Princeton University

Ph501 Midterm Examination
Electrodynamics

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Please do all work in the exam booklets provided.

You may use either Gaussian or MKSA units on this exam.

1. (10 pts.) Show that the charge induced in a small area A on a grounded conducting plane by a point charge not in that plane is proportional to the solid angle subtended at the point charge by area A.

2. (20 pts.) A hollow dielectric sphere of dielectric constant $\epsilon = 3$ has inner radius one half its outer radius. When this sphere is placed in an initially uniform electric field E_0, what is the resulting electric field strength at the center of the sphere?

3. (30 pts.) Two circular wires of radii a and b have a common center, and are free to turn on an insulating axis which is a diameter of both. Find the torque about this diameter required to hold the two wire loops at rest when their planes are at right angles and they are carrying currents I and I', supposing that $b \ll a$. Give both the leading term, and the first correction in a power of the small ratio b/a.

   Hint: This requires evaluating the first correction to both the axial and transverse magnetic field components near the center of the larger loop. Recall that the torque about a point is $\vec{\tau} = \mathbf{r} \times \mathbf{F}$ where force $\mathbf{F}$ is applied at distance $\mathbf{r}$.

Solutions

1. Let charge q be at perpendicular distance a from the grounded conducting plane. The small area A has its center at distance r from the foot of the perpendicular to charge q. The charge q' induced in the area A is related by

$$q' = \sigma A = \frac{EA}{4\pi}, \tag{1}$$

where E is the electric field strength at the surface of the conducting plane.

We calculate E using the image method, supposing that charge $-q$ is located at distance a on the other side of the conducting plane from charge q. Then,

$$E = -\frac{2q}{R^2}\frac{a}{R} = -2q\frac{\cos\theta}{R^2}, \tag{2}$$

where $R = \sqrt{a^2 + r^2}$ is the distance from charge q to area A, and θ is the angle between vector $\mathbf{R}$ and the perpendicular from q to the plane.

Combining eqs. (1) and (2), we have

$$q' = -\frac{2qA\cos\theta}{4\pi R^2} = -\frac{q\Omega}{2\pi}, \tag{3}$$

where $\Omega = A\cos\theta/R^2$ is the solid angle subtended by area A at charge q. For the whole plane, $\Omega = 2\pi$ and $q' = -q$.

2. This problem is closely related to that of a dielectric sphere in an otherwise uniform electric field. We choose the z axis antiparallel to the initial field $\mathbf{E}_0$, with the origin at the center of the dielectric sphere, where the potential is taken to be zero.

The potential of the initial field is then

$$\phi_0 = E_0 z = E_0 r\cos\theta = E_0 r P_1(\theta), \tag{4}$$

where θ is the polar angle with respect to the z axis and P_1 is the Legendre polynomial of order 1.

We recall from the case of a uniform dielectric sphere that the potential contains terms only in P_1, and we expect the same here.

Writing the inner radius of the sphere as a and the outer radius as b, we expect that the potential will have the form

$$\phi_1 = E_0 r P_1 + A\frac{r}{a}P_1, \qquad (0 < r < a) \tag{5}$$

$$\phi_2 = E_0 r P_1 + B\frac{r}{a}P_1 + C\frac{b^2}{r^2}P_1, \qquad (a < r < b) \tag{6}$$

$$\phi_3 = E_0 r P_1 + D\frac{b^2}{r^2}P_1, \qquad (a < r < b) \tag{7}$$

since the perturbation to field E_0 must be finite at $r = 0$ and ∞.

The potential is continuous at $r = a$ and b, so that

$$A = B + C\frac{b^2}{a^2}, \tag{8}$$

$$B\frac{b}{a} + C = D. \tag{9}$$

Also, the normal component of the electric displacement $\mathbf{D} = \epsilon\mathbf{E}$ is continuous at the boundaries, since $\nabla \cdot \mathbf{D} = 0$. Hence,

$$\frac{\partial\phi_1(a)}{\partial r} = \epsilon\frac{\partial\phi_2(a)}{\partial r}, \tag{10}$$

and

$$\epsilon\frac{\partial\phi_2(b)}{\partial r} = \frac{\partial\phi_3(b)}{\partial r}, \tag{11}$$

which yields

$$E_0 + \frac{A}{a} = \epsilon E_0 + \epsilon\frac{B}{a} - 2\epsilon\frac{Cb^2}{a^3}, \tag{12}$$

and

$$\epsilon E_0 + \epsilon\frac{B}{a} - 2\epsilon\frac{C}{b} = E_0 - 2\frac{D}{b}. \tag{13}$$

