LAPLACE'S EQUATION IN CYLINDRICAL COORDINATES

$$\phi = \phi(r,\theta,z) \qquad \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$ (BECKER 13.4a)

WITH $\phi = R(r)\,\Theta(\theta)\,Z(z)$ THE SEPARATED EQUATIONS ARE

$$\frac{d^2\Theta}{d\theta^2} + n^2\,\Theta = 0$$

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2r^2 - n^2\right)R = 0$$

THE RADIAL EQUATION IS CALLED BESSEL'S EQUATION. IT HAS TWO FAMILIES OF SOLUTIONS CALLED BESSEL FUNCTIONS $J_n(kr)$ AND $Y_n(kr)$ FOR WHICH POWER SERIES EXPANSIONS CAN BE OBTAINED DIRECTLY FROM THE DIFFERENTIAL EQUATIONS.

THE J_n ARE WELL BEHAVED AT $r=0$, WHILE THE Y_n ARE SINGULAR THERE. WE CONSIDER ONLY SOLUTIONS WITH THE J_n HERE.

IN OUR BRIEF DISCUSSION, WE WILL RESTRICT OURSELVES TO PROBLEMS WITH AZIMUTHAL SYMMETRY $\Rightarrow$ $\Theta = \text{CONST}$ $\Rightarrow n = 0$

WE FURTHER RESTRICT THE DISCUSSION TO THE CASE WHERE k IS REAL

$$\Rightarrow Z = e^{\pm kz}$$

EVEN SO THERE REMAIN TWO CATEGORIES OF SOLUTIONS:

a) THERE IS NO BOUNDARY CONDITION IN r OR z TO CONSTRAIN k TO TAKE ON QUANTISED VALUES. THEN k IS A CONTINUOUS PARAMETER, AND

$$\phi(r,z) = \int_0^\infty e^{\pm kz} f(k)\, J_0(kr)\, dk$$

FUNCTION TO BE DETERMINED FROM THE BOUNDARY CONDITIONS

0 SINCE WE ASSUME AZIMUTHAL SYMMETRY

b) THERE IS A RADIAL BOUNDARY CONDITION OF THE TYPE
$\phi(r=a)=0$

THE FUNCTION $J_0(kr)$ IS OSCILLATORY AND HAS MANY ZEROES

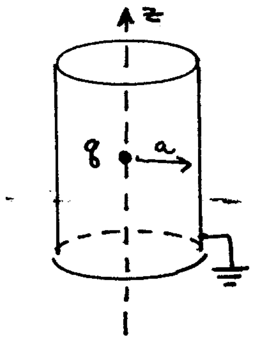
THEN k TAKES ON ONLY SUCH VALUES k_ℓ THAT $J_0(k_\ell a)=0$

AND $$\phi = \sum_\ell A_\ell\, e^{\pm k_\ell z} J_0(k_\ell r)$$

TO EVALUATE $f(k)$ OR A_ℓ WE NEED VARIOUS INTEGRAL RELATIONS AMONG THE J_n, OF WHICH THERE ARE A TRULY VAST NUMBER. SOME OF THESE ARE APPENDED TO THIS LECTURE.

IT IS HARD TO FIND AN ELEMENTARY AND INTERESTING EXAMPLE OF THE USE OF THE BESSEL FUNCTION EXPANSION. [THE PROBLEM OF A CIRCULAR HOLE IN A CONDUCTING PLANE CAN BE SOLVED BY INTEGRAL EQUATIONS WITHOUT BESSEL FUNCTIONS: R. FRIEDBERG, AM. J. PHYS. 61, 1084 (9

EXAMPLE POINT CHARGE ON THE AXIS OF A GROUNDED CONDUCTING TUBE.



THE TUBE HAS RADIUS a AND IS TAKEN AS INFINITELY LONG.

