Poynting's Theorem with Magnetic Monopoles

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(March 23, 2013; updated January 6, 2020)

1 Problem

Deduce a version of Poynting's theorem [11] in macroscopic electrodynamics supposing that magnetic charges (monopoles) exist in Nature. Discuss its relation to the Lorentz force law for magnetic charges.[1]

2 Solution

2.1 Maxwell's Equations

When Heaviside first presented Maxwell's equations in vector notation [14] he assumed that in addition to electric charge and current densities, ρ_e and $\mathbf{J}_e$, there existed magnetic charge and current densities, ρ_m and $\mathbf{J}_m$, although there remains no experimental evidence for the latter.[2,3] Maxwell's equations for microscopic electrodynamics are then (in SI units),

$$\boldsymbol{\nabla}\cdot\epsilon_0\mathbf{E} = \rho_e, \quad \boldsymbol{\nabla}\cdot\frac{\mathbf{B}}{\mu_0} = \rho_m, \quad -c^2\boldsymbol{\nabla}\times\epsilon_0\mathbf{E} = \frac{\partial}{\partial t}\frac{\mathbf{B}}{\mu_0} + \mathbf{J}_m, \quad \boldsymbol{\nabla}\times\frac{\mathbf{B}}{\mu_0} = \frac{\partial\epsilon_0\mathbf{E}}{\partial t} + \mathbf{J}_e, \tag{1}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum. In macroscopic electrodynamics we consider media that contain volume densities of electric- and Ampèrian magnetic-dipole moments, $\mathbf{P}_e$ and $\mathbf{M}_e$, respectively (often called the densities of polarization and magnetization). Supposing that magnetic charges exists, the media could also contain volume densities of (Gilbertian) electric- and magnetic-dipole moments, $\mathbf{P}_m$ and $\mathbf{M}_m$, respectively. These densities can be associated with bound charge and current densities, which together with the "free" charge and current densities $\tilde{\rho}_e$, $\tilde{\mathbf{J}}_e$, $\tilde{\rho}_m$ and $\tilde{\mathbf{J}}_m$ comprise the total charge and current

[1]Only in three spatial dimensions do the electric and magnetic fields of electric and magnetic charges have the same character, such that a single electric and a single magnetic field could describe the effects of both types of charges [116].

[2]Heaviside seems to have regarded magnetic charges as "fictitious", as indicated on p. 25 of [21].

[3]If the interaction of magnetic charges with magnetic moments due to electrical currents is to conserve energy, the magnetic charges must be at the end of "strings" of magnetic flux, as first postulated by Dirac [54, 118].

As reviewed in Appendix D.1.1 below, Coulomb noted that long, thin magnetic needles (rigid strings) can the thought of as having equal and opposite magnetic poles (monopoles) at their two ends [2, 3]. But the relative positions of these two poles remains fixed so long as the needle is unbroken.

In recent developments with so-called spin-ice systems, chains of spins (magnetic dipoles) can exist in configurations with effective monopoles at their two ends, and the positions of these ends can be varied with respect to one another with relative ease [111, 120]. This has led to claims in the popular press that magnetic monopoles have been discovered, which some people have misinterpreted as evidence that classical electrodynamics must be modified to include nonzero ρ_m as in our eq. (1).

densities, and are related by,

$$\rho_e = \tilde{\rho}_e - \boldsymbol{\nabla} \cdot \mathbf{P}_e, \qquad \mathbf{J}_e = \tilde{\mathbf{J}}_e + \frac{\partial \mathbf{P}_e}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M}_e, \tag{2}$$

$$\rho_m = \tilde{\rho}_m - \boldsymbol{\nabla} \cdot \mathbf{M}_m, \qquad \mathbf{J}_m = \tilde{\mathbf{J}}_m + \frac{\partial \mathbf{M}_m}{\partial t} - c^2 \boldsymbol{\nabla} \times \mathbf{P}_m. \tag{3}$$

It is customary in macroscopic electrodynamics to use versions of Maxwell's equations in which only "free" charge and current densities appear. For this we introduce the fields,[4]

$$\mathbf{D}_e = \epsilon_0 \mathbf{E} + \mathbf{P}_e, \qquad \mathbf{H}_e = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}_e, \qquad \mathbf{D}_m = \frac{\mathbf{E}}{\mu_0} - c^2 \mathbf{P}_m, \qquad \mathbf{H}_m = \frac{\mathbf{B}}{\mu_0} + \mathbf{M}_m, \tag{4}$$

such that $\mathbf{D}_e$ and $\mathbf{H}_m$, and also $\mathbf{H}_e$ and $\mathbf{D}_m$, have similar forms, and,

$$\boldsymbol{\nabla} \cdot \mathbf{D}_e = \tilde{\rho}_e, \qquad \boldsymbol{\nabla} \cdot \mathbf{H}_m = \tilde{\rho}_m, \qquad -\boldsymbol{\nabla} \times \mathbf{D}_m = \frac{\partial \mathbf{H}_m}{\partial t} + \tilde{\mathbf{J}}_m, \qquad \boldsymbol{\nabla} \times \mathbf{H}_e = \frac{\partial \mathbf{D}_e}{\partial t} + \tilde{\mathbf{J}}_e, \tag{5}$$

where in the absence of magnetic charges $\mathbf{D}_e$ and $\mathbf{H}_e$ are the familiar fields $\mathbf{D}$ and $\mathbf{H}$.[5,6]

2.2 Force Laws

In static situations with no "free" currents $\tilde{\mathbf{J}}_e$ or $\tilde{\mathbf{J}}_m$ the curls of both $\mathbf{D}_m$ and $\mathbf{H}_e$ are zero and these fields can be deduced from scalar potentials V_e and V_m,

$$\boldsymbol{\nabla} \times \mathbf{D}_m = 0 \quad \Leftrightarrow \quad \mathbf{D}_m = -\boldsymbol{\nabla} V_e, \qquad \boldsymbol{\nabla} \times \mathbf{H}_e = 0 \quad \Leftrightarrow \quad \mathbf{H}_e = -\boldsymbol{\nabla} V_m. \tag{6}$$

We can associate potential energies,

$$U_e = \mu_0 q_e V_e, \qquad U_m = \mu_0 q_m V_m, \tag{7}$$

[4] The forms (4) were suggested to the author by David Griffiths in a comment on an early draft of this note. Such "double" $\mathbf{D}$ and $\mathbf{H}$ fields were anticipated by Heaviside [19], who wrote $\mathbf{H}$ for $\mathbf{H}_e$ and $\mathbf{h}_0$ for $\mathbf{H}_m$ near his eq. (88). Our eq. (4) appears as eq. (5.9) of [93], in Gaussian units, where $\mathbf{P} \to \mathbf{P}_e$, $\mathbf{M} \to \mathbf{M}_e$, $\mathbf{P}^\star \to \mathbf{P}_m$, $\mathbf{M}^\star \to \mathbf{M}_m$, $\mathbf{D} \to \mathbf{D}_e$, $\mathbf{H} \to \mathbf{H}_e$, $\mathbf{E}^\star \to \mathbf{D}_m$, $\mathbf{B}^\star \to \mathbf{H}_m$. See also in sec. 4 of [106], with the identifications that $\bar{\mathbf{e}} \to \mathbf{E}$, $\bar{\mathbf{b}} \to \mathbf{B}$, $\mathbf{p} \to \mathbf{P}_e$, $\mathbf{m} \to \mathbf{M}_e$, $\mathbf{m}^\star \to \mathbf{P}_m$, $\mathbf{p}^\star \to \mathbf{M}_m$, $\mathbf{D} \to \mathbf{D}_e$, $\mathbf{H} \to \mathbf{H}_e$, $\mathbf{E} \to \mathbf{D}_m$, $\mathbf{B} \to \mathbf{H}_m$. See Appendix D.2 for a justification of eq. (4) via the concept of electromagnetic duality

[5] The relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ (or $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ in Gaussian units) seems to have been first introduced by W. Thomson in 1871, eq. (r), p. 401 of [12], and appears in sec. 399 of Maxwell's *Treatise* [8].

[6] Neither Thomson nor Maxwell enunciated a concept of the polarization density $\mathbf{P}$ of electric dipoles, and only regarded the relation between $\mathbf{D}$ and $\mathbf{E}$ as $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is now called the (relative) dielectric constant and/or the (relative) permittivity. See Art. 111 of [7] for Maxwell's use of the term polarization.

In 1885, Heaviside introduced the concept of an electret as the electrical analog of a permanent magnet [15], and proposed that the electrical analog of magnetization (density) be called electrization. He did not propose a symbol for this, nor did he write an equation such as $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$.

The density of electric dipoles was called the polarization by Lorentz (1892) in sec. 102, p. 465 of [13], and assigned the symbol $\mathbf{M}$.

Larmor (1895), p. 738 of [23], introduced the vector (f', g', h') for what is now written as the polarization density $\mathbf{P}$, and related it to the electric field $\mathbf{E} = (P, Q, R)$ as $(f', g', h') = (K - 1)(P, Q, R)/4\pi$, *i.e.*, $\mathbf{P} = (\epsilon - 1)\mathbf{E}/4\pi = (\mathbf{D} - \mathbf{E})/4\pi$. Larmor's notation was mentioned briefly on p. 91 of [26] (1898).

The symbol $\mathbf{M}$ for dielectric polarization was changed to $\mathbf{P}$ by Lorentz on p. 263 of [31] (1902), and a relation equivalent to $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ was given in eq. (22), p. 265. See also p. 224, and eq. (147), p. 240 of [35] (1903), which latter subsequently appeared as eq. (142), p. 155 of the textbook [36] (1904) by Abraham.

with electric and magnetic "test" charges q_e and q_m in scalar potentials due to other charges. If those other charges are held fixed, the forces on the "test" charges can be written as,

$$\mathbf{F}_e = -q_e \boldsymbol{\nabla} V_e = \mu_0 q_e \mathbf{D}_m, \qquad \mathbf{F}_m = -q_m \boldsymbol{\nabla} V_m = \mu_0 q_m \mathbf{H}_e. \tag{8}$$

The magnetic version of eq. (8) was introduced by Poisson [5], and Maxwell [8] reflected this tradition by calling $\mathbf{H}_e = \mathbf{H}$ the magnetic force (per unit magnetic charge) and $\mathbf{B}$ the magnetic induction. Note that eq. (8) holds in media with nonzero, static densities $\mathbf{P}_e$, $\mathbf{P}_m$, $\mathbf{M}_e$ and $\mathbf{M}_m$; the forces on charges inside static electromagnetic media are not $q_e\mathbf{E}$ or $q_m\mathbf{B}/\mu_0$.[7,8] This contrasts with force calculations for the effective magnetic-charge density, $\rho_{m,\text{eff}} = -\boldsymbol{\nabla}\cdot\mathbf{M}_e$, which represent effects of Ampèrian currents, as discussed in Appendix A.

As noted in [92] and on p. 429 of [97], if a magnetic charge q_m could be made to move around a loop some or all of which lies inside an Ampèrian magnetic material where $\mathbf{B}$ does not equal $\mu_0\mathbf{H}_e$ (and hence $\boldsymbol{\nabla}\times\mathbf{B}$ is nonzero around the loop), then energy could be extracted from the system each cycle if the force were $q_m\mathbf{B}/\mu_0$, and we would have a perpetual-motion machine. Similarly, if an electric charge q_e could be made to move around a loop some or all of which lies inside a Gilbertian magnetic material where $\mathbf{E}$ does not equal $\mu_0\mathbf{D}_m$ (and hence $\boldsymbol{\nabla}\times\mathbf{E}$ is nonzero around the loop), then energy could be extracted from the system each cycle if the force were $q_e\mathbf{E}$, and we would again have a perpetual-motion machine.

The electromagnetic force on a moving electric charge q_e and magnetic charge q_m, each with velocity $\mathbf{v}$, is, in microscopic electrodynamics,[9,10,11,12]

$$\mathbf{F}_e = q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \mu_0 q_e\left(\mathbf{D}_m + \mathbf{v}\times\mathbf{H}_m\right), \tag{9}$$

$$\mathbf{F}_m = q_m\left(\mathbf{B} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}\right) = \mu_0 q_m(\mathbf{H}_e - \mathbf{v}\times\mathbf{D}_e). \tag{10}$$

[7] The notion of the force on a static "test" charge inside a macroscopic medium is somewhat contradictory, in that the macroscopic fields are based on averages over volumes larger than atoms/molecules. People often suppose the test charge to be inside a cavity whose volume is at least as large as an atom/molecule, but then the magnitude of the force depends on the shape of the cavity. A more meaningful issue is the force on a "test" charge that moves through the medium, thereby sampling the microscopic fields in a way that can be well approximated in terms of the macroscopic fields. See also sec. 8 of [90].

