Power Received by a Small Antenna

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1 Problem

Deduce an approximate expression for the maximum power that can be received by a small antenna with a load that includes a resistance R (as well as a possible reactance) when the antenna is in a linearly polarized incident plane wave of wavelength λ and (time-average) power $P_{\rm in}$ per unit area. Show that the maximum power received is approximately,

$$P_{\rm max} \approx P_{\rm in} \frac{\lambda^2}{4\pi}\,, \tag{1}$$

independent of the physical size of the small antenna.

Antennas used for reception of signals associated with a bandwidth Δ about a carrier frequency ω_0 should have a Q no larger than $\omega_0/2\Delta$. Discuss the resulting limit on the power received by the antenna compared to the maximum (1).

2 Solution

We first note that the (time-average) incident power per unit area, assuming that the medium surrounding the antenna has unit relative permittivity and unit relative permeability, can be written as,

$$P_{\rm in} = \langle S \rangle = \frac{1}{2} E_0 H_0 = \frac{E_0^2}{2Z_0}\,, \tag{2}$$

where c is the speed of light in vacuum, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector, E_0 and H_0 are the amplitudes of the incident electric and magnetic fields, and $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$ ($= 4\pi/c$ in Gaussian units).[1,2]

We also note that the small antennas are generally well described by their electric and magnetic dipole moments.[3] So, we consider here the cases of a small linear dipole antenna, for which only its electric dipole moment is significant, and a small loop antenna, for which only its magnetic dipole moment is significant.

2.1 Effective Height and Effective Area

The present analysis builds on that presented in [2]. So far as the receiver circuit is concerned, the antenna can be considered as a two-terminal device which can be characterized, according to Thévenin, by a voltage source $V_{\rm oc}$ and a series impedance Z_A.

[1] Some people use E_0, I_0 and V_0 to denote RMS field, current and voltage, in which case various expressions involving the squares of these quantities differ by a factor of 2.

[2] The assumption in eq. (2) that $H_0 = E_0/Z_0$ is only valid, in general, when the receiving antenna is in the "far zone" of the source.

[3] An interesting exception is a small, counterwound, helical, toroidal antenna [1].

The strength of the voltage source is the (open-circuit) voltage across the terminals of the antenna when it is not connected to anything. We can write the open-circuit voltage in terms of an effective height $h_{\rm eff}$ as,

$$V_{\rm oc} = E_0 h_{\rm eff}, \tag{3}$$

where E_0 is the amplitude of the incident electric field.

The impedance Z_A equals $V_{\rm oc}/I_{\rm sc}$ where $I_{\rm sc}$ is the current between the terminals of the antenna when they are short-circuited. The short-circuit current is difficult to estimate accurately by analytic techniques, but for a short antenna it is almost purely out of phase with respect to the incident field. Then, the impedance Z_A is almost purely imaginary, corresponding to the large capacitive reactance of the gap between the antenna terminals. The impedance Z_A relevant to a receiving antenna is the same as the terminal impedance of the antenna when used for transmission, according to an antenna reciprocity theorem. For transmitting antennas the real part of the impedance can be written $R_{\rm Ohmic} + R_{\rm rad}$, where $R_{\rm Ohmic}$ is the effective resistance of the antenna due to the finite conductivity of its conductors, and $R_{\rm rad}$ is the so-called radiation resistance of the antenna.

If the load on the receiving antenna is described by an impedance $Z_L = R_L + iX_L$, then the (complex) current I through the load has amplitude,

$$I = \frac{V_{\rm oc}}{Z_A + Z_L}\,, \tag{4}$$

and the power delivered into the load resistor R_L is,

$$\begin{aligned} P &= \frac{1}{2}\left|I\right|^2 R = \frac{V_{\rm oc}^2}{2}\frac{R_L}{\left|Z_A + Z_L\right|^2} = \frac{E_0^2 h_{\rm eff}^2}{2}\frac{R}{\left|Z_A + Z_L\right|^2} \\ &= P_{\rm in}\frac{R_L Z_0}{\left|Z_A + Z_L\right|^2} h_{\rm eff}^2 \equiv P_{\rm in} A_{\rm eff}, \end{aligned} \tag{5}$$

where,

$$A_{\rm eff} = \frac{R_L Z_0}{\left|Z_A + Z_L\right|^2} h_{\rm eff}^2. \tag{6}$$

$A_{\rm eff}$ is the effective area of the antenna system.

The process of maximizing the effective area of the receiving antenna is called matching. A first step is to make the reactance X_L of the load equal and opposite to the reactance X_A of the antenna. When this is done, the effective area becomes,

$$A_{\rm eff} = \frac{R_L Z_0}{(R_{\rm Ohmic} + R_{\rm rad} + R_L)^2} h_{\rm eff}^2 \qquad \text{(matching reactance)}. \tag{7}$$

It is now clear (if it wasn't before) that the antenna will perform better if the effective Ohmic resistance of its conductors is negligible, which we will assume to be the case from now on. Then,

$$A_{\rm eff} = \frac{R_L Z_0}{(R_{\rm rad} + R_L)^2} h_{\rm eff}^2 \qquad \text{(matching reactance, } R_{\rm Ohmic} \ll R_{\rm rad}\text{)}. \tag{8}$$