THIS IS AN EXAMPLE OF CASE b) ABOVE. SO

$$\phi_\pm(r,z) = \sum_\ell A_\ell\, e^{\mp k_\ell z} J_0(k_\ell r)$$

WHERE $J_0(k_\ell a)=0$ AND $\begin{cases}\phi_+ & \text{HOLDS FOR } z>0\\ \phi_- & \text{FOR } z<0\end{cases}$

THE 'BOUNDARY CONDITION' AT $z=0$ IS OBTAINED FROM GAUSS' LAW FOR RING SHAPED PILLBOXES:

$$\int \left(E_z\big|_+ - E_z\big|_-\right) 2\pi r\, dr = 4\pi\, q_{\text{INSIDE}}$$

$$\Rightarrow \quad E_z\big|_+ - E_z\big|_- = -\frac{\partial \phi_+(0)}{\partial z} + \frac{\partial \phi_-(0)}{\partial z} = \frac{2q}{r}\,\delta(r)$$

FROM $\phi_\pm$ ABOVE, WE GET $$\sum_\ell A_\ell k_\ell J_0(k_\ell r) = q\frac{\delta(r)}{r}$$

OR
$$\sum_\ell A_\ell k_\ell r J_0(k_\ell r) = q\,\delta(r)$$

THE FAMILY $\{J_0(k_\ell r)\}$ IS A SET OF ORTHOGONAL FUNCTIONS.

FROM SECTION 5.296 OF THE PAGES APPENDED WE LEARN THAT

$$\int_0^a J_0(k_\ell r) J_0(k_m r)\, r\, dr = \frac{\delta_{\ell m}}{2} a^2 \left[J_0'(k_\ell a)\right]^2 = \frac{\delta_{\ell m}}{2} a^2 \left[J_1(k_\ell a)\right]^2$$

SEE SEC 5.302

HENCE WE LEARN THAT $\frac{A_\ell k_\ell a^2}{2}\left[J_1(k_\ell a)\right]^2 = qJ_0(0) = q$

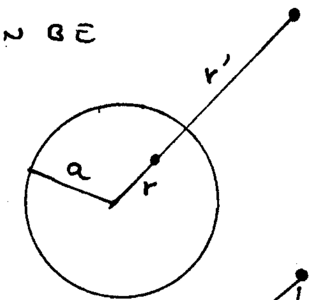
AND SO $$\phi_\pm(r,z) = \frac{2q}{a^2}\sum_\ell \frac{e^{\mp k_\ell z} J_0(k_\ell r)}{k_\ell\left[J_1(k_\ell a)\right]^2}$$

IT'S LOTS OF FUN, ESPECIALLY IF YOU LIKE RUMMAGING THRU BOOKS ON 'SPECIAL FUNCTIONS'. WE SHOW HOW TO TRANSFORM THIS RESULT TO ANOTHER OF POSSIBLE INTEREST IN THE NEXT SECTION.

SOLUTION BY INVERSION

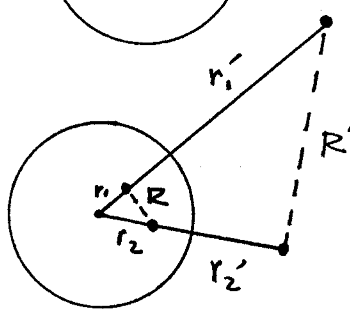
AN AMUSING TRANSFORMATION OF SOLUTIONS CAN BE OBTAINED BY 'INVERSION' WHICH RELATES POINTS ALONG A RADIUS VECTOR BY

$$r r' = a^2 ,$$



r IS SAID TO BE THE INVERSE OF r' WITH RESPECT TO A SPHERE OF RADIUS a - AND VICE VERSA.

TO GET AN IDEA OF HOW POTENTIALS WILL TRANSFORM, CONSIDER THE POTENTIAL AT r_2 DUE TO CHARGE q AT r_1.



THEN $\phi_{AT\ r_2} = \frac{q}{R}$.

SIMILARLY, IF q' IS AT r_1', THEN

$$\phi'_{AT\ r_2'} = \frac{q'}{R'}$$

BY THE SIMILAR TRIANGLES $\frac{R'}{R} = \frac{r_1'}{r_2} = \frac{r_2'}{r_1}$

$$\text{SO } \phi' = \phi \cdot \frac{q'}{q} \cdot \frac{R}{R'} = \phi \frac{q'}{q} \cdot \frac{r_1}{r_2'} \quad \text{ETC}$$

TO MAKE ANY MONEY FROM THIS WE NEED SOMETHING MORE: THE RELATION BETWEEN TWO 'INVERSE' CHARGES q AND q'

IF $\phi' = \text{FACTOR} \cdot \phi$, SURFACES AT $\phi = 0$ MUST TRANSFORM TO THOSE WITH $\phi' = 0$. IN PARTICULAR CONSIDER THE SPHERE OF RADIUS a.