[8] In sec. 400 of [8], Maxwell noted that (in Gaussian units) the $\mathbf{H}$ field inside a disk-shaped cavity with axis parallel to $\mathbf{B}$ and $\mathbf{H}$ inside a magnetic medium has $\mathbf{H}_{\text{cavity}} = \mathbf{B}_{\text{cavity}} = \mathbf{B}_{\text{medium}} = \mathbf{H}_{\text{medium}} + 4\pi\mathbf{M}$, so that in this case one could say that the force on a magnetic charge q_m in the cavity is $\mathbf{F}_m = q_m\mathbf{H}_{\text{cavity}} = q_m\mathbf{B}_{\text{medium}}$. The led Maxwell to the characterization of $\mathbf{B}$ as the "actual magnetic force", which this author finds misleading.

[9] Lorentz advocated the form $\mathbf{F}_e = \mu_0 q_e(\mathbf{D}_e + \mathbf{v}\times\mathbf{H}_e)$ in eq. (V), p. 21, of [24], although he seems mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [43]. That is, Lorentz considered $\mathbf{D}_e$ and $\mathbf{H}_e$, rather than $\mathbf{E}$ and $\mathbf{B}$, to be the microscopic electromagnetic fields.

[10] It is generally considered that Heaviside first gave the Lorentz force law (9) for electric charges in [17], but the key insight is already visible for the electric case in [14] and for the magnetic case in [16]. The form of $\mathbf{F}_m$ in terms of $\mathbf{B}$ and $\mathbf{E}$ is implicit in eq. (7) of [62] and explicit in sec. 28B of [64]. See also [96, 107].

[11] For the macroscopic equations to appear as in eq. (5), as given, for examples, in sec. 7.3.4 and prob. 7.60 of [102], the Lorentz force law must have the form (10) for magnetic charges. One could also redefine the strength of magnetic charges, $\rho_m \to \rho_m/\mu_0$, $\mathbf{J}_m \to \mathbf{J}_m/\mu_0$, which leads to the forms given, for example, in sec. 6.11 of [103]. These alternative definitions echo a debate initiated by Clausius in 1882 [9].

[12] The electromagnetic interactions between electric and magnetic charges violate space inversion (parity). See also Appendix D.1.19 below.

Consistency of the Lorentz force law with special relativity requires that either $\mathbf{E}$ and $\mathbf{B}$ or $\mathbf{D}_e$ and $\mathbf{H}_e$ or $\mathbf{D}_m$ and $\mathbf{H}_m$ appear in $\mathbf{F}_e$ and in $\mathbf{F}_m$ (see Appendix B). In macroscopic electrodynamics the Lorentz force law for the force density $\mathbf{f}$ on "free" charge and current densities takes the forms,[13,14]

$$\mathbf{f}_e = \tilde{\rho}_e \mathbf{E} + \tilde{\mathbf{J}}_e \times \mathbf{B}, \quad \text{or} \quad \mu_0(\tilde{\rho}_e \mathbf{D}_m + \tilde{\mathbf{J}}_e \times \mathbf{H}_m), \tag{11}$$

$$\mathbf{f}_m = \tilde{\rho}_m \mathbf{B} - \frac{\tilde{\mathbf{J}}_m}{c^2} \times \mathbf{E}, \quad \text{or} \quad \mu_0(\tilde{\rho}_m \mathbf{H}_e - \tilde{\mathbf{J}}_m \times \mathbf{D}_e). \tag{12}$$

It has been verified that $\mathbf{B}$ not $\mathbf{H}_e$ deflects high-energy electrically charged particles as they pass through magnetized iron (with no magnetic charges) [60], which confirms either form of $\mathbf{f}_e$ in eq. (11).[15] See also [61, 77, 78]. The above argument about perpetual motion then favors the second forms of eqs. (11)-(12).

Further confirmation of this comes via consideration of energy flow in the electromagnetic fields.

2.3 Poynting's Theorem

Poynting's argument [11] relates the rate of work done by electromagnetic fields on "free" electric and magnetic currents to both flow of energy and to rate of change of stored energy. This argument has delicacies of interpretation, discussed, for example, in sec. 2.19 of [58]. The density of the time rate of change of work on ("free") electric currents is, from eq. (11),

$$\frac{d\tilde{w}_e}{dt} = \mathbf{f}_e \cdot \mathbf{v}_e = \tilde{\mathbf{J}}_e \cdot \mathbf{E}, \quad \text{or} \quad \mu_0 \tilde{\mathbf{J}}_e \cdot \mathbf{D}_m. \tag{13}$$

Thus, either

$$\begin{aligned}
\frac{d\tilde{w}_e}{dt} &= \tilde{\mathbf{J}}_e \cdot \mathbf{E} = \mathbf{E} \cdot \left(\boldsymbol{\nabla} \times \mathbf{H}_e - \frac{\partial \mathbf{D}_e}{\partial t} \right) \\
&= -\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}_e) + \mathbf{H}_e \cdot \boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{D}_e}{\partial t} \\
&= -\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}_e) - \mathbf{E} \cdot \frac{\partial \mathbf{D}_e}{\partial t} - \mathbf{H}_e \cdot \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{J}_m \cdot \mathbf{H}_e,
\end{aligned} \tag{14}$$

where the total current $\mathbf{J}_m$ rather than the "free" current $\tilde{\mathbf{J}}_m$ appears in the last line, or

$$\frac{d\tilde{w}_e}{dt} = \mu_0 \tilde{\mathbf{J}}_e \cdot \mathbf{D}_m = \mu_0 \mathbf{D}_m \cdot \left(\boldsymbol{\nabla} \times \mathbf{H}_e - \frac{\partial \mathbf{D}_e}{\partial t} \right)$$

[13] A subtlety is that the field $\mathbf{B}$ in the first form of eq. (11) is not the total field, but rather the field at the location of the free current that would exist in its absence. See, for example, [108], especially sec. 4.

[14] In 1908-10, Einstein argued that the Lorentz force law should take the form $\mathbf{f}_e = \mu_0(\tilde{\rho}_e \mathbf{D}_e + \tilde{\mathbf{J}}_e \times \mathbf{H}_e)$ inside materials [41, 47], perhaps based on a misunderstanding discussed in [114], or that discussed in sec. 2.3.1 of [117]. This misunderstanding underlies the recent "paradox" of Mansuripur [113].

[15] The magnetization of materials such as iron depends on the character of the magnetic moment of electrons. An argument due to Fermi [53] that the hyperfine interaction depends on the magnetic field at the origin, and so can distinguish between Ampèrian and Gilbertian moments of "nuclei". For the case of positronium (e^+e^-) the data imply that the moment of the electron is Ampèrian, as discussed in [91].

$$\begin{aligned} &= -\mu_0 \boldsymbol{\nabla} \cdot (\mathbf{D}_m \times \mathbf{H}_e) + \mu_0 \mathbf{H}_e \cdot \boldsymbol{\nabla} \times \mathbf{D}_m - \mu_0 \mathbf{D}_m \cdot \frac{\partial \mathbf{D}_e}{\partial t}. \\ &= -\mu_0 \boldsymbol{\nabla} \cdot (\mathbf{D}_m \times \mathbf{H}_e) - \mu_0 \mathbf{D}_m \cdot \frac{\partial \mathbf{D}_e}{\partial t} - \mu_0 \mathbf{H}_e \cdot \frac{\partial \mathbf{H}_m}{\partial t} - \mu_0 \tilde{\mathbf{J}}_m \cdot \mathbf{H}_e. \end{aligned} \tag{15}$$

Similarly, for "free" magnetic currents in macroscopic electrodynamics we have, from eq. (12),

$$\frac{d\tilde{w}_m}{dt} = \mathbf{f}_m \cdot \mathbf{v}_m = \tilde{\mathbf{J}}_m \cdot \mathbf{B} \quad \text{or} \quad \mu_0 \tilde{\mathbf{J}}_m \cdot \mathbf{H}_e. \tag{16}$$

A requirement of simplicity of Poynting's theorem when magnetic charges are included favors that the time rate of change of the work done on "free" electric and magnetic currents be the second forms in eqs. (13) and (16), and that the Lorentz force law on macroscopic electric and magnetic charge and current densities be,

$$\mathbf{f}_e = \mu_0(\tilde{\rho}_e \mathbf{D}_m + \tilde{\mathbf{J}}_e \times \mathbf{H}_m) \to \mu_0 \tilde{\rho}_e(\mathbf{D}_m + \mathbf{v}_e \times \mathbf{H}_m), \tag{17}$$

$$\mathbf{f}_m = \mu_0(\tilde{\rho}_m \mathbf{H}_e - \tilde{\mathbf{J}}_m \times \mathbf{D}_e) \to \mu_0 \tilde{\rho}_m(\mathbf{H}_e - \mathbf{v}_m \times \mathbf{D}_e), \tag{18}$$

(as also required not to have magnetic perpetual-motion machines).[16,17]

Then,

$$\frac{d\tilde{w}}{dt} = \frac{d\tilde{w}_e}{dt} + \frac{d\tilde{w}_m}{dt} = -\mu_0 \boldsymbol{\nabla} \cdot (\mathbf{D}_m \times \mathbf{H}_e) - \mu_0 \mathbf{D}_m \cdot \frac{\partial \mathbf{D}_e}{\partial t} - \mu_0 \mathbf{H}_e \cdot \frac{\partial \mathbf{H}_m}{\partial t} \equiv -\left(\boldsymbol{\nabla} \cdot \mathbf{S} + \frac{\partial u}{\partial t}\right), \tag{19}$$

which is the same form as if only electric charges exist, and hence the usual version of Poynting's theorem still applies if magnetic charges exist. That is, the Poynting vector,

$$\mathbf{S} = \mu_0 \mathbf{D}_m \times \mathbf{H}_e \qquad \text{(all media)}, \tag{20}$$

is interpreted as describing the flow of energy in the electromagnetic field, and for isotropic, linear media in which $\mathbf{D}_e$ and $\mathbf{D}_m$ are both proportional to $\mathbf{E}$, and $\mathbf{H}_e$ and $\mathbf{H}_m$ are both proportional to $\mathbf{B}$,[18] the density u of stored energy associated with the electromagnetic fields is,

$$u = \mu_0 \frac{\mathbf{D}_e \cdot \mathbf{D}_m + \mathbf{H}_e \cdot \mathbf{H}_m}{2} \qquad \text{(isotropic, linear media)}. \tag{21}$$

[16] The form (18) is also affirmed in [106] via considerations of a magnetic current in a "wire" surrounded by a dielectric medium. The issues here are somewhat different from those for the force on individual moving charges, but are similar to those considered in [109] for an electrical current in a wire inside a magnetic medium.

[17] It is argued in [93] that a slowly moving magnetic charge perturbs electric polarization of a dielectric medium in such a way that the velocity-dependent force is $-q_m \mathbf{v} \times \epsilon_0 \mathbf{E}$, where $\mathbf{E} = \mathbf{D}/\epsilon$ is the electric field in the absence of the moving magnetic charge. The argument of [93] seems to this author to be a variant of sec. 400 of [8] in which it is supposed that the charge resides in a "cavity" whose surface details affect the fields experienced by the charge. Such arguments assume that the charge occupies a volume at least equal to one atom/molecule of the medium, which might have seemed reasonable to Maxwell but is not consistent with our present understanding of the size of elementary charges. The results of [60] show that a moving electric charges does not create a "cavity" inside a magnetic medium wherein the average $\mathbf{B}$ field differs from the macroscopic average $\mathbf{B}$ field in the absence of the charge. We infer that a moving magnetic charge would experience an average $\mathbf{D}$ inside a dielectric medium equal to the macroscopic average $\mathbf{D}$ field in the absence of the charge.

[18] See eqs. (99)-(102) for discussion of the linear relations $\mathbf{E} = \mathbf{D}_e/\epsilon_e = \mathbf{D}_m/\epsilon_m$ and $\mathbf{B} = \mu_e \mathbf{H}_e = \mu_m \mathbf{H}_m$.

Following a general argument of Poincaré [28] and Abraham [33], we could suppose that the density of momentum is related to the Poynting vector by $\mathbf{S}/c^2$, in which case we would consider the density of field momentum to be,

$$\mathbf{p}_{\text{field}}^{(\text{A})} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{D}_m \times \mathbf{H}_e}{\epsilon_0} \qquad \text{(Abraham).} \tag{22}$$

That Poynting's theorem retains its usual form when magnetic charges are present is discussed by Heaviside in sec. 19 of [19]. That the form of the Lorentz force law for magnetic charge and current densities is given by eqs. (17)-(18) is consistent with Heaviside's argument; for example, his eq. (88), but is not explicitly stated. See also sec. 50, p. 49 of [21].