Finally, we maximize the effective area by choosing a load resistor R_L equal in value to the radiation resistance $R_{\rm rad}$. This may not be practical, but in principle we obtain,

$$A_{\rm eff,max} = \frac{Z_0}{4R_{\rm rad}} h_{\rm eff}^2 \qquad \text{(matching reactance, } R_{\rm Ohmic} \ll R_L = R_{\rm rad}). \tag{9}$$

2.2 Small Linear Dipole Antenna

As shown, for example, in sec. 2 of [2] the effective height of a small linear antenna is,

$$h_{\rm eff} = h \qquad \text{(small linear antenna)}, \tag{10}$$

where the length of each of its two arms is $h \ll \lambda$. The radiation resistance of a small linear dipole antenna is (see, for example, p. 192 of [3]),

$$R_{\rm rad} = \frac{2\pi}{3} Z_0 \left(\frac{h}{\lambda}\right)^2 \qquad \text{(small linear antenna)}. \tag{11}$$

Hence, according to eq. (9) the maximum effective area of a small linear dipole antenna is,

$$A_{\rm eff,max} = \frac{Z_0}{4R_{\rm rad}} h_{\rm eff}^2 = \frac{3\lambda^2}{8\pi} \qquad \text{(matching inductor, } R_{\rm Ohmic} \ll R_L = R_{\rm rad}), \tag{12}$$

noting that the reactance of a small linear antenna is capacitive, so the matching element must be an inductor.

The result (12) is often written to include a factor of D, the directivity of the antenna, defined to be the maximum of the angular function $f(\theta, \phi)$ such that the angular distribution of the power radiated by the antenna (when used as a transmitter) is,

$$P(\theta, \phi) = \frac{P_{\rm total}}{4\pi} f(\theta, \phi). \tag{13}$$

For a small dipole antenna (either linear or loop), the function f is $(3/2)\sin^2\theta$ for polar angle θ to the relevant axis. Hence, $D = 3/2$ is the directivity for a small dipole antenna, and we can write,

$$A_{\rm eff,max} = D\frac{\lambda^2}{4\pi} = \pi \lambdabar^2 D \qquad \text{(matching inductor, } R_{\rm Ohmic} \ll R_L = R_{\rm rad}), \tag{14}$$

where $\lambdabar = \lambda/2\pi$ is the reduced wavelength.

2.3 Small Dipole Loop Antenna

The radiation resistance of a small loop antenna of radius h is (see, for example, p. 192 of [3]),

$$R_{\rm rad} = \frac{8\pi^5}{3} Z_0 \left(\frac{h}{\lambda}\right)^4 \qquad \text{(small loop antenna)}. \tag{15}$$

According to Faraday's law, the open-circuit voltage for a small loop antenna equals the time rate of change of the magnetic flux through the loop,[4]

$$V_{\rm oc} = \pi h^2 \omega B_0 = \frac{2\pi^2 h^2}{\lambda} E_0, \tag{16}$$

so the effective height of a small loop antenna is,

$$h_{\rm eff} = \frac{2\pi^2 h^2}{\lambda} \qquad \text{(small loop)}. \tag{17}$$

Thus, the maximum effective area of the small loop antenna follows from eq. (14) as,

$$A_{\rm eff,max} = \frac{3\lambda^2}{8\pi} = \frac{D\lambda^2}{4\pi} \qquad (\text{matching capacitor, } R_{\rm Ohmic} \ll R = R_{\rm rad}), \tag{18}$$

recalling that the directivity of a small loop (dipole) antenna is 3/2, and that since the reactance of a loop is inductive the matching element must be a capacitor.[5] This result is the same as (14) for a small linear dipole antenna.

3 An Effective-Area Theorem

The results (14) and (18) are examples of the fact that for any antenna,

$$A_{\rm eff,max} = \frac{D\lambda^2}{4\pi} \qquad (\text{matching element, } R_{\rm Ohmic} \ll R = R_{\rm rad}), \tag{19}$$

such that the maximum power received by the load resistor is $P_{\rm max} = A_{\rm eff,max} P_{\rm in}$.

If it is desired that the effective area be large compared to λ^2, then the directivity D must be large. This is consistent with the law of diffraction that if a beam is to have a characteristic angular spread of θ (directivity $D \approx 1/\theta^2$), then its narrowest cross section has radius $\approx \lambda/\theta$, and hence area $\approx D\lambda^2$.[6,7] We show that eq. (19) describes the maximum effective area for any antenna, arguing from an antenna reciprocity theorem (Appendix A.6).

Consider any two antennas, labeled i and j, that are separated by a distance r such that each antenna is in the far zone of the other. Each antenna is oriented such that the other is

[4]The assumption in eq. (16) that $B_0 = E_0/c$ holds, in general, only in the "far zone" of the source.

[5]The small loop might have N turns, and might be would around a ferrite rod of effective (relative) permeability μ, which multiplies the magnetic moment of the system by $N\mu$ (compared to that of a single-turn loop) and the radiation resistance $R_{\rm rad}$ by $N^2\mu^2$. In this case, the open circuit voltage $V_{\rm oc}$ (16) and the effective height $h_{\rm eff}$ (17) are multiplied by $N\mu$, but the maximum effective area $A_{\rm eff,max}$ of the small antenna remains as given in eq. (18) [if the receiving circuit is "matched", as confirmed by eq. (21)].

[6]Receiving antennas do not necessarily have their conductors all in a plane perpendicular to the direction of the incident wave, but if they do the effective area of the conductors must be $\approx D\lambda^2$ if the antenna is to have directivity D.

