rm in}|$ is called the effective height of the antenna.[1]

Also deduce the current $I_{\rm sc}$ that would flow between the terminals if they were short circuited. Then, according to Thévenin's theorem the receiving antenna acts on any load connected to it like a voltage source $V_{\rm oc}$ with internal impedance $Z_A = V_{\rm oc}/I_{\rm sc}$.

You may assume that the antenna conductors have a diameter small compared to the height h, and that they are perfect conductors. The gap between the terminals is also small compared to h.

By dimensional analysis, the no-load voltage has amplitude of order $E_0 h$, where E_0 is the amplitude of the incident wave. The problem is to show that to a good approximation the voltage is actually $E_0 h$. This problem can be addressed using techniques that are simplifications of those appropriate for antennas comprised of thick wires with complex geometries.

2 Solution

The spirit of the solution is due to Pocklington [4], who extended the insights of Lorenz [5] and Hertz [6] that electromagnetic fields can be deduced from the retarded vector potential, by consideration of the boundary condition that the tangential component of the electric field must vanish at the surface of a good/perfect conductor. Furthermore, Pocklington noted that to a good first approximation for conductors that are thin wires, the vector potential at the surface of a wire depends only on the current in the wire at that point. Pocklington deduced an integral equation for the currents in the conductors, which equation has been elaborated upon by L.V. King [7], E. Hallén [8] and R.W.P. King [9, 10, 11, 12] to become the basis of numerical electromagnetic codes such as NEC4 [13]. See also [14], on which this solution is based.

In the present example an incident electromagnetic wave with electric field

$$\mathbf{E}_{\rm in} = E_0 e^{-i(kx-\omega t)}\,\hat{\mathbf{z}} \tag{1}$$

excites an oscillating current distribution $\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})e^{i\omega t}$ in the conductors of the receiving antenna. If this current distribution is known, then the retarded vector potential $\mathbf{A}(\mathbf{r},t) =$

[1] Schelkunoff [1, 2] has defined an effective length vector $\mathbf{H}_{\rm eff}$ for transmitting antennas in terms of their far-zone electric field $\mathbf{E}_0(\theta,\phi)e^{-i(kr-\omega t)}/r$ as $\mathbf{H}_{\rm eff}(\theta,\phi) = ic\mathbf{E_0}/kI_0$, where the current at the antenna terminals is $I_0 e^{i\omega t}$, and $\mathbf{E}_0$ and I_0 are complex quantities in general. For a linear antenna, the magnitude $H_{\rm eff}(\theta = 90^\circ)$ equals the effective height of eq. (15), as can be confirmed using eqs. (10) and (61) of [3].

$\mathbf{A}(\mathbf{r})e^{i\omega t}$ of the response fields can be calculated as

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t' = t - \mathcal{R}/c)}{\mathcal{R}}\, d\mathrm{Vol}' = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\frac{e^{ik\mathcal{R}}}{\mathcal{R}}\, d\mathrm{Vol}'\, e^{i\omega t} = \mathbf{A}(\mathbf{r})e^{i\omega t}, \qquad (2)$$

where $\mathcal{R} = |\mathbf{r} - \mathbf{r}'|$, c is the speed of light, ω is the angular frequency, $k = \omega/c$ is the wave number, and the medium outside the conductors is vacuum whose permittivity is μ_0. In the present example the conductors are thin wires along the z axis, and we suppose that the current density $\mathbf{J}(\mathbf{r})$ is independent of azimuth in a cylindrical coordinate system (ρ, ϕ, z) and is well approximated by a current $I(z)$. Then, the vector potential has only a z component,

$$A_z(\mathbf{r}) = \frac{\mu_0}{4\pi}\int I(z')\frac{e^{-ik\mathcal{R}}}{\mathcal{R}}\, dz'. \qquad (3)$$

Since we work in the Lorenz gauge where

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0, \qquad (4)$$

the scalar potential $V(\mathbf{r},t) = V(\mathbf{r})e^{i\omega t}$ of the response fields is related to the vector potential according to

$$V(\mathbf{r}) = \frac{ic}{k}\frac{\partial A_z(\mathbf{r})}{\partial z} \equiv \frac{ic}{k}\partial_z A_z(\mathbf{r}). \qquad (5)$$

The response fields $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{i\omega t}$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r})e^{i\omega t}$ can then be calculated from the vector potential $A_z(\mathbf{r})$ as