A peculiar argument that the "ordinary" form of Poynting's theorem implies the existence of magnetic charges is given in sec. 7.10 of [72]; thus misunderstanding is clarified in [74].

The extension of Poynting's theorem to momentum flow, with the implication that $\mathbf{p}_{\text{field}}^{(\text{M})} = \mu_0 \mathbf{D}_e \times \mathbf{H}_m$ *is the density of stored momentum, as argued by Minkowski [42], remains valid if the Lorentz force law for magnetic charges is given by eqs. (17)-(18), but not for other forms, as discussed in sec. V of [93]. See also Appendix C.*

In a search for an isolated magnetic charge q_m *in media that otherwise contain only electric charges and currents,* $\mathbf{D}_e \to \mathbf{D}$, $\mathbf{D}_m \to \mathbf{E}/\mu_0$, $\mathbf{H}_e \to \mathbf{H}$, $\mathbf{H}_m \to \mathbf{B}/\mu_0$, *the Lorentz force law reduces to,*

$$\mathbf{F}_e = q_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}), \qquad \mathbf{F}_m \;=\; \mu_0 q_m(\mathbf{H} - \mathbf{v}_m \times \mathbf{E}). \tag{23}$$

A Appendix: Effective Magnetic Charge Density $\rho_{m,\text{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{M}$

So far as is presently known, magnetic charges do not exist, and all magnetic effects can be associated with electrical currents, as first advocated by Ampère [4].[19] For materials with magnetization density $\mathbf{M}_e = \mathbf{M}$ the associated (macroscopic) electrical current density is,

$$\mathbf{J}_e = \boldsymbol{\nabla} \times \mathbf{M}, \tag{24}$$

and on the surface of such materials there is the surface-current density,

$$\mathbf{K}_e = \hat{\mathbf{n}} \times \mathbf{M}, \tag{25}$$

where $\hat{\mathbf{n}}$ is the outward unit vector normal to the surface.

Alternatively, we can suppose the magnetization is associated with densities of effective magnetic charges. Some care is required to use this approach, since a true (Gilbertian) magnetic charge density ρ_m would obey $\boldsymbol{\nabla}\cdot\mathbf{B} = \mu_0 \rho_m$ as in eq. (1), and the static force density on these charges would be $\mathbf{F}_m = \mu_0 \rho_m \mathbf{H}_e$. However, in Nature $\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \boldsymbol{\nabla} \cdot \mu_0(\mathbf{H} + \mathbf{M})$, so we can write,

$$\boldsymbol{\nabla} \cdot \mathbf{H} = -\boldsymbol{\nabla} \cdot \mathbf{M} = \rho_{m,\text{eff}}, \tag{26}$$

[19] For discussion of the experimental evidence that "permanent" magnetization is Ampèrian, see [117].

and identify,

$$\rho_{m,\text{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{M} \tag{27}$$

as the volume density of effective (Ampèrian) magnetic charges.

Inside isotropic, linear magnetic media, where $\mathbf{B} = \mu \mathbf{H}$, the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ then implies that $\rho_{m,\text{eff}} = 0$. However, a surface density $\sigma_{m,\text{eff}}$ of effective magnetic charges can exist on an interface between two media, and we see that Gauss' law for the field $\mathbf{H}$ implies that,

$$\sigma_{m,\text{eff}} = (\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{n}}, \tag{28}$$

where unit normal $\hat{\mathbf{n}}$ points across the interface from medium 1 to medium 2. The magnetic surface charge density can also be written in terms of the magnetization $\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H}$ as,

$$\sigma_{m,\text{eff}} = (\mathbf{M}_1 - \mathbf{M}_2) \cdot \hat{\mathbf{n}}, \tag{29}$$

since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ insures that the normal component of $\mathbf{B}$ is continuous at the interface.

The force on the surface density of effective magnetic charges is,

$$\mathbf{F} = \sigma_{m,\text{eff}} \mathbf{B}, \tag{30}$$

since the effective magnetic charges, which are a representation of effects of electrical currents, couple to the magnetic field $\mathbf{B}$, as in eq. (9).[20]

The total force on a linear medium is, in this view, the sum of the force on the conduction current plus the force on the effective magnetic surface charges. Care is required to implement such a computation of the force, as discussed in [109], where eq. (30) is affirmed by example.

The key result of this Appendix is that while "true" (Gilbertian, and nonexistent in Nature) magnetic charges q_m obey the force law $\mathbf{F}_{m,\text{true}} = \mu_0 q_m \mathbf{H}$, the effective (Ampèrian) magnetic charges (which are a representation of effects of electrical currents) obey $\mathbf{F}_{m,\text{eff}} = q_{m,\text{eff}} \mathbf{B}$.

For "effective" Ampèrian magnetic charges the magnetic fields obey $\boldsymbol{\nabla} \cdot \mathbf{B}/\mu_0 = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{H}_e = \rho_{m,\text{eff}}$ inside magnetic materials, while for "true" Gilbertian magnetic charges the fields obey $\boldsymbol{\nabla} \cdot \mathbf{B}/\mu_0 = \rho_{m,\text{true}}$ and $\boldsymbol{\nabla} \cdot \mathbf{H}_m = 0$ inside magnetic materials where there are no "free", "true" magnetic charges. Hence, the roles of $\mathbf{B}/\mu_0$ and $\mathbf{H}$ are reversed in magnetic materials that contain "true" or "effective" magnetic charges. We illustrate this below for the fields of a uniformly magnetized sphere.

A.1 Fields of a Uniformly Magnetized Sphere

In this subappendix we deduce the static magnetic fields associated with uniform spheres of radius a with either uniform Gilbertian magnetization density M_m or uniform Ampèrian (effective) magnetization density M_e.

[20] Equation (30) is in agreement with prob. 5.20 of [103], recalling the different convention for factors of μ_0 used there. However, the Coulomb Committee in their eq. (1.3-4) [63], and Jefimenko in his eq. (14-9.9a,b) [95], recommends that the field $\mathbf{H}/\mu_0$ be used rather than $\mathbf{B}$ when using the method of effective magnetic charges, which would imply a force μ_0/μ times that of eq. (30) for isotropic, linear media.

A.1.1 Uniform Ampèrian Magnetization Density M_e

The total magnetic moment of the sphere is,

$$\mathbf{m}_e = \frac{4\pi \mathbf{M}_e a^3}{3}. \tag{31}$$

We speed up the derivation by noting that the fields inside the sphere are uniform, and the fields outside the sphere are the same at those of a point magnetic dipole of strength $\mathbf{m}_e$,

$$\frac{\mathbf{B}(r > a)}{\mu_0} = \mathbf{H}_e(r > a) = \mathbf{H}_m(r > a) = \frac{3(\mathbf{m}_e \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{4\pi r^3} = \frac{M_e a^3 (2\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})}{3r^3}, \tag{32}$$

in a spherical coordinate system with origin at the center of the sphere and z-axis parallel to $\mathbf{M}_e$.

To characterize the fields inside the sphere, we note use the method of effective magnetic charges (Appendix A). Since $\mathbf{M}_e$ is constant inside the sphere, there is no net effective magnetic charge density there, $\rho_{e,\mathrm{eff}}(r < a) = -\boldsymbol{\nabla} \cdot \mathbf{M}_e(r < a) = 0$, while there is a nonzero surface density of effective magnetic charge,

$$\sigma_{e,\mathrm{eff}}(r = a) = \mathbf{M}_e \cdot \hat{\mathbf{r}} = M_e \cos\theta. \tag{33}$$

The boundary condition on the magnetic field $\mathbf{H}_e$ at the surface of the sphere is that,

$$H_{e,r}(r = a^+) - H_{e,r}(r = a^-) = \sigma_{e,\mathrm{eff}}(r = a), \tag{34}$$

and hence,

$$\begin{aligned} H_{e,r}(r = a^-) &= H_{e,r}(r < a) = H_e(r < a)\cos\theta = H_{e,r}(r = a^+) - \sigma_{e,\mathrm{eff}}(r = a) \\ &= \frac{2M_e\cos\theta}{3} - M_e\cos\theta = -\frac{M_e\cos\theta}{3}, \end{aligned} \tag{35}$$

$$\mathbf{H}_e(r < a) = -\frac{\mathbf{M}_e}{3}, \qquad \mathbf{H}_m(r < a) = \frac{\mathbf{B}(r < a)}{\mu_0} = \mathbf{H}_e(r < a) + \mathbf{M}_e(r < a) = \frac{2\mathbf{M}_e}{3}. \tag{36}$$

The result (36) for $\mathbf{B}/\mu_0$ implies that the magnetic field for the idealization of a “point”, “effective” (Ampèrian) magnetic dipole $\mathbf{m}_e$ would be,

$$\frac{\mathbf{B}}{\mu_0} = \frac{3(\mathbf{m}_e \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{4\pi r^3} + \frac{2\mathbf{m}_e}{3}\delta^3(\mathbf{r}). \tag{37}$$

A.1.2 Uniform Gilbertian Magnetization Density M_m

The total magnetic moment of the sphere for this case is,

$$\mathbf{m}_m = \frac{4\pi \mathbf{M}_m a^3}{3}. \tag{38}$$

As in sec. A.1.1, we speed up the derivation by noting that the fields inside the sphere are uniform, and the fields outside the sphere are the same at those of a point magnetic dipole of strength $\mathbf{m}_m$,

$$\frac{\mathbf{B}(r > a)}{\mu_0} = \mathbf{H}_e(r > a) = \mathbf{H}_m(r > a) = \frac{3(\mathbf{m}_m \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{4\pi r^3} = \frac{M_m a^3 (2\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})}{3r^3}. \tag{39}$$

To characterize the fields inside the sphere, we note use the method of effective magnetic charges (Appendix A). Since $\mathbf{M}_m$ is constant inside the sphere, there is no net true magnetic charge density there, $\rho_m(r < a) = -\boldsymbol{\nabla} \cdot \mathbf{M}_m(r < a) = 0$, while there is a nonzero surface density of true magnetic charge,

$$\sigma_m(r = a) = \mathbf{M}_m \cdot \hat{\mathbf{r}} = M_m \cos\theta. \tag{40}$$

The boundary condition on the magnetic field $\mathbf{B}$ at the surface of the sphere is that,

$$B_r(r = a^+) - B_r(r = a^-) = \mu_0 \sigma_m(r = a), \tag{41}$$

and hence,

$$\begin{aligned} \frac{B_r(r = a^-)}{\mu_0} &= \frac{B_r(r < a)}{\mu_0} = \frac{B(r < a)\cos\theta}{\mu_0} = \frac{B_r(r = a^+)}{\mu_0} - \sigma_r(r = a) \\ &= \frac{2M_m \cos\theta}{3} - M_m \cos\theta = -\frac{M_m \cos\theta}{3}, \end{aligned} \tag{42}$$

$$\frac{\mathbf{B}(r < a)}{\mu_0} = \mathbf{H}_e(r < a) = -\frac{\mathbf{M}_m}{3}, \qquad \mathbf{H}_m(r < a) = \frac{\mathbf{B}(r < a)}{\mu_0} + \mathbf{M}_e(r < a) = \frac{2\mathbf{M}_e}{3}. \tag{43}$$

Comparing with eqs. (35)-(36) we see that the roles of $\mathbf{B}$ and $\mathbf{H}$ are reversed in the case of uniform true and effective magnetization. In particular, the sign of $\mathbf{B}$ inside the magnetized sphere is opposite for the cases of Ampèrian and Gilbertian magnetization, although $\mathbf{B}$ is the same outside the sphere in the two cases.[21]

The result (43) for $\mathbf{B}/\mu_0$ implies that the magnetic field for the idealization of a "point," "true" (Gilbertian) magnetic dipole $\mathbf{m}_m$ would be,

$$\frac{\mathbf{B}}{\mu_0} = \frac{3(\mathbf{m}_m \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_m}{4\pi r^3} - \frac{\mathbf{m}_m}{3}\delta^3(\mathbf{r}). \tag{44}$$

B Appendix: Lorentz Transformations of the Fields

The various electromagnetic fields can be embedded in antisymmetric 4-tensors (six vectors) that obey Lorentz transformations. The microscopic fields $\mathbf{E}$ and $\mathbf{B}$ can be written as

[21] For the case of a cylinder with uniform transverse magnetization, see [104], where the interior $\mathbf{B}$ field is equal and opposite for Ampèrian and Gilbertian magnetization.