[7]In the quantum view the function of a receiving antenna is to absorb photons from the incident wave. If the antenna is "matched" it can absorb essentially all photons that strike some part of it. A photon has cross-sectional area $\approx \lambda^2$, so the effective area of an antenna that lies entirely in a plane perpendicular to the direction of the incident wave is the larger of λ^2 or the actual area of the antenna. This maximum equals $D\lambda^2$, according to the previous footnote.

at the maximum of the angular distributed of power radiated when the antenna is used as a transmitter. Each antenna includes a matching network such that the terminal impedance of the antenna plus network is $R_{\rm rad}$, the (real) radiation resistance of the antenna.

If antenna j is used as a receiver, the open-circuit (no load) voltage $V_{{\rm oc},j}$ induced across its terminals by the radiation from antenna i with drive current I_i is related to the open-circuit voltage $V_{{\rm oc},i}$ induced across the terminals of antenna i (when used as a receiver) by the radiation from antenna j (when used as a transmitter with drive current I_j) according to the reciprocity relation (Appendix A),

$$\frac{V_{{\rm oc},j}}{I_i} = \frac{V_{{\rm oc},i}}{I_j} \,. \tag{20}$$

The maximum time-average power received by a load resistor R_j connected to antenna j when it is used as a receiver occurs for $R_j = R_{{\rm rad},j}$ (and the receiver includes a matching network whose reactance cancels that of the antenna itself, so that $R_j = Z_{A,j}$), follows from eq. (5) as,

$$P_{j,{\rm receive,max}} = \frac{V_{{\rm oc},j}^2}{8R_{{\rm rad},j}} \,. \tag{21}$$

The time-average power per unit area incident on antenna j due to current $I_i e^{-i\omega t}$ delivered to the terminals of antenna i is,

$$\frac{dP_{ij}}{dA} = \frac{D_i}{4\pi r^2} P_{i,{\rm transmit}} = \frac{D_i}{8\pi r^2} R_{{\rm rad},i} I_i^2, \tag{22}$$

where D_i is the directivity of antenna i. The (maximum) effective area of antenna j is the ratio of the (maximum) power received to the incident power per unit area,

$$A_{{\rm eff,max},j} = \frac{P_{j,{\rm receive,max}}}{dP_{ij}/dA} = \frac{V_{{\rm oc},j}^2}{8R_{{\rm rad},j}} \frac{8\pi r^2}{D_i R_{{\rm rad},i} I_i^2} \tag{23}$$

Thus,

$$A_{{\rm eff,max},j} D_i = \frac{V_{{\rm oc},j}^2}{I_i^2} \frac{\pi r^2}{R_{{\rm rad},i} R_{{\rm rad},j}} = \frac{V_{{\rm oc},i}^2}{I_j^2} \frac{\pi r^2}{R_{{\rm rad},i} R_{{\rm rad},j}} = A_{{\rm eff,max},i} D_j, \tag{24}$$

using the reciprocity relation (20). Hence, the ratio $A_{\rm eff,max}/D$ is the same for any two antennas. From secs. 2.1 and 2.2 we see that this ratio is $\lambda^2/4\pi$, which confirms the general validity of eq. (19).

This section follows sec. 2.14 of [4], which is the earliest discussion of the effective-area theorem that I have found. However, the theorem is likely much older. Rayleigh discusses the effective area of an acoustic "resonator" in [5] and reports a value close to $\lambda^2/4\pi$.

4 Received Power for a Specified Signal Bandwidth

For an antenna (system) to receive maximum power (1) the total reactance X must be zero and the resistance equal to the radiation resistance $R_{\rm rad}$, which latter is very small for small antennas. As a consequence, the antenna system has a very narrow bandwidth 2δ about the nominal (angular) frequency ω_0 (such that the power received at frequencies $\omega_0 \pm \delta$ is one half that the maximum at ω_0). This is not a desirable feature if the antenna is to receive signals of bandwidth $2\Delta > 2\delta$. Increasing the bandwidth of the receiving antenna results in a reduction in the power received compared to the maximum possible (*i.e.*, a reduction in efficiency, and in the capture area), as discussed in this section.

As shown, for example, in [6], regarding the antenna system as a series R-L-C circuit permits us to characterize the system by its Q, related by,

$$Q \equiv \frac{\omega_0}{\text{bandwidth}} = \frac{\omega_0}{2\delta} \approx \frac{2\omega_0 \times \text{stored energy at resonance}}{\text{power delivered to the load at resonance}} \approx \frac{\text{reactance of } L \text{ or of } C \text{ at resonance}}{R}. \tag{25}$$

The requirement of a bandwidth 2Δ implies that the load resistance R should be,

$$R \approx R_{\rm rad} \frac{Q_{\rm max}}{Q}, \qquad \text{where} \qquad Q_{\rm max} = \frac{\text{reactance of } L \text{ or of } C \text{ at resonance}}{R_{\rm rad}}. \tag{26}$$

The capture area of the receiving is reduced by the factor $R_{\rm rad}/R = Q/Q_{\rm max}$ compared to the maximum (18), and the intercepted power is reduced compared to the maximum (1) by the same ratio.