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}V(\mathbf{r}) - i\omega\mathbf{A}(\mathbf{r}) = -\frac{ic}{k}[\partial^2_{rz}A_z(\mathbf{r})\,\hat{\boldsymbol{\rho}} + (\partial_z^2 + k^2)A_z(\mathbf{r})\,\hat{\mathbf{z}}], \qquad (6)$$

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) = -\partial_\rho A_z(\mathbf{r})\,\hat{\boldsymbol{\phi}}. \qquad (7)$$

The key relation between the incident electric field $\mathbf{E}_{\mathrm{in}}$ and the response field $\mathbf{E}$ is that the tangential component of the total electric field $\mathbf{E}_{\mathrm{in}} + \mathbf{E}$ must vanish at the surface of the conductors. In the thin-wire approximation for wire radius a much less that the antenna half height h, the constraint is essentially on the z-component of the response electric field on the z axis,

$$E_z(0,0,z) = -E_{\mathrm{in}} = -E_0, \qquad (8)$$

for the intervals $[-h, -d/2]$ and $[d/2, h]$ that contain the conductors of the antenna, where the gap between the terminals of the antenna has width $d \ll h$. From eq. (6), we obtain a differential equation for the vector potential on these intervals,

$$(\partial_z^2 + k^2)A_z(0,0,z) = \frac{ik}{c}E_z(0,0,z) = -\frac{ik}{c}E_0. \qquad (9)$$

Two solutions to the homogeneous differential equation $(\partial_z^2 + k^2)A_z(0,0,z) = 0$ are, of course, $\cos kz$ and $\sin kz$. A solution to the particular equation is simply the constant $-iE_0/kc$. Hence, a general solution to eq. (9) on the interval $[d/2, h]$ can now be written as

$$A_z(0,0,d/2 \le z \le h) = C_1\cos kz + C_2\sin kz - \frac{iE_0}{kc}. \qquad (10)$$

since $kd \ll 1$. We expect that the vector potential will be symmetric about $z = 0$, so the solution on the interval $[-h, -d/2]$ can be written as

$$A_z(0,0,-h \le z \le -d/2) = C_1 \cos kz - C_2 \sin kz - \frac{iE_0}{kc}. \tag{11}$$

To evaluate the constants of integration C_1 and C_2 we need additional conditions on the system. In particular, we note that for a no-load (open circuit) receiving antenna, the current $I(z)$ must vanish at the ends of the conductors, *i.e.*, at $z = -h$, $-d/2$, $d/2$ and h. In the thin-wire approximation, the vector potential on the wire is proportional to the current in the wire at that point, because of the $1/R$ dependence in eq. (3). In this approximation, the needed conditions on the vector potential are that it also vanishes at the ends of the conductors. From this we find,

$$C_1 = \frac{iE_0}{kc}, \qquad C_2 = -\frac{iE_0}{kc}\frac{1-\cos kh}{\sin kh}. \tag{12}$$

Finally, from eq. (5) we obtain the open-circuit voltage across the terminals,

$$\begin{aligned} V_{\rm oc} &= V(0,0,d/2) - V(0,0,-d/2) = \frac{ic}{k}(A'_z(0,0,d/2) - A'_z(0,0,-d/2)) \\ &= 2icC_2 = -\frac{2E_0}{k}\frac{1-\cos kh}{\sin kh}. \end{aligned} \tag{13}$$

For a short antenna with $kh \ll 1$, the open-circuit voltage is

$$V_{\rm oc} = -E_0 h \qquad (kh \ll 1), \tag{14}$$

in agreement with the estimate via dimensional analysis.

The effective height of the antenna is

$$H_{\rm eff} = \left|\frac{V_{\rm oc}}{E_0}\right| = \frac{\lambda}{\pi}\frac{1-\cos(2\pi h/\lambda)}{\sin(2\pi h/\lambda)}. \tag{15}$$

3 Remarks

This example required a determination of the vector potential only at the antenna itself, and so is somewhat simpler than the task of determining the response fields in all space around the antenna. However the method used here is readily extended to a full solution of the antenna problem.

In particular, equations (3), (8) and (9) can be combined into an integral equation that relates the incident electric field at the conductors to the response currents in those conductors,

$$\int I(z')(\partial_z^2 + k^2)\frac{e^{-ik\mathcal{R}}}{\mathcal{R}}\, dz' = -\frac{4\pi ik}{Z_0}E_{\rm in}(z), \tag{16}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$. This is Pocklington's integral equation [4], whose solution is implemented numerically in codes such as NEC4 [13].