FROM THE IMAGE METHOD WE KNOW THAT CHARGES q_1 AT DISTANCE d_1 AND $q_2 = -q_1\frac{a}{d_1}$ AT $d_2 = a^2/d_1$ PUT THE SPHERE AT $\phi = 0$

NOTE THAT $d_1' = \frac{a^2}{d_1} = d_2$ IS THE INVERSE OF d_1

$$d_2' = \frac{a^2}{d_2} = d_1 \quad \text{. . . - - - . . } d_2$$

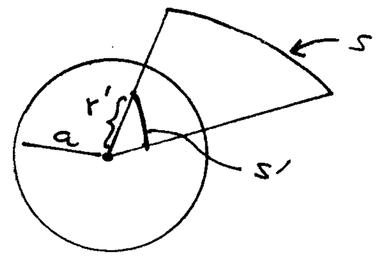
HENCE WE SHOULD TAKE $q_1' = \pm q_1 \frac{a}{d_1}$ AND $q_2' = \pm q_2 \frac{a}{d_2}$

WHICH LEAVES THE SPHERE AT $\phi' = 0$ IN THE INVERSE PROBLEM.

WE CHOOSE TO ALWAYS USE THE + CASE. THEN

$$\left.\begin{aligned} d' &= a^2/d \\ q' &= q\frac{a}{d} \\ \phi' &= \phi\frac{q'}{q}\frac{r_1}{r_2'} = \phi\frac{a}{r_2'} = \phi\frac{r_2}{a} \end{aligned}\right\} \text{INVERSION OF DISTANCE, CHARGE AND POTENTIAL}$$

AS AN APPLICATION OF THESE RESULTS CONSIDER THE INVERSE OF A CONDUCTING SURFACE S WHICH IS AT POTENTIAL V.



THE INVERSE SURFACE S' IS NOT AT CONSTANT POTENTIAL, BUT RATHER THE POTENTIAL AT A DISTANCE r' FROM THE ORIGIN IS

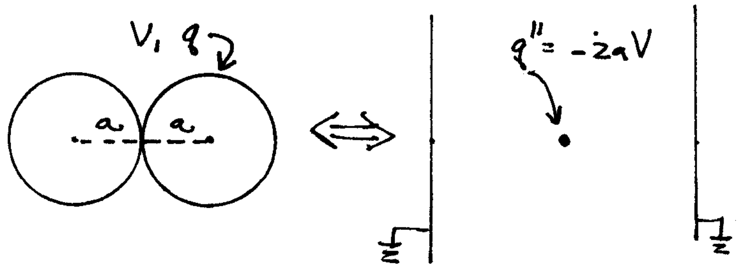
$$V' = V\frac{a}{r'}$$

HOWEVER SUPPOSE WE ADDED CHARGE $q'' = -Va$ TO THE ORIGIN IN THE <u>INVERSE</u> SITUATION. NOW THE POTENTIAL ON SURFACE S' WOULD BE

$$V' = \frac{q''}{r'} + \frac{Va}{r'} = 0$$

THE USE OF THIS IS BY INVERTING AGAIN! THE PROBLEM OF A CONDUCTING SURFACE AT POTENTIAL V CAN BE SOLVED BY CONSIDERING THE INVERSE SURFACE AT ZERO POTENTIAL BUT WITH A CHARGE $-Va$ AT THE ORIGIN (a = RADIUS OF SPHERE OF INVERSION.)

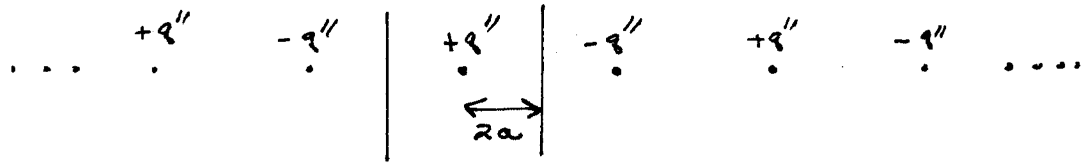
<u>EXAMPLE</u> THE CAPACITANCE OF TWO TANGENT CONDUCTING SPHERES OF RADIUS a



(PROB 7 SET 2)

INVERT USING A SPHERE OF RADIUS $2a$ ⇒ SPHERES BECOME PLANES.