The case of a small loop antenna is easier to analyze (and relevant to typical AM radios). We consider the loop to consist of N turns of radius r comprising a coil of length l wound around a rod of effective relative permeability $\mu_{\rm eff}$. Then the inductance L is given approximately by,

$$L = \frac{N^2 \mu_{\rm eff} \mu_0 \pi r^2}{l}, \tag{27}$$

The radiation resistance $R_{\rm rad}$ of the coil is given by,[8]

$$R_{\rm rad} = \frac{\pi N^2 \mu_{\rm eff}^2 \mu_0 c}{6} \frac{(2\pi r)^4}{\lambda^4} = \frac{\pi N^2 \mu_{\rm eff}^2 Z_0}{6} \frac{(2\pi r)^4}{\lambda^4}, \tag{28}$$

noting as in footnote 3 that the presence of N turns and the effective permeability $\mu_{\rm eff}$ inside the coil multiplies the radiation resistance by $N^2 \mu_{\rm eff}^2$. For maximum power reception, the load resistance R equals $R_{\rm rad}$ and the maximum Q is given by (independent of N),

$$Q_{\rm max} = \frac{\omega_0 L}{R_{\rm rad}} = \frac{2\pi c L}{\lambda R_{\rm rad}} = \frac{3\lambda^3}{8\pi^3 \mu_{\rm eff} l r^2}. \tag{29}$$

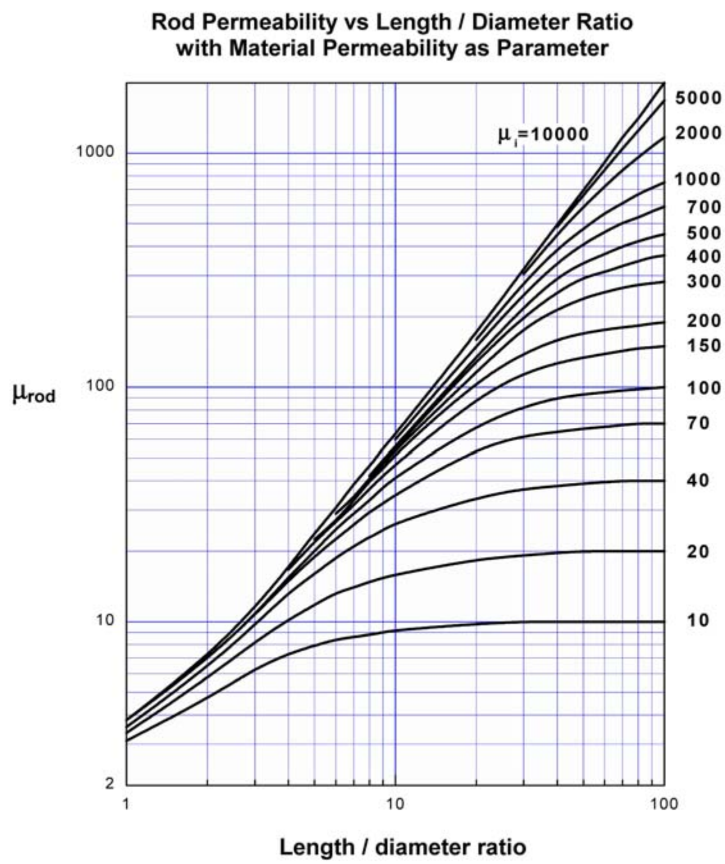
For example, at frequency $f_0 = \omega_0/2\pi = 1$ MHz, $\lambda = 300$ m, so for $l = r = 1$ cm and, say, $\mu_{\rm eff} = 100$, we find $Q_{\rm max} \approx 3 \times 10^9$. However, for reception of audio signals modulated onto this carrier frequency, we might desire the bandwidth to be 30 kHz, and the Q to be no more than 30. Hence, we must lower the Q by a factor of 10^8, which is accomplished by increasing

[8] See, for example, eq. (26) of [7].

the load resistance to $R = 10^8 R_{\rm rad}$. For a coil with $N = 100$ turns and the above l and r the radiation resistance is $R_{\rm rad} \approx 1.5 \times 10^{-4}\ \Omega$, so the load resistance should be $R \approx 15$ kΩ.[9]

If it were desired for the AM radio to extract the maximum possible power from the wave, perhaps for a crystal radio set, while also $Q_{\rm max} = 30$, then according to eq. (29), $\mu_{\rm eff}^2 l r^2 \approx \lambda^3/80\pi^3 \approx 10^4$ for $\lambda = 300$ m. Then, an air-core coil with $l \approx r \approx 22$ m would be required. The length of the wire of the air-core coil exceeds $\lambda/4$, which would be the appropriate length for a linear monopole antenna with similar performance. As such, large linear antennas, rather than air-core loop antennas are typically used with crystal radio sets.

A high-performance ferrite rod could be used, with $l = 100r$ so that $\mu_{\rm eff} \approx \mu \approx 10^4$ [8]. In this case, the radius of the rod/coil would need to be only 1 cm, with length $l = 1$ m, which is not impractical. The radiation resistance is then $R_{\rm rad} \approx 4 \times 10^{-5} N^2\ \Omega$, according to eq. (28), so if it is desired that, say, $R_{\rm rad} = 100\ \Omega$, then the number of turns should be $N \approx 1200$. *The effective resistance of the ferrite rod would need to be less than* $100\ \Omega$ *at* 1 *MHz for this scheme to work. Also, the inductance* L *would be about 10 H according to eq. (27), which would require a tuning capacitor with* $C = 1/\omega_0^2 L \approx 1/400$ *pF at* $f_0 = \omega_0/2\pi = 1$ *MHz, which is problematic.*



Finally, we note that is a very narrow bandwidth is acceptable, small superconducting antennas can be operated with $R_{\rm load} = R_{\rm rad}$. See, for example, [9].