3.1 Short-Circuit Current in a Receiving Antenna

For the receiving antenna, it is of interest to calculate the current across its terminals when they are shorted. Then, using Thévenin's theorem [15], we could characterize the behavior of the antenna as part of the receiving circuit. An accurate calculation of the short-circuit current (or its equivalent, the antenna terminal impedance Z_A) can/must be made by solving the integral equation (16).

Here, we illustrate the limitations of the thin-wire approximation in estimating the antenna impedance. When the antenna terminals are shorted, the constraint (8) on the response field that the total, tangential electric field vanish at the surface of the wire now applies over the entire interval $[-h, h]$. Then, a symmetric solution to the differential equation (9) for the vector potential on this interval is

$$A_z(0, 0, -h \le z \le h) = C \cos kz - \frac{iE_0}{kc}. \tag{17}$$

To determine the constant C we again require that the current, and hence the vector potential in the thin-wire approximation, vanish at the ends of the wire, $z = \pm h$, such that

$$A_z(0, 0, -h \le z \le h) = \frac{iE_0}{kc} \frac{\cos kz - \cos kh}{\cos kh}. \tag{18}$$

The thin-wire approximation to eq. (3) is that $A_z(0, 0, z) \approx \mu_0 I(z)/4\pi$, so we estimate that the short-circuit current at the terminals of the receiving antenna is

$$I_{\rm sc} = \frac{4\pi}{\mu_0} \frac{iE_0}{kc} \frac{1 - \cos kh}{\cos kh} = \frac{4\pi i E_0}{kZ_0} \frac{1 - \cos kh}{\cos kh}, \tag{19}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$.

The internal impedance Z_A of the antenna is then, according to Thévenin's analysis,

$$Z_A = \frac{V_{\rm oc}}{I_{\rm sc}} \approx -\frac{iZ_0}{2\pi} \cot kh. \tag{20}$$

This correctly indicates that the reactance of a short linear antenna is capacitive and that the reactance vanishes for $h \approx \lambda/2$, but the predicted divergence of the reactance for $kh \ll 1$ is unphysical. Furthermore, the real part of the current, and also of the impedance, is neglected in the thin-wire approximation, so if the antenna were used as a transmitter, this analysis indicates that it would not consume any energy from the rf power source, *i.e.*, the antenna would not radiate.[2]

3.2 Receiving Antenna with Load Impedance Z_L

If the receiving antenna has a load impedance Z_L attached to it, then according to Thévenin's analysis the current I_L through the load will be

$$I_L = \frac{V_{\rm oc}}{Z_A + Z_L}. \tag{21}$$

[2] The real part of the antenna impedance, the so-called radiation resistance $R_{\rm rad}$, can be well calculated using the relation $\langle P \rangle = I_0^2 R_{\rm rad}/2$, where $\langle P \rangle$ is the time-average radiated power in the far zone as deduced from an appropriate thin-wire approximation to the current distribution caused by the rf power source. See, for example, [3].

A difficulty with this approach is that the receiving antenna also scatters the incident electromagnetic field, so that the total power "dissipated" by the receiving antenna system includes the power in the load plus the scattered power. The scattered electromagnetic fields $\mathbf{E}_{\rm scat}$ and $\mathbf{B}_{\rm scat}$ can be computed from the current I_L which is also the terminal current of the antenna, now considering the antenna to be a transmitter. However, the power transmitted to "infinity" is described by the Poynting vector of the sum of the incident and scattered fields, so if we write

$$\mathbf{S} = \mathbf{S}_{\rm in} + \mathbf{S}_{\rm scat}, \tag{22}$$

then

$$\mathbf{S}_{\rm scat} = \frac{\mathbf{E}_{\rm scat} \times \mathbf{B}_{\rm scat}}{\mu_0} + \frac{\mathbf{E}_{\rm scat} \times \mathbf{B}_{\rm in} + \mathbf{E}_{\rm in} \times \mathbf{B}_{\rm scat}}{\mu_0}, \tag{23}$$

which depends on details of the incident wave as well as of the receiving antenna.[3] Hence, the scattered power is not well accounted for in a Thévenin analysis, as discussed on p. 43 of [16] and in [17, 18, 19, 20, 21, 22].[4,5]