components of the tensor F, and of its dual $\mathsf{F}^\star$ obtained by the transformation $\mathbf{E} \to c\,\mathbf{B}$, $c\,\mathbf{B} \to -\mathbf{E}$,[22]

$$\mathsf{F} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}, \qquad \mathsf{F}^\star = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{pmatrix}, \tag{45}$$

such that the fields in the $'$ frame where the frame of eq. (45) has velocity $\mathbf{v}$ are,

$$\mathbf{E}' = \gamma\left(\mathbf{E} - \frac{\mathbf{v}}{c} \times c\mathbf{B}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{E})\hat{\mathbf{v}}, \tag{46}$$

$$c\mathbf{B}' = \gamma\left(c\mathbf{B} + \frac{\mathbf{v}}{c} \times \mathbf{E}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot c\mathbf{B})\hat{\mathbf{v}}. \tag{47}$$

The densities of Ampèrian electric and magnetic dipole moments $\mathbf{P}_e$ and $\mathbf{M}_e$ comprise the tensor P_e, as first noted by Lorentz [48],

$$\mathsf{P}_e = \begin{pmatrix} 0 & -P_{e,x} & -P_{e,y} & -P_{e,z} \\ P_{e,x} & 0 & M_{e,z}/c & -M_{e,y}/c \\ P_{e,y} & -M_{e,z}/c & 0 & M_{e,x}/c \\ P_{e,z} & M_{e,y}/c & -M_{e,x}/c & 0 \end{pmatrix}, \tag{48}$$

such that the fields in the $'$ frame where the frame of eq. (48) has velocity $\mathbf{v}$ are,

$$\mathbf{P}'_e = \gamma\left(\mathbf{P}_e + \frac{\mathbf{v}}{c} \times \frac{\mathbf{M}_e}{c}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{P}_e)\hat{\mathbf{v}}, \tag{49}$$

$$\frac{\mathbf{M}'_e}{c} = \gamma\left(\frac{\mathbf{M}_e}{c} - \frac{\mathbf{v}}{c} \times \mathbf{P}_e\right) - (\gamma - 1)\left(\hat{\mathbf{v}} \cdot \frac{\mathbf{M}_e}{c}\right)\hat{\mathbf{v}}, \tag{50}$$

The macroscopic fields $\mathbf{D}_e = \epsilon_0\mathbf{E} + \mathbf{P}_e$ and $\mathbf{H}_e = \mathbf{B}/\mu_0 - \mathbf{M}_e$ can be written as components of the tensor G_e,

$$\mathsf{G}_e = \epsilon_0\mathsf{F} + \mathsf{P}_e = \begin{pmatrix} 0 & -D_{e,x} & -D_{e,y} & -D_{e,z} \\ D_{e,x} & 0 & -H_{e,z}/c & H_{e,y}/c \\ D_{e,y} & H_{e,z}/c & 0 & -H_{e,x}/c \\ D_{e,z} & -H_{e,y}/c & H_{e,x}/c & 0 \end{pmatrix}, \tag{51}$$

such that the fields in the $'$ frame where the frame of eq. (51) has velocity $\mathbf{v}$ are,

$$\mathbf{D}'_e = \gamma\left(\mathbf{D}_e - \frac{\mathbf{v}}{c} \times \frac{\mathbf{H}_e}{c}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{D}_e)\hat{\mathbf{v}}, \tag{52}$$

$$\frac{\mathbf{H}'_e}{c} = \gamma\left(\frac{\mathbf{H}_e}{c} + \frac{\mathbf{v}}{c} \times \mathbf{D}_e\right) - (\gamma - 1)\left(\hat{\mathbf{v}} \cdot \frac{\mathbf{H}_e}{c}\right)\hat{\mathbf{v}}. \tag{53}$$

[22] The terminology that the electromagnetic field tensor $\mathsf{F}^\star$ is the dual of the field tensor F was introduced by Minkowski, eq. (35) of [42].

The densities of Gilbertian electric and magnetic dipole moments $\mathbf{P}_m$ and $\mathbf{M}_m$ comprise the tensor P_m,

$$\mathsf{P}_m = \begin{pmatrix} 0 & P_{m,x} & P_{m,y} & P_{m,z} \\ -P_{m,x} & 0 & M_{m,z}/c & -M_{m,y}/c \\ -P_{m,y} & -M_{m,z}/c & 0 & M_{m,x}/c \\ -P_{m,z} & M_{m,y}/c & -M_{m,x}/c & 0 \end{pmatrix} \tag{54}$$

such that the fields in the $'$ frame where the frame of eq. (54) has velocity $\mathbf{v}$ are,

$$\mathbf{P}'_m = \gamma\left(\mathbf{P}_m + \frac{\mathbf{v}}{c} \times \frac{\mathbf{M}_m}{c}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{P}_m)\hat{\mathbf{v}}, \tag{55}$$

$$\frac{\mathbf{M}'_m}{c} = \gamma\left(\frac{\mathbf{M}_m}{c} - \frac{\mathbf{v}}{c} \times \mathbf{P}_m\right) - (\gamma - 1)\left(\hat{\mathbf{v}} \cdot \frac{\mathbf{M}_m}{c}\right)\hat{\mathbf{v}}, \tag{56}$$

Finally, the macroscopic fields $\mathbf{D}_m = \mathbf{E}/\mu_0 - c^2\mathbf{P}_m$ and $\mathbf{H}_m = \mathbf{B}/\mu_0 + \mathbf{M}_m$ can be written as components of the tensor G_m,

$$\mathsf{G}_m = \frac{\mathsf{F}}{\mu_0} - c^2\mathsf{P}_m = \begin{pmatrix} 0 & D_{m,x} & D_{m,y} & D_{m,z} \\ -D_{m,x} & 0 & -cH_{m,z} & cH_{m,y} \\ -D_{m,y} & cH_{m,z} & 0 & -cH_{m,x} \\ -D_{m,z} & -cH_{m,y} & cH_{m,x} & 0 \end{pmatrix}, \tag{57}$$

such that the fields in the $'$ frame where the frame of eq. (57) has velocity $\mathbf{v}$ are,

$$\mathbf{D}'_m = \gamma\left(\mathbf{D}_m - \frac{\mathbf{v}}{c} \times c\mathbf{H}_m\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{D}_m)\hat{\mathbf{v}}, \tag{58}$$

$$c\mathbf{H}'_m = \gamma\left(c\mathbf{H}_m + \frac{\mathbf{v}}{c} \times \mathbf{D}_m\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot c\mathbf{H}_m)\hat{\mathbf{v}}. \tag{59}$$

If we accept that the forces on electric and magnetic charges q_e and q_m in their rest frame are,

$$\mathbf{F}_e = \mu_0 q_e \mathbf{D}_m, \qquad \mathbf{F}_m = \mu_0 q_m \mathbf{H}_e, \tag{60}$$

as in eq. (8), then we see by inverting eqs. (53) and (58) that the Lorentz forces in a frame where the charges have velocity $\mathbf{v}$ are as in eqs. (17)-(18).

C Appendix: Momentum Density and Stress Tensor

We can extend an argument of Minkowski [42] as to field momentum by considering the total force density on electromagnetic media, following eqs. (17)-(18),

$$\mathbf{f} = \mathbf{f}_e + \mathbf{f}_m = \mu_0(\tilde{\rho}_e\mathbf{D}_m + \tilde{\mathbf{J}}_e \times \mathbf{H}_m) + \mu_0(\tilde{\rho}_m\mathbf{H}_e - \tilde{\mathbf{J}}_m \times \mathbf{D}_e) = \frac{d\mathbf{p}_{\rm mech}}{dt}, \tag{61}$$

where $\mathbf{p}_{\rm mech}$ is the density of mechanical momentum in the media. Using the Maxwell equations (5) for the macroscopic fields,

$$\begin{aligned}
\frac{d\mathbf{p}_{\rm mech}}{dt} &= \mu_0\left[\mathbf{D}_m(\boldsymbol{\nabla}\cdot\mathbf{D}_e) - \mathbf{H}_m\times(\boldsymbol{\nabla}\times\mathbf{H}_e) + \mathbf{H}_m\times\frac{\partial\mathbf{D}_e}{\partial t}\right] \\
&\quad +\mu_0\left[\mathbf{H}_e(\boldsymbol{\nabla}\cdot\mathbf{H}_m) - \mathbf{D}_e\times(\boldsymbol{\nabla}\times\mathbf{D}_m) - \mathbf{D}_e\times\frac{\partial\mathbf{H}_m}{\partial t}\right] \\
&= -\frac{\partial}{\partial t}(\mu_0\mathbf{D}_e\times\mathbf{H}_m) + \mu_0\left[\mathbf{D}_m(\boldsymbol{\nabla}\cdot\mathbf{D}_e) + \mathbf{H}_e(\boldsymbol{\nabla}\cdot\mathbf{H}_m)\right. \\
&\qquad \left. -\mathbf{D}_e\times(\boldsymbol{\nabla}\times\mathbf{D}_m) - \mathbf{H}_m\times(\boldsymbol{\nabla}\times\mathbf{H}_e)\right] \\
&\equiv -\frac{\partial\mathbf{p}_{\rm EM}}{\partial t} + \boldsymbol{\nabla}\cdot\mathsf{T}_{\rm EM},
\end{aligned} \tag{62}$$

where,

$$\mathbf{p}_{\rm EM} = \mu_0\mathbf{D}_e\times\mathbf{H}_m \tag{63}$$

is the density of momentum associated with the electromagnetic field, and for isotropic, linear media,[23]

$$T_{{\rm EM},ij} = \mu_0\left[D_{m,i}D_{e,j} + H_{e,i}H_{m,j} - \delta_{ij}\frac{\mathbf{D}_e\cdot\mathbf{D}_m + \mathbf{H}_e\cdot\mathbf{H}_m}{2}\right] \tag{64}$$

is the symmetric Maxwell stress 3-tensor associated with the electromagnetic fields. To arrive at eq. (64) we note that for isotropic, linear media,

$$\begin{aligned}
[\mathbf{D}_m(\boldsymbol{\nabla}\cdot\mathbf{D}_e) - \mathbf{D}_e\times(\boldsymbol{\nabla}\times\mathbf{D}_m)]_i &= D_{m,i}\frac{\partial D_{e,j}}{\partial x_j} - D_{e,j}\frac{\partial D_{m,j}}{\partial x_i} + D_{e,j}\frac{\partial D_{m,i}}{\partial x_j} \\
&= \frac{\partial}{\partial x_j}\left[D_{m,i}D_{e,j} - \delta_{ij}\frac{\mathbf{D}_e\cdot\mathbf{D}_m}{2}\right].
\end{aligned} \tag{65}$$

D Appendix: Electromagnetic Duality

D.1 Digression on the History of Duality

In philosophy, the concept of dualism refers to the view that mind and matter are very different entities. The last philosopher-physicist to champion this view may have been Descartes.

In physics, duality has come to be a description of phenomena that appear to be different, but have essential properties in common.

D.1.1 Coulomb

In 1785, Coulomb confirmed (and made widely known) that the static force between pairs of electric charges q_1 and q_2 varies as q_1q_2/r^2 [2], and that the force between idealized magnetic poles p_1 and p_2 at the ends of long, thin magnets varies as p_1p_2/r^2 [3]. The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form. Nonetheless, the fact that both electric and magnetic forces are repulsive between like charges/poles, and vary as $1/r^2$, can be regarded as an early hint of electromagnetic duality.

[23] See eqs. (99)-(102) for discussion of the linear relations $\mathbf{E} = \mathbf{D}_e/\epsilon_e = \mathbf{D}_m/\epsilon_m$ and $\mathbf{B} = \mu_e\mathbf{H}_e = \mu_m\mathbf{H}_m$.

D.1.2 Maxwell

Another early hint of electromagnetic duality is in Arts. 630 and 632 of [8], where Maxwell noted that the densities of energy in the electric and magnetic fields have similar forms, $\mathbf{E} \cdot \mathbf{D}/8\pi$ and $\mathbf{B} \cdot \mathbf{H}/8\pi$.[24]

D.1.3 Clausius

An aspect of electromagnetic duality is that the units used to describe electromagnetism should respect this concept. Historically, the different systems of units developed to describe electric and magnetic phenomena did not do so, as perhaps first notably emphasized by Clausius (1882) [9]. The ensuing debate was influential in the development of our present SI system of units.

D.1.4 Heaviside

Heaviside (1885) used the term *duplex* to describe his version of Maxwell's equations that included magnetic charges and currents as well as electric charges and currents [14], and, in the footnote on p. 444 of [19] (1892), he wrote of the *Duplex method* that *its characteristics are the exhibition of the electric, magnetic, and electromagnetic relations in a duplex form, symmetrical with respect to the electric and magnetic sides*.