[9]In practice, there will be energy losses in the medium with effective permeability $\mu_{\rm eff}$, so that the load resistance R should be considered as the sum of an Ohmic resistance $R_{\rm Ohmic}$ and an effective resistance $R_{\rm eff}$ of the permeable medium.

A Appendix: Reciprocity Theorems

A.1 Green's Reciprocation Theorem for Electrostatics

The first reciprocity theorem is due to Green (1828, p. 39 of [10]), which states that if a set $\{i\}$ of fixed conductors is at potentials V_i when carrying charges Q_i, and at potentials V_i' when carrying charges Q_i', then

$$\sum_i V_i Q_i' = \sum_i V_i' Q_i. \tag{30}$$

To see this, we label the 3-dimensional potential distribution associated with charges Q_i by $\Phi(\mathbf{r})$, and that associated with charges Q_i' by Φ'. The space outside the conductors is charge free and with relative dielectric constant $\epsilon = 1$.

We invoke Green's theorem (p. 23 of [10])

$$\int (\Phi \nabla^2 \Phi' - \Phi' \nabla^2 \Phi)\, d\mathrm{Vol} = \oint (\Phi \boldsymbol{\nabla} \Phi' - \Phi' \boldsymbol{\nabla} \Phi) \cdot d\mathbf{S}, \tag{31}$$

where we take the bounding surface S to be that of the set of conductors. In the charge-free space outside the conductors we have $\nabla^2 \Phi = 0 = \nabla^2 \Phi'$, and the conductors are equipotentials with $\Phi = V_i$ and $\Phi' = V_i'$ on conductor i, so that

$$0 = \sum_i V_i \oint \boldsymbol{\nabla} \Phi_i' \cdot d\mathbf{S}_i - V_i' \sum \oint \boldsymbol{\nabla} \Phi_i \cdot d\mathbf{S}_i = -4\pi \left(\sum_i V_i Q_i' - \sum_i V_i' Q_i \right), \tag{32}$$

using Gauss' Law (in Gaussian units) that

$$4\pi Q_i = \oint \mathbf{E}_i \cdot d\mathbf{S}_i = -\oint \boldsymbol{\nabla} \Phi_i \cdot d\mathbf{S}_i. \tag{33}$$

A.2 Helmholtz Reciprocity

The next step in enlarging the scope of reciprocity relations appears to have been taken by Helmholtz (1859), p. 29 of [11], who stated that *Wenn in einem mit Luft gefüllten Raume, der teils von endlich ausgedehnten festen Körpern begrenzt, teils unbegrenzt ist, im Punkte A Schallwellen erregt werden, so ist das Geschwindigkeitspotential derselben in einem zweiten Punkte B ebenso groß, als es in A sein würde, wenn nicht in A, sondern in B Wellen von derselben Intensität erregt würden. Auch ist der Unterschied der Phasen des erregenden und erregten Punktes in beiden Fällen gleich.*[10]

Apparently Helmholtz considered this theorem to be "obvious," as he offered no proof. Helmholtz' theorem is for scalar waves generated by point sources, and detected by point observers. In this case we readily write that for a wave of angular frequency ω that is generated at point A with strength S_A, the wave observed at point B a distance r from A is

$$O_B = S_A \frac{e^{i(kr - \omega t)}}{r}. \tag{34}$$

[10]If in a space filled with air which is partly bounded by finitely extended fixed bodies and is partly unbounded, sound waves be excited at any point A, the resulting velocity potential at a second point B is the same both in magnitude and phase, as it would have been at A, had B been the source of the sound.

Likewise a source of strength S'_B at point B leads to a wave that is observed at point A to be

$$O'_A = S'_B \frac{e^{i(kr-\omega t)}}{r}\,. \tag{35}$$

Thus, we obtain the reciprocity relation

$$O'_A S_A = O_B S'_B = S_A S'_B \frac{e^{2i(kr-\omega t)}}{r}\,. \tag{36}$$

While Helmholtz reciprocity may be "obvious" for point sources and point observers it does not always hold for sources and observers of finite extent [12].

In 1886 Helmholtz returned to the theme of reciprocity, and gave a general argument based on Hamiltonian dynamics [13]. For a commentary in English, see [14].

A.3 Maxwell and Reciprocity

Variants of reciprocity theorems for static mechanical systems were considered by several authors in the 1860's [15], including a version by Maxwell [16].

Maxwell later discussed Green's reciprocation theorem in sec. 86 of his *Treatise* [17]. In secs. 280-281 he noted that in a linear circuit that contains only resistors, but which may contain linkages of arbitrary complexity, if a (constant) voltage V_{ij} applied between points i and j leads to a (steady) current I_{kl} between points k and l, then (constant) voltage V'_{kl} applied between point k and l leads to (steady) current I'_{ij} between points i and j (which now has no applied voltage) that obeys the reciprocity relation

$$V_{ij} I'_{ij} = V'_{kl} I_{kl}. \tag{37}$$

Maxwell's reciprocity relation (37) is perhaps "obvious" for a circuit that contains only a single loop, and seems to be little referenced (whereas his name is commonly attached to a reciprocity theorem in mechanics).

We sketch Maxwell's argument, noting that it is readily extended to include capacitors and inductors, and to include time-harmonic voltage sources, so long as radiation is neglected, and with the important restriction that there is no mutual capacitance or inductance between circuit elements.