THE INVERSE PROBLEM IS SOLVED BY A SERIES OF IMAGE CHARGES:



TO FIND THE CAPACITANCE FOR THE ORIGINAL PROBLEM, ALL WE NEED IS Q. BUT THIS IS JUST THE SUM OF ALL THE INVERSES OF THE IMAGE CHARGES — EXCLUDING THE CHARGE AT THE ORIGIN.

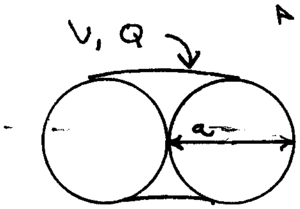
NOW WE HAVE CHARGE $(-1)^n q''$ AT DISTANCE $4na$

THIS INVERTS TO $q_n = (-1)^n q'' \dfrac{2a}{4na}$ (AT DISTANCE $\frac{a}{n}$, $n \neq 0$)

SO $Q = 2\sum_{n=1}^{\infty} q_n = q'' \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} = -q'' \ln 2 = 2aV \ln 2$

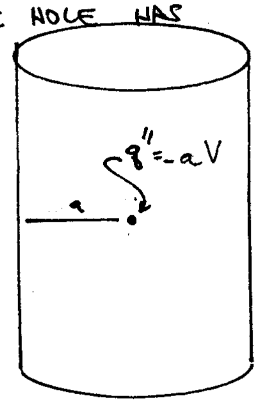
SO $C = \dfrac{Q}{V} = \underline{2a \ln 2}$

EXAMPLE THE CAPACITANCE OF A CONDUCTOR FORMED BY ROTATING A SPHERE OF DIAMETER a ABOUT A TANGENT.



THIS IS A KIND OF DONUT WHOSE HOLE HAS SHRUNK TO ZERO.

THE INVERSE OF THIS SURFACE IS A CYLINDER OF RADIUS a USING A SPHERE OF INVERSION OF RADIUS a.



ON P 63 WE SOLVED THE CYLINDER PROBLEM:

$$\phi_\pm(r,z) = \frac{2q''}{a} \sum_\ell \frac{e^{\mp k_\ell z} J_0(k_\ell r)}{k_\ell [J_1(k_\ell a)]^2}.$$

WE NEED THE CHARGE. THE SURFACE CHARGE DENSITY ON THE CYLINDER IS

$$\sigma' = -\frac{E_r(r=a)}{4\pi} = \frac{1}{4\pi}\frac{\partial \phi}{\partial r}(r=a) = \frac{q''}{2\pi a} \sum_\ell \frac{e^{\mp k_\ell z} J_0'(k_\ell a)}{[J_1(k_\ell a)]^2}$$

$$= \frac{-q''}{2\pi a} \sum_\ell \frac{e^{\mp k_\ell z}}{J_1(k_\ell a)}, \quad \text{NOTING } J_0' = -J_1.$$

THEN $dq' = 2\pi a \sigma' dz$ ON THE SURFACE

THE INVERSE CHARGE IS $dq = dq' \dfrac{a}{\sqrt{a^2+z^2}}$

HENCE THE CHARGE ON THE DONUT IS

$$Q = \int dq = 2\pi a^2 \left(\frac{-q''}{2\pi a}\right) \sum_\ell \frac{1}{J_1(k_\ell a)} \cdot 2\int_0^\infty \frac{dz}{\sqrt{a^2+z^2}}\, e^{-k_\ell z},$$

NOTING $q'' = -aV$, AND CHANGING VARIABLES $\frac{z}{a} = \sinh\phi$.

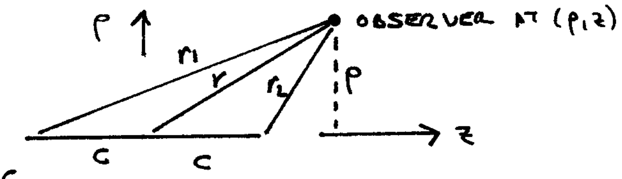
$$C = \frac{Q}{V} = 2a \sum_\ell \frac{1}{J_1(k_\ell a)} \int_0^\infty d\phi\, e^{-k_\ell a \sinh\phi}.$$

SOLUTIONS IN SPECIAL COORDINATE SYSTEMS : PROLATE ELLIPSOID

(BECKER SEC 22)

LAPLACE'S EQUATION $\nabla^2\phi = 0$ IS SEPARABLE IN SEVERAL EXOTIC COORD SYSTEMS, LEADING TO SOLUTIONS FOR SURFACES OF CONSTANT COORDINATE IN THAT SYSTEM. MOST SUCH SOLUTIONS ARE SOMEWHAT CUMBERSOME, BUT SOME FEATURES OF THE SOLUTION FOR A CHARGED, CONDUCTING PROLATE ELLIPSOID CAN BE GOTTEN QUICKLY.