The term *perfect magnetic conductor* was introduced by Heaviside (1893), p. 536 of [20]: *Although a perfect magnetic conductor is, in the absence of knowledge even of a finite degree of magnetic conductivity, a very far-fetched idea, yet it is useful in electromagnetic theory to contrast with the perfect electric conductor. A perfect magnetic conductor behaves towards displacement just as a perfect electric conductor does towards induction; that is, the displacement goes round it tangentially. It also behaves towards induction as a perfect electric conductor does towards displacement; that is, the induction meets it perpendicularly, as if it possessed exceedingly great inductivity, without magnetic conductivity. This magnetic conductor is also perfectly obstructive internally, and is a perfect reflector, though not quite in the same way as electric conductors. The tangential magnetic force and the normal electric force are zero.* Thus, Heaviside considered that the boundary conditions for a perfect magnetic conductor were for $\mathbf{D}$ and $\mathbf{B}$, in contrast to those for $\mathbf{E}$ and $\mathbf{H}$ (p. 535 of [20]) in case of a perfect electric conductor.[25]

D.1.5 Tesla

In 1891, p. 34 of [18], Tesla made what may be the first use of the term *dual* in electromagnetism: *electricity and magnetism, with their singular relationship, with their seemingly dual character, unique among the forces in nature, with their phenomena of attractions,*

[24] That the electric and magnetic parts of Maxwell's stress tensor have similar forms was noted by Larmor (1897), sec. 39, p. 253, of [25].

[25] As noted in sec. 2.4.1 of [119], it seems to the present author that if a perfect electric conductor had a magnetic current on its surface, the usual boundary condition for it would no longer hold (and likewise, if a perfect magnetic conductor supported an electric surface current, then the proposed boundary condition for that novel type of conductor would not hold).

repulsions and rotations, strange manifestations of mysterious agents, stimulate and excite the mind to thought and research.

D.1.6 Poincaré

In 1893, Poincaré began his studies of algebraic topology, which are now considered to include the concept of *Poincaré duality* [112], although Poincaré himself did not use the term *duality*. Once century later, Poincaré duality is considered by some to be related to electromagnetic duality [105].

D.1.7 Love

In 1901, Love initiated studies of vector diffraction theory (building on the scalar theory of Huygens [1] and Kirchhoff [10]), in which the electromagnetic fields within some source-free volume could be computed from surface integrals of fields, or of both electric and magnetic charge and current densities. In sec. 14, p. 12 of [29], titled *The Reciprocal Theorem*, Love spoke of *magnetic displacements* as well as *electric displacements*, which is interpreted by some people as the first important application of electromagnetic duality to a physics problem, although Love did not use this term.

Love's lead was followed by Macdonald (1902), sec. 14 of [32] and p. 95 of [49].

D.1.8 Sire de Vilar

In 1901, Sire de Vilar [30] wrote on *La Dualité en Électrotechnique*, *i.e.*, *duality* in electrical circuits, which described various relations between "electric" and "magnetic" aspects of circuits, but which did not discuss electromagnetic fields.[26]

D.1.9 Larmor

Larmor (1903), p. 10 of [34], mentioned a *magnetic current-sheet, which may be considered to arise from a varying sheet of tangential magnetization, after the analogy of the electric current-sheet of the previous case*. Here, Larmor added some clarity to Love's usage [29] of *magnetic displacements*, that what matters in particular for the surface integrals which they considered is the possibility of *magnetic currents*. This can be interpreted as an expression of electromagnetic duality, although Larmor only spoke of an *analogy*.

Larmor (1900) used the term *duality* in the interesting comment, p. 27 of [27]: *The duality arising from the assumption of two kinds of electrons, only differing chirally so that one is the reflexion of the other in a plane mirror, will present nothing strange to those physicists who regard with equanimity even the hypothesis of the possible existence of both positive and negative matter.* Here, Larmor anticipates that electrons have spin, and that there exist negative electrons = positrons.

[26] The paper of Vilar was transcribed into English as Chap. 21 of [38].

D.1.10 Silberstein

In 1907, Silberstein [40] discussed electromagnetism in terms of the bivector $\mathbf{E}_1 + i\mathbf{E}_2 = \mathbf{E} + i\mathbf{B}$, as well as its complex conjugate, noting that Maxwell's equations have the same form for either bivector. This was an early recognition of electromagnetic duality, without using that term.

D.1.11 Minkowski

The earliest use of the term *dual* in the manner of the present conception of electromagnetic duality was by Minkowski (1908), pp. 81 and 93 of [42], where he wrote of the *dual matrix* $\mathsf{F}^\star = F^\star_{\mu\nu}$ of a 4-tensor $\mathsf{F} = F_{\mu\nu}$ such as that of the electromagnetic field, eq. (45) above.

D.1.12 Sommerfeld

Sommerfeld (1910) [44, 46] (see also pp. 754-755 of [64]) recast the 6 independent, nonzero component of the electromagnetic field tensor F as a *6 vector* $(\mathbf{E}, \mathbf{H})$ with an electric and a magnetic *side*.[27] Sommerfeld, eqs. (18) and (18*) of [46], was the first to write Maxwell's equations in the form, in Gaussian units,

$$\partial_\mu \mathsf{F}_{\mu\nu} = \frac{4\pi}{c} J_\mu \,, \qquad \partial_\mu \mathsf{F}^\star_{\mu\nu} = 0, \qquad \text{where} \qquad \partial_\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right). \tag{66}$$

A survey in 1910 [45] of methods of solution of Maxwell's equation via transformations of other solutions did not include use of duality transformations (which had not yet been recognized explicitly). See also [55].

D.1.13 Rainich

In 1925, Rainich [51] discussed symmetries of antisymmetric 4-tensors, such as the stress-energy-momentum tensor of electrodynamics. On p. 113 he noted that if such a tensor is represented in a certain manner by two pairs of 3-vectors, $\{\mathbf{i}, \mathbf{j}\}$ and $\{\mathbf{k}, \mathbf{l}\}$,the physics is invariant under the transformations $\mathbf{i}' = \mathbf{i}\cos\chi - \mathbf{j}\sin\chi$, $\mathbf{j}' = \mathbf{i}\sin\chi + \mathbf{j}\cos\chi$, and $\mathbf{k}' = \mathbf{k}\cos\psi - \mathbf{l}\sin\psi$, $\mathbf{l}' = \mathbf{k}\sin\psi + \mathbf{l}\cos\psi$; Rainich left it to the reader to imagine a relation between his four 3-vectors and the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

D.1.14 Schelkunoff

Schelkunoff (1936) continued the theme of vector diffraction theory, with the comment on p. 93 of [56]: *We shall find it convenient, at least for analytical purposes, to employ the concept of magnetic current on a par with the concept of electric current.*

And on p. 69 of [59] he wrote; *On some occasions, considerable mathematical simplifications may be secured if we write Maxwell's equations in a more symmetrical form by including hypothetical magnetic charges and corresponding currents.*

[27] A 6-vector is closely related to the bivector of Hamilton, p. 665 of [6], first applied to electromagnetism by Silberstein (1907) [40].

However, Schelkunoff (and subsequent electrical engineers) seem to regard the symmetry of Maxwell's equations slightly differently than do most physicists. On p. 70 of [59] he wrote: *Maxwell's equations in the form in which we have expressed them possess considerable symmetry;* **E** *and* **H** *correspond to each other, the first being measure in volts per meter and the second in amperes per meter;* **D** *and* **B** *correspond to each other, the first being measured in ampere-seconds per square meter and the second in volt-seconds per square meter; electric and magnetic currents correspond to each other, the first being measured in amperes and the second in volts. In literature one finds arguments to the effect that "physically"* **E** *and* **B** *(and* **D** *and* **H***) are similar*[28] *and that* **B** *is more "basic" than* **H**.[29] *All such arguments seem sterile since electric and magnetic quantities are physically different; whatever similarity there is comes from the equations.*[30]

On p. 107 of [56], Schelkunoff revived Heaviside's concept [20] of a perfect magnetic conductor: *Perfect magnetic conductors are defined by analogy with perfect electric conductors-the tangential component of the magnetic intensity vanishes at the surface of the former just as the tangential component of the electric intensity vanishes at the surface of the latter. Magnetic conductors support magnetic current sheets just as electric conductors support electric current sheets. The densities of the sheets are given by the discontinuities of the tangential components of* **E** *in the former case and* **H** *in the latter.*

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[28]In 1904, Lorentz [37] gave the Lorentz transformation for electromagnetic fields in which **D** and **H** transform into one another for observers in different inertial frames of reference. This transformation was confirmed and greatly elucidated by Einstein in 1905 [39]. Such considerations had dramatic impact on the physics community, but apparently had little effect on electrical engineers.

[29]For example, see the final pages, pp. 327-328, of the 1929 physics text [52]: *There remains, finally, one relatively trivial matter which should be mentioned. Except for the linkage with more ordinary notations which occurs in the problems, use has been made of only two field vectors,* **E** *and* **B**. *When in free space, the use of one electric and one magnetic vector, rather than of the four vectors* **E**, **D**, **B** *and* **H**, *is an obviously desirable simplification; and within matter, it is useful to have in explicit evidence the electrical properties* ϵ *and* μ *of the matter. The choice of* **B** *as the fundamental magnetic vector, rather than* **H**, *rests on the occurrence of* **E** *and* **B** *in the equation of force for a charge. The subject of magnetostatics has been developed in as close analogy as possible with electrostatics; and the fundamental magnetic vector—the counterpart of* **E**—*must clearly be the vector which, in the basic law for magnetostatic action, plays the same rôle as does* **E** *in electrostatics. The choice of* **B** *rather than* **H** *is also clearly indicated by the fact that the divergence of* **E** *gives the total charge, while the curl of* **B** *(not of* **H***) give the total current. The confusion which results from the choice of* **H** *as the fundamental magnetic vector is, perhaps, most clearly illustrated by the equations which arise when one considers the relation between the so-called microscopic and macroscopic field equations. Lorentz, for example, take as microscopic equations, valid everywhere,*

$$\mathrm{div}\,\mathbf{e} = \rho, \qquad \mathrm{curl}\,\mathbf{e} = -\frac{1}{c}\frac{\partial \mathbf{h}}{\partial t}, \qquad \mathrm{div}\,\mathbf{h} = 0, \qquad \mathrm{curl}\,\mathbf{h} = \frac{1}{c}(\rho\,\mathbf{v} + \dot{\mathbf{e}}), \tag{67}$$

and finds that the average values of **e** *and* **h** *are given by,*

$$\overline{\mathbf{e}} = \mathbf{E} \qquad \overline{\mathbf{h}} = \mathbf{B}, \tag{68}$$

where **E** *and* **B** *are the ordinary macroscopic field vectors used in this volume. The last equation indicates that* **B** *is the fundamental macroscopic vector, and the that fundamental microscopic vector should be designated as* **b** *rather than* **h**.

[30]Such views are at odds with the theme of this note, that if magnetic charges and currents existed as well as electric charges and currents, then there would be two **D**-fields, $\mathbf{D}_e$ and $\mathbf{D}_m$, as well as two **H**-fields, $\mathbf{H}_e$ and $\mathbf{H}_m$, but only one **E**-field and only one **B**-field, as discussed in sec. 2.1 above.

D.1.15 Stratton and Chu

In 1939, Stratton and Chu [57] continued the theme of vector diffraction theory, and also wrote of *ficticious magnetic sources*. Such considerations were reviewed by Stratton in sec. 8.14, p. 464, of his text (1941) [58].

On p. 72 of [58], and again on p. 82, Stratton wrote of the *dual* field tensors in the manner of Minkowski, but without references, and he did not consider a duality transformation.