The system consists of a set $\{i\}$ of nodes, $i = 1, ..., n$, connected by "wires".

Any or all of the $n(n-1)/2$ pairs of nodes, ij, can be directly connected by a "wires" along which there exists one or more circuit elements (resistor, capacitor, inductor, or external voltage source).

The impedance[11] and the external voltage (if any, but which will then contribute to the impedance) along the "wire" that directly connects nodes i and j are written Z_{ij} and V_{ij}, respectively, where V_{ij} is positive when the external voltage is lower near node i.

A key feature of Maxwell's analysis is the assumption that the current I_{ij} that flows from node i to node j can be written

$$I_{ij} = \frac{V_i - V_j + V_{ij}}{Z_{ij}} = -I_{ji}, \tag{38}$$

[11]The term impedance in circuit analysis was introduced by Heaviside in 1886 [18]. See also Lodge (1888) [19].

where V_i is the voltage at node i. This assumption excludes the possibility of mutual capacitance or inductance between "wire" ij and any other "wires" in the circuit.

The sum of the currents at each node is zero,

$$0 = \sum_{j \neq i} I_{ij} = \sum_{j \neq i} \frac{V_i - V_j + V_{ij}}{Z_{ij}} = V_i \sum_{j \neq i} \frac{1}{Z_{ij}} - \sum_{j \neq i} \frac{V_j}{Z_{ij}} + \sum_{j \neq i} \frac{V_{ij}}{Z_{ij}} . \tag{39}$$

A clever trick of Maxwell was to define

$$V_{ii} = 0, \qquad \text{and} \qquad \frac{1}{Z_{ii}} = - \sum_{j \neq i} \frac{1}{Z_{ij}} , \tag{40}$$

which permits us to rewrite eq. (39) as the set of n equations

$$\sum_j \frac{V_j}{Z_{ij}} = \sum_j \frac{V_{ij}}{Z_{ij}} . \tag{41}$$

Since $V_{ji} = -V_{ij}$ while $Z_{ji} = Z_{ij}$, we have that $\sum_i \sum_j V_j/Z_{ij} = \sum_i \sum_j V_{ij}/Z_{ij} = 0$, and only $n - 1$ of these equations are independent. This is to be expected since the voltages are defined only up to an overall constant. Suppose, for example, that we define $V_n = 0$. Then we consider eqs. (41) only for $i = 1, ..., n - 1$, and it suffices to sum over index j only for $j = 1, ..., n - 1$. Thus, we obtain

$$V_i = \frac{\sum_k \Delta_{ik} \sum_l V_{kl}/Z_{kl}}{\Delta} , \tag{42}$$

where Δ is the determinant of the admittance matrix $A_{ij} = 1/Z_{ij}$, and Δ_{ij} is the minor determinant associated with element A_{ij}. Since impedance is symmetric, $Z_{ji} = Z_{ij}$, matrix A_{ij} is symmetric, and it follows that $\Delta_{ji} = \Delta_{ij}$.

Now suppose that all external voltages are zero except for $V_{kl} = -V_{lk}$. Then, the current in the "wire" from node i to node j is

$$I_{ij} = \frac{V_i - V_j}{Z_{ij}} = V_{kl} \frac{\Delta_{ik} - \Delta_{il} - \Delta_{jk} + \Delta_{jl}}{Z_{ij} Z_{kl} \Delta} . \tag{43}$$

Similarly, if all external voltages are zero except for $V'_{ij} = -V'_{ji}$, then the current in the "wire" from node k to node l is

$$I'_{kl} = \frac{V'_k - V'_l}{Z_{kl}} = V'_{ij} \frac{\Delta_{ki} - \Delta_{kj} - \Delta_{li} + \Delta_{lj}}{Z_{ij} Z_{kl} \Delta} = V'_{ij} \frac{\Delta_{ik} - \Delta_{il} - \Delta_{jk} + \Delta_{jl}}{Z_{ij} Z_{kl} \Delta} . \tag{44}$$

Comparing eqs. (43) and (44) we arrive at the reciprocity relation (37).

A.4 Rayleigh Reciprocity

In 1873 Rayleigh extended the static reciprocity theorems of Green, Maxwell and others to include that case that the "force" (derivable from a scalar potential) was periodic rather than static [20]. Although he acknowledged Helmholtz' earlier theorem for waves [11], Rayleigh's discussion did not explicitly include the possibility of energy in the form of radiation. His subsequent exposition in secs. 106-111 of *The Theory of Sound* [21] included various examples to show the generality of reciprocity in quasistatic systems, with the effect that his name is now often associated with reciprocity theorems. The statement of his theorem for electrical circuits is the same as eq. (37), but Rayleigh's argument holds even when there is mutual capacitance and inductance between circuit elements.

Rayleigh's original argument, like Maxwell's, involved minors of a relevant determinant. A shorter derivation was given in 1952 by Tellegen [22], as discussed in sec. A.7.

A.5 Lorentz Reciprocity

The generalization of reciprocity theorems to the vector fields of electromagnetism, including waves, is due to Lorentz (1896 [23]).