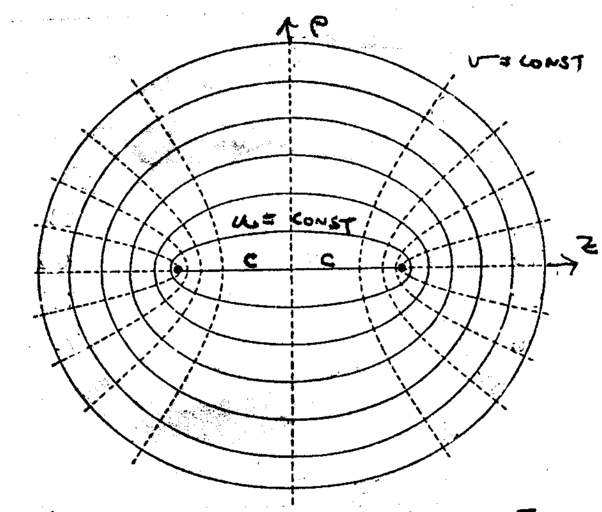
WE START IN ORDINARY CYLINDRICAL COORDS, (ρ, z), AND CONSIDER THE POTENTIAL DUE TO CHARGE q UNIFORMLY DISTRIBUTED ALONG A LINE OF LENGTH $2c$.



$$\phi = \int \frac{\rho}{r}\, d\text{vol} = \frac{q}{2c}\int_{-c}^{c} \frac{dz'}{\sqrt{\rho^2+(z-z')^2}} = \frac{-q}{2c} \ln\left(z - z' + r\right)\Big|_{-c}^{c}$$

$$= \frac{q}{2c} \ln\left(\frac{z+c+r_1}{z-c+r_2}\right) \qquad \text{WHERE} \qquad r_{1,2} = \sqrt{\rho^2 + (z \pm c)^2}$$

THE TRICK IS TO NOTE THAT THIS BECOMES SIMPLE IN PROLATE ELLIPSOIDAL COORDINATES.



THESE ARE $u = \frac{r_1 + r_2}{2}$, $v = \frac{r_1 - r_2}{2}$

SO $r_1 = u + v$, $r_2 = u - v$

AND $cz = uv$.

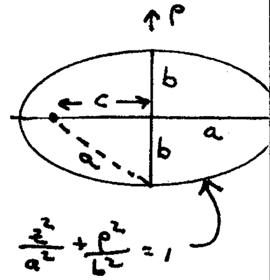
THEN $$\frac{z+c+r_1}{z-c+r_2} = \frac{uv + c^2 + uc + vc}{uv - c^2 + uc - vc}$$

$$= \frac{(u+c)(v+c)}{(u-c)(v+c)} = \frac{u+c}{u-c}$$

THE POINTS $(\rho, z) = (0, \pm c)$ ARE THE FOCI OF THE ELLIPSOIDS $u =$ CONST.

So $\phi = \frac{q}{2c} \ln \frac{u+c}{u-c}$ = CONST ON SURFACES OF CONSTANT u.

SUCH SURFACES ARE PROLATE ELLIPSOIDS - ELLIPSES ROTATED ABOUT THEIR MAJOR AXIS. THE SURFACES v = CONST ARE ORTHOGONAL TO THE ELLIPSOIDS, AND HENCE ARE ALONG THE FIELD LINES. THE FIELD IS OF COURSE, STRONGEST AT THE TIPS OF THE ELLIPSOID WHERE THE CURVATURE IS THE GREATEST.