D.1.16 Shanmugadhasan (Written June 15, 2020)

In 1952, Shanmugadhasan [65] noted that if magnetic charges existed, one could consider dual field tensors (in the sense of Minkowski [42]),

$$\mathsf{F}_e = \begin{pmatrix} 0 & -\frac{E_{e,1}}{c} & -\frac{E_{e,2}}{c} & -\frac{E_{e,3}}{c} \\ \frac{E_{e,1}}{c} & 0 & -B_{e,3} & B_{e,2} \\ \frac{E_{e,2}}{c} & B_{e,3} & 0 & -B_{e,1} \\ \frac{E_{e,3}}{c} & -B_{e,2} & B_{e,1} & 0 \end{pmatrix}, \quad \mathsf{F}_e^\star = \begin{pmatrix} 0 & -B_{e,1} & -B_{e,2} & -B_{e,3} \\ B_{e,1} & 0 & \frac{E_{e,3}}{c} & -\frac{E_{e,2}}{c} \\ B_{e,2} & -\frac{E_{e,3}}{c} & 0 & \frac{E_{e,1}}{c} \\ B_{e,3} & \frac{E_{e,2}}{c} & -\frac{E_{e,1}}{c} & 0 \end{pmatrix}, \tag{69}$$

$$\mathsf{F}_m = \begin{pmatrix} 0 & -B_{m,1} & -B_{m,2} & -B_{m,3} \\ B_{m,1} & 0 & \frac{E_{m,3}}{c} & -\frac{E_{m,2}}{c} \\ B_{m,2} & -\frac{E_{m,3}}{c} & 0 & \frac{E_{m,1}}{c} \\ B_{m,3} & \frac{E_{m,2}}{c} & -\frac{E_{m,1}}{c} & 0 \end{pmatrix}. \quad \mathsf{F}_m^\star = \begin{pmatrix} 0 & -\frac{E_{m,1}}{c} & -\frac{E_{m,2}}{c} & -\frac{E_{m,3}}{c} \\ \frac{E_{m,1}}{c} & 0 & -B_{m,3} & B_{m,2} \\ \frac{E_{m,2}}{c} & B_{m,3} & 0 & -B_{m,1} \\ \frac{E_{m,3}}{c} & -B_{m,2} & B_{m,1} & 0 \end{pmatrix}, \tag{70}$$

where $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$, such that the total field tensors are,

$$\mathsf{F} = \mathsf{F}_e + \mathsf{F}_m^\star, \qquad \mathsf{F}^\star = \mathsf{F}_e^\star + \mathsf{F}_m. \tag{71}$$

In terms of the field tensors, Maxwell's equations (1) are,[31,32]

$$\partial_\nu \mathsf{F}_{e,\nu\mu} = \mu_0 J_{e,\mu}, \qquad \partial_\nu \mathsf{F}_{m,\nu\mu} = -\mu_0 J_{m,\mu}, \qquad \partial_\nu \mathsf{F}^\star_{e,\nu\mu} = 0 = \partial_\nu \mathsf{F}^\star_{m,\nu\mu}, \tag{72}$$

where $J_\mu = (c\rho, \mathbf{J})$ is the charge-current 4-vector; in terms of the partial fields,

$$\boldsymbol{\nabla}\cdot\mathbf{E}_e = \frac{\rho_e}{\epsilon_0}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B}_e = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E}_e = -\frac{\partial\mathbf{B}_e}{\partial t}, \qquad \boldsymbol{\nabla}\times\mathbf{B}_e = \frac{1}{c^2}\frac{\partial\mathbf{E}_e}{\partial t} + \mu_0\mathbf{J}_e, \tag{73}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}_m = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{B}_m = \mu_0\rho_m, \quad \boldsymbol{\nabla}\times\mathbf{E}_m = -\frac{\partial\mathbf{B}_m}{\partial t} - \mu_0\mathbf{J}_m, \quad \boldsymbol{\nabla}\times\mathbf{B}_m = \frac{1}{c^2}\frac{\partial\mathbf{E}_m}{\partial t}. \tag{74}$$

As usual, since $\boldsymbol{\nabla}\cdot\mathbf{B}_e = 0$ we can relate $\mathbf{B}_e$ to a vector potential $\mathbf{A}_e$ such that $\mathbf{B}_e = \boldsymbol{\nabla}\times\mathbf{A}_e$. Then, $\boldsymbol{\nabla}\times\mathbf{E}_e = -\boldsymbol{\nabla}\times\partial\mathbf{A}_e/\partial t$, such that there exists a scalar potential V_e where

[31] We avoid distinguishing between covariant and contravariant vectors and tensors by use of the conventions that $\partial_\mu = (\partial/\partial ct, -\boldsymbol{\nabla})$ and $A_\mu B_\mu = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$.

[32] The Lorentz force laws for charge and currents densities that correspond to eq. (8) can be written as $f_\mu = \mathsf{F}_{\nu\mu}J_\nu$ where $J_\nu = J_{e,\mu} + J_{m,\nu}$.

$\mathbf{E}_e + \partial \mathbf{A}_e/\partial t = -\boldsymbol{\nabla} V_e$. Similarly, since $\boldsymbol{\nabla} \cdot \mathbf{E}_m = 0$ we can relate $\mathbf{E}_m$ to a vector potential $\mathbf{A}_m$ such that $\mathbf{E}_m = -\boldsymbol{\nabla} \times \mathbf{A}_m$. Then, $\boldsymbol{\nabla} \times \mathbf{B}_m = -\boldsymbol{\nabla} \times \partial \mathbf{A}_m/\partial t$, such that there exists a scalar potential V_m where $\mathbf{B}_m + \partial \mathbf{A}_m/\partial t = -\boldsymbol{\nabla} V_m$.

That is, there exist (dual) 4-potentials $A_{e,\mu} = (V_e/c, \mathbf{A}_e)$ and $A_{m,\mu} = (V_m/c, \mathbf{A}_m)$ such that,

$$\mathbf{E}_e = -\boldsymbol{\nabla} V_e - \frac{\partial \mathbf{A}_e}{\partial t}, \quad \mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e, \quad \mathbf{E}_m = -\boldsymbol{\nabla} \times \mathbf{A}_m, \quad \mathbf{B}_m = -\boldsymbol{\nabla} V_m - \frac{\partial \mathbf{A}_m}{\partial t}. \tag{75}$$

The field tensors F_e and F_m are related to the 4-potentials by,

$$\mathsf{F}_{e,\mu\nu} = \partial_\mu A_{e,\nu} - \partial_\nu A_{e,\mu}, \qquad \mathsf{F}_{m,\mu\nu} = \partial_\nu A_{m,\mu} - \partial_\mu A_{m,\nu}. \tag{76}$$

These dual fields and potentials have also been discussed in [75, 84, 85, 94, 107, 121].[33]

D.1.17 Rose

In 1955, sec. 5, p. 9, of his book [66], Rose wrote: *We again consider the free-space Maxwell equations. It is evident that if* $\mathbf{E}$ *and* $\mathbf{H}$ *are solutions, then* $\mathbf{E}'$ *and* $\mathbf{H}'$ *are also solutions if,*

$$\mathbf{E}' = \pm \mathbf{H}, \qquad \mathbf{H}' = \mp \mathbf{E}, \tag{1.31}$$

where either the upper or lower signs are to be used. The field $\mathbf{E}'$, $\mathbf{H}'$ *is* dual *to the field* $\mathbf{E}$, $\mathbf{H}$. *Clearly, apart from an irrelevant overall sign,* $\mathbf{E}$, $\mathbf{H}$ *is dual to* $\mathbf{E}'$, $\mathbf{H}'$.

This may be the first explicit statement of an electromagnetic duality transformation.

Note that Rose did not consider magnetic charges or currents to exist, and restricted his duality transformation to source-free (free-space) regions. That is, the duality described by Rose holds in the absence of magnetic charges and currents, and holds in regions that can be described by permittivities ϵ and permeabilities μ different from ϵ_0 and μ_0 but which regions cannot contain permanent electric moments, or permanent magnetic moments, of any order.

The pseudoduality described by Rose (and many followers) contrasts with what I will call full duality that would hold if magnetic charges and currents existed (and which would hold throughout all space). These two types of electromagnetic duality are almost never distinguished in the literature, which is therefore somewhat confusing.

D.1.18 Ohmura

In 1956, Ohmura, in a very brief note [67], discussed electrodynamics with both electric and magnetic charges, and with dual potentials.

D.1.19 Ramsey

In 1957, shortly after the discovery of parity violation in the weak interaction [68], Ramsey [70] noted that the electromagnetic interactions of electric and magnetic charges would violate parity (P). He speculated that these interaction would obey the combined symmetries CP and MP, where C and M are the operations of electric and magnetic charge conjugation.[34]

See also sec. 5 of [75].

[33] Potentials V_m and $\mathbf{A}_m$ were discussed for static fields in eq. (24) of [99], supposing that $\mathbf{E}_e = \boldsymbol{\nabla} \times \mathbf{A}_m$.

[34] It is now known that CP symmetry is also violated in the weak interaction [81].

D.1.20 Harrington

In 1958, Harrington, sec. 7.6, p. 177 of [73], discussed a concept of duality in which magnetic charges and currents existed, but he was not aware that in this case there would be two **D** fields and two **H** fields. As such, his identifications in Table 7-2, p. 178, that **E** is the dual of **H**, and that **D** is the dual of **B**, are not viable.

In 1961, Harrington produced another book [73], in which magnetic charges and currents have (p. 34) *no physical significance*, but are sometimes considered as *fictitious* sources in problems as a means of discussing the electromagnetic fields that hold when only electric charges and currents are present. The *duality* discussed by Harrington in sec. 3.2, p. 98, of [73], is thus very close to the *pseudoduality* of Rose [66].[35]

The Rose/Harrington version of *duality* continues to have considerable influence on the electrical-engineering literature, in contrast to the literature in physics which tends to follow the versions of Katz, Calkin, Schwinger and Jackson (although the latter two versions have issues that will be noted below).

D.1.21 Cabibbo and Ferrari

In 1962, Cabibbo and Ferrari [75] independently introduced the dual potentials discussed in sec. D.1.16 above.

D.1.22 Katz

The work of Rainich (sec. D.1.13 above) went largely unnoticed until 1964, when Katz, sec. IV of [82], stated that Rainich had shown (or implied) that Maxwell's equations are invariant under the transformations,

$$\mathbf{E}' = \mathbf{E}\cos\theta + \mathbf{B}\sin\theta, \qquad \mathbf{B}' = -\mathbf{E}\sin\theta + \mathbf{B}\cos\theta, \tag{77}$$

$$\rho_e' = \rho_e\cos\theta + \rho_m\sin\theta, \qquad \rho_m' = -\rho_e\sin\theta + \rho_m\cos\theta, \tag{78}$$

$$\mathbf{J}_e' = \mathbf{J}_e\cos\theta + \mathbf{J}_m\sin\theta, \qquad \mathbf{J}_m' = -\mathbf{J}_e\sin\theta + \mathbf{J}_m\cos\theta, \tag{79}$$

where θ is an arbitrary constant.

Katz did not use the term *duality*. He did consider the possible physical significance of the parameter θ, but concluded that no experiment could determine its value.

D.1.23 Calkin

In 1965, Calkin [83] argued that the free-space Maxwell equations are invariant under the transformation,

$$\mathbf{E}' = \mathbf{E}\cos\theta + \mathbf{B}\sin\theta, \qquad \mathbf{B}' = -\mathbf{E}\sin\theta + \mathbf{B}\cos\theta, \tag{80}$$

where θ is an arbitrary constant. He further noted, in the spirit of Noether's theorem [50], that this invariance principle is associated with the "conservation law" that the difference

[35] Another vision of electric/magnetic "analogies" from this time is [76].

between the number of left and right circularly polarized photons in the electromagnetic field is a constant.[36,37]

Calkin did not mention the term *duality*, nor did he cite Rainich [51]. Calkin's paper was submitted a few days before the publication of Katz' paper [82] (in the same journal), but this went unacknowledged.

D.1.24 Rohrlich

In 1966, Rorhlich [85] dicussed *duality invariance* that would hold if magnetic charges exists, as well a dual potentials, citing [75] on the latter.

D.1.25 Misner and Wheeler

In 1967, Misner and Wheeler, p. 529 of [69], followed a hint of Rainich [51] to consider the transformation involving fields $\mathbf{E}$ and $\mathbf{H}$ and an arbitrary angle α. This was not quite the duality transformation of secs. D.1.22 and D.1.23 below.

D.1.26 Schwinger

In 1969, Schwinger [88] gave a statement of a continuous duality symmetry of Maxwell's equations (without using the term *duality*) including both electric and magnetic charge and current densities, but was somewhat careless in not distinguishing between "free" and "total" (= "free" + "bound") charge and current densities. His equations would be correct if the densities were the "total" densities, and his $\mathbf{H}$ were replaced by $\mathbf{B}$. Then, his field transformations would agree with our eq. (80), due to Calkin [83].[38]

Schwinger also gave the (duality) transformation between electric and magnetic charge densities.[39]

D.1.27 Jackson

In the 1975 edition of his text, sec. 6.12 of [89], Jackson gave a version of Schwinger's argument, but with the field $\mathbf{E}$ replaced by $\mathbf{D}$ in some places, and also without being clear as to whether the charge and current densities were "free" or "total".

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[36] This observation suggests that while the duality symmetry of electromagnetism is elegant, its physical significance is relatively minor. Furthermore, it is unclear that Calkin's conservation law holds for interacting electromagnetic fields.