Inserting eq. (8) in (12), we get

$$\frac{\epsilon - 1}{a}B - (2\epsilon + 1)\frac{b^2}{a^3}C = (1 - \epsilon)E_0, \tag{14}$$

while using eq. (9) in (13) gives

$$\frac{\epsilon + 2}{a}B - \frac{2(\epsilon - 1)}{b}C = (1 - \epsilon)E_0. \tag{15}$$

These could be solved in general for A, B and C, but here we consider the particular case that $a = 1$, $b = 2$ and $\epsilon = 3$, for which eqs. (14) and (15) become

$$B - 14C = -E_0, \tag{16}$$

and

$$5B - 2C = -2E_0. \tag{17}$$

We quickly find that

$$B = -\frac{13}{34}E_0, \qquad C = \frac{3}{68}E_0, \tag{18}$$

and from eq. (11),

$$A = B + 4C = -\frac{7}{34}E_0. \tag{19}$$

The electric field strength at the center of the dielectric sphere is

$$E(0) = E_0 + A = \frac{27}{34}E_0. \tag{20}$$

A dielectric sphere is not as effective as a conducting sphere in shielding its interior from an external electric field.

3. (Problem 12, p. 448 of *The Mathematical Theory of Electricity and Magnetism* by J. Jeans.)

The leading term of the torque is given by $\vec{\mu} \times \mathbf{B}(0)$, where

$$\mu = \frac{\pi I' b^2}{c} \tag{21}$$

is the magnetic moment of the small loop of radius b that carries current I', and

$$B(0) = \frac{1}{c}\int \frac{\mathbf{I} \times d\mathbf{l}}{r^2} = \frac{2\pi a I}{ca^2} = \frac{2\pi I}{ca} \tag{22}$$

is the magnetic field at the center of the loops due to the current I in the loop of radius a. When the two loops are at right angles, the vectors $\vec{\mu}$ and $\mathbf{B}(0)$ are also at right angles, so the magnitude of the leading term of the torque is

$$\tau = \frac{2\pi I I' b^2}{c^2 a} \tag{23}$$

To evaluate the torque in greater detail, we consider the variation of the magnetic field over the small loop, and use the basic torque equation

$$\vec{\tau} = \int \mathbf{r} \times d\mathbf{F} = \frac{1}{c}\int \mathbf{r} \times [I' d\mathbf{l}' \times \mathbf{B}(\text{due to I})]. \tag{24}$$

We use a coordinate system in which the centers of the loops are at the origin, with the axis of loop a is along the z axis. We take the sign of current I to be such that the resulting magnetic field at the origin is in the $+z$ direction. The axis of loop b is defined to be the y axis, and the sign of current I' is such that the magnetic moment $\vec{\mu}$ is along the $+y$ axis. Then, we desire the x component of the torque $\vec{\tau}$ about the origin:

$$\begin{aligned} \tau_x &= \frac{1}{c}\int b\hat{\mathbf{r}} \times [I' b\hat{\phi} d\phi \times (B_z \hat{\mathbf{z}} + B_\rho \hat{\rho})]\bigg|_x \\ &= \frac{b^2 I'}{c}\int_0^{2\pi} d\phi \cos\phi(\cos\phi B_z + \sin\phi B_\rho), \end{aligned} \tag{25}$$

where angle ϕ is measured in the x-z plane with respect to the z axis, such that for a point on loop b, $\rho = b\sin\phi$ and $z = b\cos\phi$.