Lorentz showed that if a (time-dependent) current distribution $\mathbf{J}_1$ leads to electric and magnetic fields $\mathbf{E}_1$ and $\mathbf{B}_1$ in a linear medium, and that current distribution $\mathbf{J}_2$ leads to electric and magnetic fields $\mathbf{E}_2$ and $\mathbf{B}_2$, then

$$\int \mathbf{J}_1 \cdot \mathbf{E}_2 \, d\mathrm{Vol} = \int \mathbf{J}_2 \cdot \mathbf{E}_1 \, d\mathrm{Vol}. \tag{45}$$

A demonstration of eq. (45) invokes the vector calculus identity that

$$\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}). \tag{46}$$

Thus, for fields with time dependence $e^{-i\omega t}$,

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{E}_1 \times \mathbf{B}_2) &= \mathbf{B}_2 \cdot (\boldsymbol{\nabla} \times \mathbf{E}_1) - \mathbf{E_1} \cdot (\boldsymbol{\nabla} \times \mathbf{B_2}) \\ &= -\frac{1}{c}\mathbf{B}_2 \cdot \frac{\partial \mathbf{B}_1}{\partial t} - \mathbf{E_1} \cdot \left(\frac{4\pi}{c}\mathbf{J}_2 + \frac{1}{c}\frac{\partial \mathbf{E}_2}{\partial t} \right) \\ &= -\frac{4\pi}{c}\mathbf{E_1} \cdot \mathbf{J}_2 + \frac{i\omega}{c}(\mathbf{B}_2 \cdot \mathbf{B}_1 + \mathbf{E_1} \cdot \mathbf{E}_2). \end{aligned} \tag{47}$$

Similarly,

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{B}_1 \times \mathbf{E}_2) &= \mathbf{E}_2 \cdot (\boldsymbol{\nabla} \times \mathbf{B}_1) - \mathbf{B_1} \cdot (\boldsymbol{\nabla} \times \mathbf{E_2}) \\ &= \mathbf{E}_2 \cdot \left(\frac{4\pi}{c}\mathbf{J}_1 + \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t} \right) - \frac{1}{c}\mathbf{B_1} \cdot \frac{\partial \mathbf{B}_2}{\partial t} \\ &= -\frac{4\pi}{c}\mathbf{E_2} \cdot \mathbf{J}_1 + \frac{i\omega}{c}(\mathbf{E}_2 \cdot \mathbf{E}_1 + \mathbf{B_1} \cdot \mathbf{B}_2). \end{aligned} \tag{48}$$

Hence,

$$\boldsymbol{\nabla} \cdot (\mathbf{E}_1 \times \mathbf{B}_2 - \mathbf{B}_1 \times \mathbf{E}_2) = \frac{4\pi}{c}(\mathbf{J_1} \cdot \mathbf{E}_2 - \mathbf{J_2} \cdot \mathbf{E}_1), \tag{49}$$

and so,

$$\int (\mathbf{J_1} \cdot \mathbf{E}_2 - \mathbf{J_2} \cdot \mathbf{E}_1) \, d\mathrm{Vol} = \frac{c}{4\pi} \oint (\mathbf{E}_1 \times \mathbf{B}_2 - \mathbf{E}_1 \times \mathbf{B}_2) \cdot d\mathbf{Area}. \tag{50}$$

A small delicacy in the argument is that for sources $\mathbf{J}_1$ and $\mathbf{J}_2$ contained in a bounded region, the asymptotic radiation fields vary as $1/r$ and have the form $\mathbf{B}_{1(2)} = \hat{\mathbf{r}} \times \mathbf{E}_{1(2)}$, so the surface integral vanishes, and we obtain the reciprocity relation (45), independent of the angular frequency ω.

The assumption of linear media seems necessary only to insure that sources at frequency ω lead to fields only at this frequency. See [24] for a review of Lorentz reciprocity in the time domain.

A.6 An Antenna Reciprocity Theorem

The earliest application of Lorentz' reciprocity relation (45) to antennas that I have found is in the papers of Carson [25, 26, 27], but he appears not to have explicitly deduced eq. (54). A result close to this was given by Ballantine (1929, [28]). I first find CEQ. (54) in sec. 11.10 of [29].

Consider two antennas, A and B, which contain the only currents in our system. Then CEQ. (45) can be written

$$\int_A \mathbf{J}_{1A} \cdot \mathbf{E}_{2A}\, d\mathrm{Vol} + \int_B \mathbf{J}_{1B} \cdot \mathbf{E}_{2B}\, d\mathrm{Vol} = \int_A \mathbf{J}_{2A} \cdot \mathbf{E}_{1A}\, d\mathrm{Vol} + \int_B \mathbf{J}_{2B} \cdot \mathbf{E}_{1B}\, d\mathrm{Vol}. \tag{51}$$

In situation 1, antenna A is the transmitter, and antenna B is the receiver, operated with no load (open circuit), while their roles are reversed in situation 2. Then the currents $\mathbf{J}_{1B}$ and $\mathbf{J}_{2A}$ exist only in the conductors of the receiving antennas, and not in the gap between the terminals of these antennas. Of course, these currents flow along the conductors. In the approximation of perfect conductors, the electric fields in/on these conductors have no component parallel to the conductors. Hence,

$$\mathbf{J}_{1B} \cdot \mathbf{E}_{2B} = 0 = \mathbf{J}_{2A} \cdot \mathbf{E}_{1A}, \tag{52}$$

while the remaining integrals in CEQ. (51) have contributions only from the gap between the terminals (where the idealized power sources for the transmitting antennas are located):

$$\int_{A,\mathrm{gap}} \mathbf{J}_{1A} \cdot \mathbf{E}_{2A}\, d\mathrm{Vol} = \int_{B,\mathrm{gap}} \mathbf{J}_{2B} \cdot \mathbf{E}_{1B}\, d\mathrm{Vol}. \tag{53}$$