3.3 Transmitting Antenna

In the case of a transmitting antenna, the incident electric field is taken to be the internal field $E_{\rm in} = V_{\rm in}/d$ of an rf generator that is located in the gap of width d between the terminals of the antenna. This internal field is the negative of the (response) field E_z in the gap in the more realistic case that the rf generator is located some distance from the antenna and connected to it via a transmission line. The incident electric field is zero elsewhere on the conductors of the antenna. Then, the relation (8) can be extended for a center-fed linear dipole antenna to read

$$E_z(0,0,z) = -E_{\rm in}(z) = \begin{cases} 0 & (-h < z < -d/2), \\ -V_{\rm in}/d & (-d/2 < z < d/2), \\ 0 & (d/2 < z < h). \end{cases} \tag{24}$$

However, use of the extreme form of the thin-wire approximation to solve eq. (16) for the transmitting antenna leads to response currents that imply response fields with nonzero E_z along the antenna conductors. While this approximation turns out to be good at predicting the response fields in the far zone, greater care is required for a good understanding of the response fields close to the antenna [23].

[3] If the cross terms in eq. (23) could be ignored, then the (time-average) scattered power would simply be the power associated with the scattered fields, $|I_L|^2 \, Re Z_A/2$, which is the power associated with impedance Z_A in the Thévenin analysis.

[4] The current (21) in the load can also be computed in a Norton analysis where the receiving antenna is regarded as a current source of strength $I_{\rm sc} = V_{\rm oc}/Z_A$ that is in parallel with the impedance Z_A as well as with the load impedance Z_L. However, the Norton analysis suggests that the current into the antenna terminals is $I_L Z_L/Z_A$ which is not correct unless $Z_A = Z_L$. For this reason the author prefers the Thévenin analysis of a receiving antenna to the Norton analysis.

[5] If no load is attached to the receiving antenna (equivalently, $Z_L = \infty$), the antenna is being operated "open circuit" but there is still power scattered by the antenna, whereas the Thévenin analysis predicts $I_L = I_A = 0$ and hence no power dissipated by the antenna. Meanwhile, the Norton analysis predicts $I_A = I_{\rm sc}$ (and nonzero power dissipation by the antenna) even though the antenna is actually "open circuit".

A Appendix: "Electric" and "Magnetic" Antennas

Antennas are sometimes characterized as "electric" or "magnetic" depending on whether, when transmitting, their electric field E is greater or less than the magnetic field cB (SI units, or just B in Gaussian units) in the near zone (less than one wavelength from the center of the antenna).[6] For example, small linear dipole antennas are "electric", while small loop antennas are "magnetic".[7]

When considering antennas as receivers of electromagnetic waves, we note that only the incident electric field is associated with a force on the conduction electrons that can change their velocity/energy, since the magnetic force $q\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$ (whose direction is along the wires of the antenna).

A.1 Small Loop Antenna

The terminal voltage of a small, loop, receiving antenna is conveniently computed via the incident magnetic field $\mathbf{B}$, which should be perpendicular to the plane of the loop for best reception. Then, the integral form of Faraday's law tells us that the magnitude of the terminal voltage in a loop of radius r is, for, say, $\mathbf{B} = B_0\, e^{i(kx-\omega t)}\, \hat{\mathbf{z}}$ with $cB_0 = E_0$,

$$V_{\rm loop} = \left| -\frac{d\Phi_B}{dt} \right| \approx \left| i\pi r^2 \omega B_0 \right| = \frac{\pi r^2 \omega E_0}{c} = E_0 r \frac{2\pi^2 r}{\lambda}\,. \tag{25}$$

This is much smaller than the terminal voltage $E_0 r$ for a small linear antenna with arms of length r, as discussed in sec. 2 above.

Faraday's law can also be written as

$$\begin{aligned} V_{\rm loop} &= -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{Area} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{Area} = -\int \boldsymbol{\nabla}\times\mathbf{E}\cdot d\mathbf{Area} \\ &= \oint_{\rm loop} \mathbf{E}\cdot d\mathbf{l}, \end{aligned} \tag{26}$$

which latter form gives emphasis to $\mathbf{E}$ rather than $\mathbf{B}$. That is, Faraday's law, by itself, does not allow us to say that the terminal voltage is "caused" by $\mathbf{E}$ or by $\mathbf{B}$. Thus, it is misleading to imply that a "magnetic" receiving antenna couples only to the incident magnetic field, as at https://en.wikipedia.org/wiki/Loop_antenna#Magnetic_vs._electrical_antennas.