[37] Calkin was partly inspired by the discovery, reported in [79], of other conserved quantities for free electromagnetic fields. See also [80].

[38] The discrete duality transformation (81) is given on p. 18 of Schwinger's posthumous text [101].

[39] Schwinger's main theme was a possible extension of electromagnetic duality to include the weak interaction, in which case there might be vector bosons with magnetic charge, in addition to the (since discovered) electrically charged gauge-bosons $W^{\pm}$. Schwinger did not cite the then-recent electroweak model of Weinberg [86] and Salam [87]. There has been surprisingly little followup to Schwinger's suggestion, but one example is [110].

In the meantime, the most cited paper in elementary-particles physics is the so-called gauge-gravity duality of Maldacena [100] (with Weinberg's paper having the secondmost citations).

Again, a more correct treatment would be to use the "total" charge and current densities, and only the fields $\mathbf{E}$ and $\mathbf{B}$. If it is desired to consider the fields $\mathbf{D}$ and $\mathbf{H}$, then one must use all four of $\mathbf{D}_e$, $\mathbf{D}_m$, $\mathbf{H}_e$ and $\mathbf{H}_m$ as in sec. D.3 below.

D.2 Microscopic Electrodynamics

As noted at the beginning of Appendix B, Minkowski [42] implicitly introduced the duality transformation,

$$\mathbf{E} \to c\,\mathbf{B}, \qquad c\,\mathbf{B} \to -\mathbf{E}, \tag{81}$$

as a useful construct when discussing Maxwell's equation in 4-tensor notation. The microscopic form of Maxwell's equation, eq. (1), is symmetric (invariant) under this transformation if in addition all electric charges are replaced by magnetic charges, and *vice versa*.

We now consider the appropriate duality transformation for other quantities than the fields $\mathbf{E}$ and $\mathbf{B}$. An important constraint is that such a transformation should relate quantities that have the same dimensions. Now, in SI units electric and magnetic charges do not have the same dimensions, so their duality transformation is not simply $q_e \leftrightarrow q_m$.

First, we note that in SI units the microscopic density u of electromagnetic field energy is,

$$u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\,. \tag{82}$$

That is, $\sqrt{\epsilon_0}\,\mathbf{E}$ and $\mathbf{B}/\sqrt{\mu_0}$ have the same dimension, such that $\mathbf{E}$ and $\mathbf{B}/\sqrt{\epsilon_0\mu_0} = c\,\mathbf{B}$ have the same dimensions. Hence, Minkowski's duality transformation (81) is indeed between quantities with the same dimensions.[40]

We then rewrite the first two Maxwell equations (1) as,

$$\boldsymbol{\nabla}\cdot\sqrt{\epsilon_0}\,\mathbf{E} = \frac{\rho_e}{\sqrt{\epsilon_0}}\,, \quad \boldsymbol{\nabla}\cdot\frac{\mathbf{B}}{\sqrt{\mu_0}} = \sqrt{\mu_0}\,\rho_m, \tag{83}$$

which indicates that in SI units $\rho_e/\sqrt{\epsilon_0}$ has the same dimensions as $\sqrt{\mu_0}\,\rho_m$, *i.e.*, that q_e and q_m/c have the same dimensions. Thus, the duality transformations for charges, and charge and current densities, are,[41]

$$q_e \to \frac{q_m}{c}\,, \qquad \frac{q_m}{c} \to -q_e, \qquad \rho_e \to \frac{\rho_m}{c}\,, \qquad \frac{\rho_m}{c} \to -\rho_e, \qquad \mathbf{J}_e \to \frac{\mathbf{J}_m}{c}\,, \qquad \frac{\mathbf{J}_m}{c} \to -\mathbf{J}_e. \tag{84}$$

These relations are consistent with the microscopic force laws (9)-(10),

$$\mathbf{F}_e = q_e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times c\,\mathbf{B}\right), \qquad \mathbf{F}_m = \frac{q_m}{c}\left(c\,\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}\right), \tag{85}$$

[40] This contrasts with Schelkunoff's view, sec. D.1.14 above, that the duality transformation should be between $\mathbf{E}$ and $\mathbf{H}$ because these have "analogous" dimensions, volts/m and amperes/m.

[41] If only electric charges exist in Nature (as is the case so far as we know), the duality transformations (81) and (84) have the somewhat trivial application that we could redefine electric fields to be magnetic fields, and *vice versa*, call all charges magnetic, and say that electric charges do not exist.

whose duality transformation is,

$$\mathbf{F}_e \leftrightarrow \mathbf{F}_m. \tag{86}$$

For completeness, we note that the third and fourth Maxwell equations (1) can be written as,

$$-\boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial}{\partial ct}(c\,\mathbf{B}) + \mu_0 c\frac{\mathbf{J}_m}{c}, \qquad \boldsymbol{\nabla} \times c\,\mathbf{B} = \frac{\partial \mathbf{E}}{\partial ct} + \mu_0 c\,\mathbf{J}_e, \tag{87}$$

which are consistent with the duality transformations (81) and (84).

One should not (in this author's view) say that the constants ϵ_0 and μ_0 (which have different dimensions) are duals of one another. Indeed, since $\epsilon_0\mu_0 = 1/c^2$, they are not independent quantities, and electrodynamics could be formulated using only one of them (along with the universal constant c). For example, the static force between two like electric charges separated by distance r is $F_e = q_e^2/4\pi\epsilon_0 r^2$, while the force between two magnetic charges is $F_m = \mu_0 q_m^2/4\pi r^2 = \mu_0 c^2 (q_m/c)^2/4\pi r^2 = (q_m/c)^2/4\pi\epsilon_0 r^2$, such that the duality relation (84) leads to the relation (86) without any "duality relation" between ϵ_0 and μ_0.

D.2.1 Dual Potentials (Written June 15, 2020)

As perhaps first noted in [65], if magnetic charges existed one could consider (dual) 4-potentials $A_{e,\mu} = (V_e/c, \mathbf{A}_e)$ and $A_{m,\mu} = (V_m/c, \mathbf{A}_m)$ such that (in our notation) $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$ where,

$$\mathbf{E}_e = -\boldsymbol{\nabla} V_e - \frac{\partial \mathbf{A}_e}{\partial t}, \quad \mathbf{E}_m = -\boldsymbol{\nabla} \times \mathbf{A}_m, \quad \mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e, \quad \mathbf{B}_m = -\boldsymbol{\nabla} V_m - \frac{1}{c^2}\frac{\partial \mathbf{A}_m}{\partial t}. \tag{88}$$

The duality transformations for the potentials, and for the partial fields, are,

$$V_e \to cV_m, \qquad cV_m \to -V_e, \qquad c\mathbf{A}_e \to \mathbf{A}_m, \qquad \mathbf{A}_m \to -c\mathbf{A}_e, \tag{89}$$

$$\mathbf{E}_e \to c\mathbf{B}_m, \qquad \mathbf{E}_m \to c\mathbf{B}_e, \qquad c\mathbf{B}_e \to -\mathbf{E}_m, \qquad c\mathbf{B}_m \to -\mathbf{E}_e. \tag{90}$$

D.3 Macroscopic Electrodynamics

In macroscopic electrodynamics we consider media with densities $\mathbf{P}_e$ of electric dipoles and $\mathbf{M}_e$ of (Ampèrian) magnetic moments due to electrical currents. As the dual of electric charge q_e is magnetic charge divided by c, q_m/c, the dual of $\mathbf{P}_e$ is a density $\mathbf{M}_m/c$ of (Gilbertian) magnetic dipoles due to pairs of opposite magnetic charges, divided by c. Similarly, the dual of $\mathbf{M}_e$ is proportional to a density $\mathbf{P}_m$ of (Gilbertian) electric dipoles due to currents of magnetic charges.

Historically, the electric densities $\mathbf{P}_e$ and $\mathbf{M}_e$ were incorporated in the macroscopic fields $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}_e$ and $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}_e$ such that $\mathbf{H}/c = \epsilon_0\, c\mathbf{B} - \mathbf{M}_e/c$. While we say that $c\,\mathbf{B}$ is the dual of $\mathbf{E}$, it is not the case that $\sqrt{\mu_0/\epsilon_0}\mathbf{H}$ is the dual of $\mathbf{E}$ (nor is $\mathbf{B}$ the dual of $\sqrt{\mu_0/\epsilon_0}\mathbf{D}$, as claimed in sec. 6.11 of [103]). Rather, we see that duality requires the introduction of a second $\mathbf{D}$ and a second $\mathbf{H}$ field according to eq. (4),

$$\mathbf{D}_e = \epsilon_0\mathbf{E} + \mathbf{P}_e, \qquad \mathbf{H}_e = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}_e, \qquad \mathbf{D}_m = \frac{\mathbf{E}}{\mu_0} - c^2\,\mathbf{P}_m, \qquad \mathbf{H}_m = \frac{\mathbf{B}}{\mu_0} + \mathbf{M}_m, \tag{91}$$

with the duality transformations,

$$c\,\mathbf{P}_e \to \mathbf{M}_m, \qquad \mathbf{M}_e \to -c\,\mathbf{P}_m, \qquad c\,\mathbf{P}_m \to \mathbf{M}_e, \qquad \mathbf{M}_m \to -c\,\mathbf{P}_e, \tag{92}$$

$$c\,\mathbf{D}_e \to \mathbf{H}_m, \qquad c\,\mathbf{H}_e \to -\mathbf{D}_m, \qquad \mathbf{D}_m \to c\,\mathbf{H}_e, \qquad \mathbf{H}_m \to -c\,\mathbf{D}_e. \tag{93}$$

A consequence of the minus signs in eq. (92) is that while the Ampèrian magnetic dipole moment $\mathbf{m}_e$ associated with electric current density $\mathbf{J}_e$ is,

$$\mathbf{m}_e = \int \frac{\mathbf{r} \times \mathbf{J}_e}{2}\, d\mathrm{Vol} \to I_e\, \mathbf{Area}, \tag{94}$$

the Gilbertian electric dipole moment $\mathbf{p}_m$ associated with magnetic current density $\mathbf{J}_m$ is,

$$\mathbf{p}_m = -\int \frac{\mathbf{r} \times \mathbf{J}_m}{2}\, d\mathrm{Vol} \to -I_m\, \mathbf{Area}. \tag{95}$$

The macroscopic force densities and their duality relation are,

$$\mathbf{f}_e = \mu_0(\tilde{\rho}_e \mathbf{D}_m + \tilde{\mathbf{J}}_e \times \mathbf{H}_m), \qquad \mathbf{f}_m = \mu_0(\tilde{\rho}_m \mathbf{H}_e - \tilde{\mathbf{J}}_m \times \mathbf{D}_e), \qquad \mathbf{f}_e \leftrightarrow \mathbf{f}_m. \tag{96}$$

The Poynting vector is,

$$\mathbf{S} = \mu_0 \mathbf{D}_m \times \mathbf{H}_e \qquad \text{(all media)}, \tag{97}$$

and for isotropic, linear media in which $\mathbf{D}_e$ and $\mathbf{D}_m$ are both proportional to $\mathbf{E}$, and $\mathbf{H}_e$ and $\mathbf{H}_m$ are both proportional to $\mathbf{B}$, the density u of stored energy associated with the electromagnetic fields is,

$$u = \mu_0 \frac{\mathbf{D}_e \cdot \mathbf{D}_m + \mathbf{H}_e \cdot \mathbf{H}_m}{2} \qquad \text{(isotropic, linear media)}. \tag{98}$$

Both $\mathbf{S}$ and u are self-dual.

For isotropic, linear media with polarization densities $\mathbf{P}_e$ and $\mathbf{M}_e$ based on electric charges and currents, we can write,

$$\mathbf{P}_e = \epsilon_0 \chi_{D_e} \mathbf{E}, \qquad \mathbf{D}_e = \epsilon_e \mathbf{E}, \qquad \epsilon_e = \epsilon_0(1 + \chi_{H_e}), \tag{99}$$

$$\mathbf{M}_e = \chi_{D_e} \mathbf{H}_e, \qquad \mathbf{B} = \mu_e \mathbf{H}_e, \qquad \mu_e = \mu_0(1 + \chi_{H_e}), \tag{100}$$

such that ϵ_e and μ_e revert to the familiar permittivity ϵ and permeability μ in the absence of magnetic charges and currents. To have the corresponding relations for polarization densities $\mathbf{P}_m$ and $\mathbf{M}_m$ based on magnetic charges and currents obey forms similar to eqs. (99)-(100) for isotropic, linear media, we use,

$$\mathbf{P}_m = \epsilon_0 \mu_0 \chi_{D_m} \mathbf{D}_m, \qquad \mathbf{D}_m = \epsilon_m \mathbf{E}, \qquad \epsilon_m = \frac{\epsilon_0 c^2}{1 + \chi_{D_m}} = \frac{1}{\mu_0(1 + \chi_{D_m})}, \tag{101}$$

$$\mathbf{M}_m = \chi_{H_m} \frac{\mathbf{B}}{\mu_0}, \qquad \mathbf{B} = \mu_m \mathbf{H}_m, \qquad \mu_m = \frac{\mu_0}{1 + \chi_{H_m}}. \tag{102}$$

However, the permittivities ϵ, the permeabilities μ and the susceptibilities χ do not obey simple duality relations.