We can write the currents in the gap as $\mathbf{J} = I\, d\mathbf{l}/d\mathrm{Vol}$, so that CEQ. (53) becomes

$$I_{1A} \int_{A,\mathrm{gap}} \mathbf{E}_{2A} \cdot d\mathbf{l} = I_{1A} V_{2A}^{\mathrm{oc}} = I_{2B} V_{1B}^{\mathrm{oc}} = I_{2B} \int_{B,\mathrm{gap}} \mathbf{E}_{1B} \cdot d\mathbf{l}, \tag{54}$$

where V^{oc} is measured between the terminals of the receiving antennas. This is the form of the reciprocity theorem used in sec. 2.3 above, which holds for both “linear” and “loop” antennas.

A.7 Tellegen's Theorem

Tellegen [22] has given a network theorem that then leads to a kind of reciprocity theorem. See also [30].

Consider a network with nodes, and links between some or all pairs of nodes. The network can consist of parts with no links between different parts.

"Currents" flow along links between pairs with nodes, with the same scalar value for the "current" at both nodes. A "current I_{jk} is defined to be positive if it flows from node j to node k. Then, $I_{ji} = -I_{ij}$. The only "physical" assumption underlying Tellegen's theorem is that

$$\sum_{\text{nodes } k \text{ directly linked to node } j} I_{jk} = 0 \tag{55}$$

for all nodes j.

We also suppose that every node can be assigned a scalar "voltage" V_j. However, there is not necessarily any "physical" relation between "current" and "voltage."

It follows immediately from eq. (55 that

$$\sum_j V_j \sum_{\text{nodes } k \text{ directly linked to node } j} I_{jk} = 0. \tag{56}$$

The "current" in a link appears exactly twice in the sum (56), in the form

$$V_j I_{jk} = V_k I_{kj} = (V_j - V_k) I_{jk} \equiv V_{jk} I_{jk}. \tag{57}$$

Summing over all links, we obtain Tellegen's theorem,

$$\sum_{\text{links}} V_{jk} I_{jk} = 0. \tag{58}$$

Another consequence of the definition of the "voltage drop" $V_{jk} = V_j - V_k$ is that the directed sum of "voltage drops" around any closed loop of links is zero.

We can obtain a kind of reciprocity theorem from eq. (58) by considering a second set of "voltages" V'_j and the corresponding "voltage drops" V'_{jk}. Since eq. (58) does not require there to be any "physical" relation between "current" and "voltage." we also have that $\sum_{\text{links}} V'_{jk} I_{jk} = 0$. Likewise, we can consider another set of "currents" I'_{jk} that are not necessarily related to either the V_{jk} or the V'_{jk} (other than applying to the same network topology), for which we can write $\sum_{\text{links}} V_{jk} I'_{jk} = 0$. Hence, we obtain Tellegen's reciprocity relation

$$\sum_{\text{links}} V'_{jk} I_{jk} = \sum_{\text{links}} V_{jk} I'_{jk} = 0. \tag{59}$$

However, we cannot deduce from this that $V'_{jk} I_{jk} = V_{mn} I'_{mn}$ for a pair of links jk and mn.

The "nonphysical" character of Tellegen's theorem clarifies how the reciprocity theorems are somewhat abstract "bookkeeping" constructs, rather than a manifestation of "cause and effect."

We can, however, obtain the antenna reciprocity theorem of Appendix 6 from Tellegen's Theorem, but only for "linear" antennas. For this, we consider a network of two disconnected parts, antennas A and B, with three links, $A1$, $A2$, $A3$, *etc.* in each part in the case of a "linear" antenna. Link 2 is the gap between the physical conductors of the antenna (between its terminals).

In the unprimed situation 1, antenna A is the transmitter, and antenna B is the receiver, operated with no load (open circuit), while their roles are reversed in the primed situation. Then the "currents" in this example are the physical electrical currents, so that I_{B2} and I'_{A2} are zero, since the receiving antennas are operated "open circuit."[12] We define the "voltage difference" between the ends of a link to be $\int \mathbf{E}\cdot d\mathbf{l}$. Hence, $V_{A1} = V_{A3} = V_{B1} = V_{B1} = V'_{A1} = V'_{A3} = V'_{B1} = V'_{B1} = 0$, in the approximation that the conductors of the antenna are perfect. The voltages V_{jA} at the four nodes of antenna A are then $V_{1A} = V_{2A}$ and $V_{3A} = V_{4A}$.

Thus, of the 12 terms in eq. (59) for the pair of "linear" antennas only two are nonzero, and we have

$$V'_{A2} I_{A2} = V_{B2} I'_{B2}, \qquad (60)$$

as in eq. (20), where again the "voltages" V'_{A2} and V_{B2} are "open circuit."

However, a similar argument for a "loop" antenna, representing it by two links connected to two nodes, fails. If link 2 again represent the gap between the terminals, then the "voltage" $V_{1A} = \int_{\text{link } 1A} \mathbf{E}\cdot d\mathbf{l} = 0$, but $V_{2A} = \int_{\text{link } 2A} \mathbf{E}\cdot d\mathbf{l}$ is nonzero in general, so a unique scalar "voltage" cannot be assigned to the two nodes of antenna A.

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