If we don't recall the results of problem 7, set 4, the magnitude of B_ρ can be estimated quickly using the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ and a "pillbox" surface of radius ρ and thickness dz whose axis is along the z axis:

$$\begin{aligned} 0 &= \int \nabla \cdot \mathbf{B} d\text{Vol} = \int \mathbf{B} \cdot d\mathbf{S} \\ &\approx \pi\rho^2(B_z(0, z + dz) - B_z(0, z)) + 2\pi\rho dz B_\rho(\rho, z). \\ &\approx \pi\rho^2 dz \frac{\partial B_z(0, z)}{\partial z} + 2\pi\rho dz B_\rho(\rho, z). \end{aligned} \tag{26}$$

Hence,

$$B_\rho(\rho, z) \approx -\frac{\rho}{2}\frac{\partial B_z(0,z)}{\partial z}. \tag{27}$$

Then, near the center of loop a its magnetic field obeys $\nabla \times \mathbf{B} = 0$, and in particular

$$\frac{\partial B_z(\rho, z)}{\partial \rho} = \frac{\partial B_r(\rho, z)}{\partial z} \approx -\frac{\rho}{2}\frac{\partial^2 B_z(0,z)}{\partial z^2}, \tag{28}$$

using eq. (27). We can integrate this to find

$$B_z(\rho, z) \approx B_z(0,z) - \frac{\rho^2}{4}\frac{\partial^2 B_z(0,z)}{\partial z^2}, \tag{29}$$

in agreement with the results of Problem 7, Set 4.

For points along the z axis the magnetic field due to loop a is

$$B_z(0,z) = \frac{1}{c}\int \frac{\mathbf{I} \times d\mathbf{l}}{r^2}\bigg|_z = \frac{2\pi a^2 I}{c(a^2+z^2)^{3/2}} \approx \frac{2\pi I}{ca}\left(1 - \frac{3z^2}{2a^2}\right), \tag{30}$$

where the approximation can be used when we evaluate the field on loop b for which $|z| \le b \ll a$. Thus,

$$\frac{\partial B_z(0,z)}{\partial z} = -\frac{6\pi a^2 z I}{c(a^2+z^2)^{5/2}} \approx -\frac{6\pi z I}{ca^3}, \tag{31}$$

and

$$\frac{\partial^2 B_z(0,z)}{\partial z^2} = -\frac{6\pi a^2 I(a^2 - 4z^2)}{c(a^2+z^2)^{7/2}} \approx -\frac{6\pi I}{ca^3}, \tag{32}$$

Using eqs. (27) and (31), the transverse magnetic field at a point on loop b is

$$B_\rho(\rho, z) \approx \frac{3\pi I \rho z}{ca^3} = \frac{3\pi b^2 I \cos\phi \sin\phi}{ca^3}, \tag{33}$$

and eqs. (29), (30) and (32) give the axial field as

$$B_z(\rho, z) \approx \frac{2\pi I}{ca}\left(1 - \frac{3z^2}{2a^2}\right) + \frac{3\pi I \rho^2}{2ca^3} = \frac{2\pi I}{ca}\left(1 - \frac{3b^2\cos^2\phi}{2a^2}\right) + \frac{3\pi b^2 I \sin^2\phi}{2ca^3}. \tag{34}$$

Combining eqs. (25), (33) and (34) we find

$$\begin{aligned}
\tau_x &\approx \frac{\pi b^2 I I'}{c^2 a}\int_0^{2\pi} d\phi \left(2\cos^2\phi - \frac{3b^2\cos^4\phi}{a^2} + \frac{3b^2\cos^2\phi\sin^2\phi}{2a^2} + \frac{3b^2\cos^2\phi\sin^2\phi}{a^2}\right) \\
&= \frac{\pi b^2 I I'}{c^2 a}\int_0^{2\pi} d\phi \left(2\cos^2\phi - \frac{3b^2\cos^2\phi}{a^2} + \frac{15b^2\sin^2 2\phi}{8a^2}\right) \\
&= \frac{2\pi^2 b^2 I I'}{c^2 a}\left(1 - \frac{9b^2}{16a^2}\right).
\end{aligned} \tag{35}$$

[The answer in MKSA units is obtained on setting $c = 1$ in the magnetic force equation, and replacing $1/c$ by $\mu_0/4\pi$ in the Biot-Savart law, so $2\pi^2/c^2 \to \pi\mu_0/2$.]