D.3.1 Why Does $\mathbf{p}_m = -\int \mathbf{r} \times \mathbf{J}_m \, d\text{Vol}/2c^2$?

As noted in eqs. (94)-(95), the duality transformation (92) contains the prescription that the dual of a magnetic-dipole moment due to electric currents is the negative of an electric-dipole moment due to magnetic currents,

$$\mathbf{m}_e \to -c\,\mathbf{p}_m, \tag{103}$$

which minus sign is perhaps counterintuitive. Now,

$$\mathbf{m}_e = \int \frac{\mathbf{r} \times \mathbf{J}_e}{2} \, d\text{Vol} \to \int \frac{\mathbf{r} \times \mathbf{J}_m}{2} \, d\text{Vol}, \tag{104}$$

so eqs. (103)-(104) imply that,

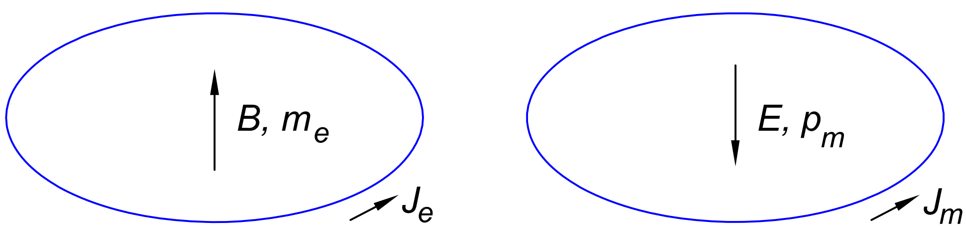
$$\mathbf{p}_m = -\int \frac{\mathbf{r} \times \mathbf{J}_m}{2c^2} \, d\text{Vol}, \tag{105}$$

which also is perhaps surprising.

We recall that for a magnetic dipole $\mathbf{m}_e$ associated with an electric-current density $\mathbf{J}_e$ that flows in a loop, say in a static situation, the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}_e c$ implies that,

$$\mu_0 \oint_{\text{loop}} \mathbf{J}_e \cdot d\mathbf{l} = \oint_{\text{loop}} \boldsymbol{\nabla} \times \mathbf{B} \cdot d\mathbf{l} = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area}. \tag{106}$$

That is, the direction of the magnetic field $\mathbf{B}$ at the center of the loop is related to the direction of $\mathbf{J}_e$ by the righthand rule, as sketched in the left figure below.



This is consistent with the usual relation,

$$\mathbf{m}_e = \int \frac{\mathbf{r} \times \mathbf{J}_e}{2} \, d\text{Vol}. \tag{107}$$

In the case of a loop of magnetic current density $\mathbf{J}_m$, again in a static situation, the Maxwell equation (24), $\boldsymbol{\nabla} \times \mathbf{E} = -\mu_0 \mathbf{J}_m$, implies that,

$$\mu_0 \oint_{\text{loop}} \mathbf{J}_m \cdot d\mathbf{l} = -\oint_{\text{loop}} \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{l} = \int_{\text{loop}} \mathbf{E} \cdot d\mathbf{Area}. \tag{108}$$

That is, the direction of the magnetic field $\mathbf{E}$ at the center of the loop is related to the direction of $\mathbf{J}_m$ by the lefthand rule, as sketched in the right figure above. This is consistent with the relation (105), which is in turn consistent with the duality relation (61), $\mathbf{m}_e \to -c\,\mathbf{p}_m$.

Thus, the difference in sign between the relations (105) and (107) is due to the difference in signs of the terms in the current densities in the Maxwell equations,

$$c^2 \boldsymbol{\nabla} \times \epsilon_0 \mathbf{E} = -\left(\frac{\partial}{\partial t} \frac{\mathbf{B}}{\mu_0} + \mathbf{J}_m \right), \qquad \boldsymbol{\nabla} \times \frac{\mathbf{B}}{\mu_0} = \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} + \mathbf{J}_e. \tag{109}$$

The equation for $\boldsymbol{\nabla} \times \mathbf{E}$ with magnetic currents was first discussed by Heaviside in 1885 [14]. He argued (p. 448 of [14]) that just as in the equation for $\boldsymbol{\nabla} \times \mathbf{B}$ where the current density $\mathbf{J}_e$ and the "displacement-current density" $\partial \epsilon_0 \mathbf{E}/\partial t$ have the same sign, the current density $\mathbf{J}_m$ and the "magnetic displacement-current density" $\partial(\mathbf{B}/\mu_0)/\partial t$ should have the same sign in the equation for $\boldsymbol{\nabla} \times \mathbf{E}$.[42]

D.4 Gaussian Units

We have already remarked (footnote 11, p. 3 above) that some people write the second Maxwell equation as $\boldsymbol{\nabla} \cdot \mathbf{B} = \rho_m$ in SI units. The macroscopic fields $\mathbf{D}_m$ and $\mathbf{H}_m$ could also be defined differently than here. In particular, the definition $\mathbf{D}_m = \epsilon_0 \mathbf{E} - \mathbf{P}_m$ (which is our $\mathbf{D}_m$ divided by c^2) could be considered, although then the third and fourth Maxwell equations of eq. (5) differ more in their forms.

Gaussian units were developed to have greater symmetry between electric and magnetic quantities, which will carry over into the duality relations in these units. Here, we summarize the main electrodynamic relations in Gaussian units.

The microscopic Maxwell equations in Gaussian units are,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi \rho_e, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 4\pi \rho_m, \quad -c \boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + 4\pi \mathbf{J}_m, \quad c \boldsymbol{\nabla} \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}_e, \tag{110}$$

for which the duality relations are,

$$q_e \to q_m, \qquad q_m \to -q_e, \qquad \mathbf{E} \to \mathbf{B}, \qquad \mathbf{B} \to -\mathbf{E}. \tag{111}$$

In Gaussian units, electric and magnetic charges have the same dimensions, and electric and magnetic fields have the same dimensions.

The macroscopic fields are,

$$\mathbf{D}_e = \mathbf{E} + 4\pi \mathbf{P}_e, \qquad \mathbf{H}_e = \mathbf{B} - 4\pi \mathbf{M}_e, \qquad \mathbf{D}_m = \mathbf{E} - 4\pi \mathbf{P}_m, \qquad \mathbf{H}_m = \mathbf{B} + 4\pi \mathbf{M}_m, \tag{112}$$

with the duality transformations,

$$\mathbf{P}_e \to \mathbf{M}_m, \qquad \mathbf{M}_e \to -\mathbf{P}_m, \qquad \mathbf{P}_m \to \mathbf{M}_e, \qquad \mathbf{M}_m \to -\mathbf{P}_e, \tag{113}$$

$$\mathbf{D}_e \to \mathbf{H}_m, \qquad \mathbf{H}_e \to -\mathbf{D}_m, \qquad \mathbf{D}_m \to \mathbf{H}_e, \qquad \mathbf{H}_m \to -\mathbf{D}_e. \tag{114}$$

The macroscopic Maxwell equations are,

$$\begin{gathered} \boldsymbol{\nabla} \cdot \mathbf{D}_e = 4\pi \tilde{\rho}_e, \qquad \boldsymbol{\nabla} \cdot \mathbf{H}_m = 4\pi \tilde{\rho}_m, \\ -c \boldsymbol{\nabla} \times \mathbf{D}_m = \frac{\partial \mathbf{H}_m}{\partial t} + 4\pi \tilde{\mathbf{J}}_m, \qquad c \boldsymbol{\nabla} \times \mathbf{H}_e = \frac{\partial \mathbf{D}_e}{\partial t} + 4\pi \tilde{\mathbf{J}}_e. \end{gathered} \tag{115}$$

[42] See also [98].

The microscopic force law on charges is,

$$\mathbf{F}_e = q_e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \qquad \mathbf{F}_m = q_m \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \right), \qquad \mathbf{F}_e \leftrightarrow \mathbf{F}_m, \tag{116}$$

while the macroscopic force densities and their duality relations are,

$$\mathbf{f}_e = \tilde{\rho}_e \mathbf{D}_m + \frac{\tilde{\mathbf{J}}_e}{c} \times \mathbf{H}_m, \qquad \mathbf{f}_m = \tilde{\rho}_m \mathbf{H}_e - \frac{\tilde{\mathbf{J}}_m}{c} \times \mathbf{D}_e, \qquad \mathbf{f}_e \leftrightarrow \mathbf{f}_m. \tag{117}$$

The Poynting vector is,

$$\mathbf{S} = \frac{\mathbf{D}_m \times \mathbf{H}_e}{4\pi c} \qquad \text{(all media)}, \tag{118}$$

and for linear media in which $\mathbf{D}_e$ and $\mathbf{D}_m$ are both proportional to $\mathbf{E}$, and $\mathbf{H}_e$ and $\mathbf{H}_m$ are both proportional to $\mathbf{B}$, the density u of stored energy associated with the electromagnetic fields is,

$$u = \frac{\mathbf{D}_e \cdot \mathbf{D}_m + \mathbf{H}_e \cdot \mathbf{H}_m}{8\pi} \qquad \text{(linear media)}. \tag{119}$$

Both $\mathbf{S}$ and u are self-dual.

For isotropic, linear media with polarization densities $\mathbf{P}_e$ and $\mathbf{M}_e$ based on electric charges and currents, we can write,[43]

$$\mathbf{P}_e = \chi_{D_e} \mathbf{E}, \qquad \mathbf{D}_e = \epsilon_e \mathbf{E}, \qquad \epsilon_e = 1 + 4\pi\chi_{D_e}, \tag{120}$$

$$\mathbf{M}_e = \chi_{H_e} \mathbf{H}_e, \qquad \mathbf{B} = \mu_e \mathbf{H}_e, \qquad \mu_e = 1 + 4\pi\chi_{H_e}. \tag{121}$$

such that ϵ_e and μ_e revert to the usual ϵ and μ in the absence of magnetic charges and currents. To have the corresponding relations for polarization densities $\mathbf{P}_m$ and $\mathbf{M}_m$ based on magnetic charges and currents obey forms similar to eqs. (120)-(121) for linear media, we might define,[44]

$$\mathbf{P}_m = \chi_{D_m} \mathbf{D}_m, \qquad \mathbf{D}_m = \epsilon_m \mathbf{E}, \qquad \epsilon_m = \frac{1}{1 + 4\pi\chi_{D_m}}, \tag{122}$$

$$\mathbf{M}_m = \chi_{H_m} \mathbf{B}, \qquad \mathbf{B} = \mu_m \mathbf{H}_m, \qquad \mu_m = \frac{1}{1 + 4\pi\chi_{H_m}}. \tag{123}$$

The fields $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$ are related to potentials $A_{e,\mu} = (V_e, \mathbf{A}_e)$ and $A_{m,\mu} = (V_m, \mathbf{A}_m)$ as,

$$\mathbf{E}_e = -\boldsymbol{\nabla} V_e - \frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t}, \quad \mathbf{E}_m = -\boldsymbol{\nabla} \times \mathbf{A}_m, \quad \mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e, \quad \mathbf{B}_m = -\boldsymbol{\nabla} V_m - \frac{1}{c}\frac{\partial \mathbf{A}_m}{\partial t} \tag{124}$$

The duality transformations for the potentials are,

$$V_e \to V_m, \qquad V_m \to -V_e, \qquad \mathbf{A}_e \to \mathbf{A}_m, \qquad \mathbf{A}_m \to -\mathbf{A}_e. \tag{125}$$

[43]For discussion of the effect of the factors of 4π in the conversion of $\mathbf{D}$ and $\mathbf{H}$ between SI and Gaussian units, see [115].

[44]If we defined $\mathbf{P}_m = \chi_{D_m} \mathbf{E}$ and $\mathbf{M}_m = \chi_{H_m} \mathbf{H}_m$, we might readily have negative values of ϵ_m and μ_m.

Acknowledgment

Thanks to Stephen Barnett, David Griffiths and Ömer Zor for e-discussions of this problem.

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