To show that the terminal voltage in the loop can be computed from the incident electric field, we suppose the loop is in the x-y plane with its leftmost point at the origin, as in the figure on the next page. The magnetic field is in the z-direction, perpendicular to the

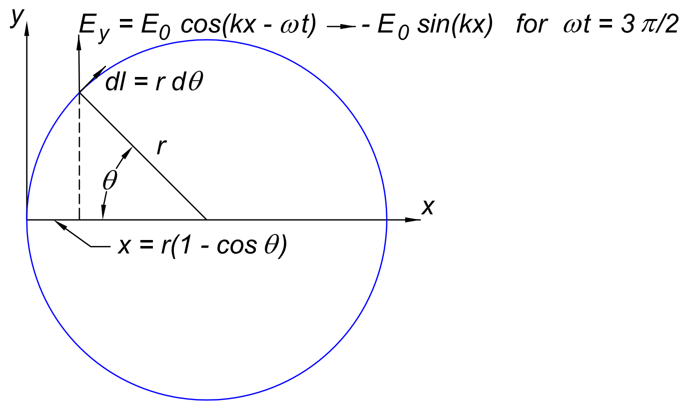
[6]The energy stored in the near field of an "electric" antenna is dominantly that in the electric field, such that the antenna's terminal reactance is capacitative (impedance $Z = R + iX$ with negative X), while the energy stored in the near field of a "magnetic" antenna is primarily in the magnetic field and the reactance is inductive ($X > 0$).

For comments on subtleties associated with antenna reactance, see [24].

[7]This is illustrated in the figure in sec. 2.2 of [25].

Self resonant antennas (with zero reactance) are neither "electric" nor '"magnetic". Both linear and loop antennas of appropriate dimensions can be self resonant, such that not all linear antennas are "electric", and not all loop antennas are "magnetic".

loop, and the electric field is in the y-direction, $E_y = E_0 \cos(kx - \omega t)$. The induced $\mathcal{EMF}$ is greatest when the time derivative of the incident magnetic (and electric) field is the greatest, such as for $\omega t = 3\pi/2$, when the electric field varies in space as $E_y = -E_0 \sin(kx)$.

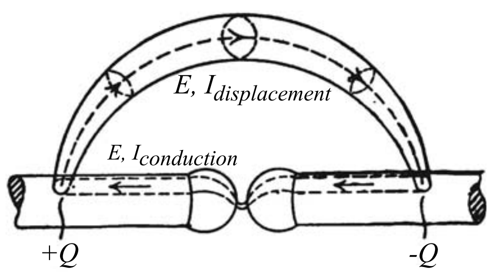


Then, for $r \ll \lambda$ ($kr \ll 1$), and noting that $x = r(1 - \cos\theta)$ for a point on the loop,

$$
\begin{aligned}
V_{\rm loop} &= \oint \mathbf{E}\cdot d\mathbf{l} = \int_0^{2\pi} r\, d\theta\, E_y \cos\theta = -rE_0 \int_0^{2\pi} d\theta\, \sin(kx)\cos\theta \\
&= -rE_0 \int_0^{2\pi} d\theta\, \sin[kr(1-\cos\theta)]\cos\theta \approx -rE_0 \int_0^{2\pi} d\theta\, kr(1-\cos\theta)\cos\theta \\
&= \pi k r^2 E_0 = E_0 r \frac{2\pi^2 r}{\lambda}\,, \qquad (27)
\end{aligned}
$$

as previously found more simply in eq. (25).[8]

[8]While the terminal voltage for a small loop antenna can be computed as above, the terminal voltage for a small linear antenna is not correctly computed by consideration of a "virtual" semicircular circuit to which Faraday's law could be applied. Such a "virtual" circuit, consisting of the two arms of the antenna plus a half circle that connects the tips of these arms, was implicit in Fig. 86 of [26] by Poincaré, as shown below.



Taking the plane of the "virtual" circuit to be perpendicular to $\mathbf{B}$ of the incident wave, we could compute that

$$
V_{\rm virtual\ circuit} = \left| -\frac{d\Phi_B}{dt} \right| \approx \left| \frac{-i\pi h^2 \omega B_0}{2} \right| = \frac{\pi h^2 \omega E_0}{2c} = E_0 h \frac{\pi^2 h}{\lambda} \ll E_0 h. \qquad (28)
$$

However, in sec. 2 above we argued that the terminal voltage due to an incident electromagnetic wave with $\mathbf{E}$ parallel to the arms of the antenna has peak value $V_{\rm linear} = E_0 h$. That is, the physics of the terminal voltage of a linear dipole antenna is different from that of the "virtual" circuit, in which the current on the semicircle is Maxwell's displacement current rather than a conduction current.

The "virtual" circuit concept is, however, useful when estimating the capacitance of a small linear antenna, as in sec. 2.1.1 of [27].

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