CONSIDER AN ELLIPSOID OF SEMI-MAJOR AXIS a

THEN $a^2 = b^2 + c^2$ WHERE b = SEMI-MINOR AXIS

ALSO $u = \frac{r_1 + r_2}{2} = a$ ON THE SURFACE OF THE ELLIPSOID

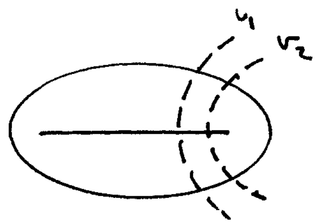
THE POTENTIAL IS $\phi = \frac{q}{2c} \ln \frac{a+c}{a-c}$.

A CONDUCTING SURFACE AT THIS POTENTIAL WILL CONTAIN CHARGE q (BY GAUSS' LAW). HENCE THE CAPACITY IS $C = q/\phi = \frac{2c}{\ln\left(\frac{a+c}{a-c}\right)}$

IN THE LIMIT $b/a \lll 1$ THE ELLIPSOID BECOMES A NEEDLE - OR A PIECE OF WIRE OF LENGTH $2a$, DIAMETER $2b$.

THEN $c \to a - \frac{b^2}{2a}$; $\phi \to \frac{q}{2a} \ln \frac{4a^2}{b^2} = \frac{q}{a} \ln \frac{2a}{b} \Rightarrow C = \frac{a}{\ln(2a/b)}$

A NICE EXPRESSION FOR THE SURFACE CHARGE DENSITY CAN BE OBTAINED BY MORE DETAILED CONSIDERATION OF GAUSS' LAW + GEOMETRY.



WITH $\bar{E}$ BEING ALONG LINES OF CONSTANT v, WE SEE THAT ALL CHARGE ON THE AXIS BETWEEN v_1 AND v_2 MAPS ONTO THE CHARGE DENSITY CONTAINED WITHIN THE BAND BETWEEN v_1 AND v_2 ON THE SURFACE OF THE ELLIPSOID $u = a$.

NOW $z = \frac{uv}{c}$. ON THE ORIGINAL AXIS OF CHARGE FROM $-c$ TO $+c$,

$u = c$ SO $z = v \Rightarrow dq = \frac{q}{2c} dv$ = CHARGE AS A FUNCTION OF v.

ON THE ELLIPSOID, $u = a$, $z = \frac{a}{c} v$ SO $dz = \frac{a}{c} dv$

HENCE $dq = \frac{q}{2a} dz$ I.E. CHARGE IS STILL DISTRIBUTED UNIFORMLY AS A FUNCTION OF z.

HOWEVER THE SURFACE CHARGE DENSITY IS NOT UNIFORM:

THE SURFACE AREA OF THE BAND FROM z TO $z + dz$ IS $dS = 2\pi \rho \sqrt{1 + \rho'^2}\, dz = 2\pi b \sqrt{1 - \left(\frac{cz}{a^2}\right)^2}\, dz$

Hence $\sigma = \frac{dq}{dS} = \frac{q}{4\pi a b\sqrt{1 - \left(\frac{cz}{a^2}\right)^2}}$

In the limit of a needle with $b \ll a$, this becomes $\sigma \sim \frac{q}{4\pi b\sqrt{a^2 - z^2}}$

The field at the surface is $E_\perp = 4\pi\sigma = \frac{q}{b\sqrt{a^2-z^2}}$

Another useful result is the charge distribution projected onto the plane $z = 0$.

Combine $\frac{dq}{dz} = \frac{q}{2a}$, $\frac{z^2}{a^2} + \frac{\rho^2}{b^2} = 1$, $dz = \frac{a^2}{b^2}\frac{\rho\, d\rho}{z}$

to find $\sigma(\rho) = \frac{dq}{2\pi\rho\, d\rho} = \frac{q}{4\pi b\sqrt{b^2 - \rho^2}}$

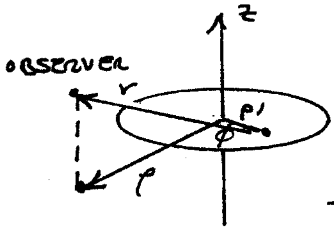
This is independent of a & c!

It is also true for a sphere of radius b

Again this demonstrates the presence of high field and charge density on sharp projections of conductors.
[For discussion of a paradox, see J.D. Jackson, Am. J. Phys. 68, 789 (2000).]

Solution via Integral Equations – Charge Distribution on a Conducting Disc

The problem of a conducting disc can be considered as a limiting case of an oblate ellipsoid (or spheroid), and can be solved in oblate spheroidal coordinates. Historically it was first solved by the technique of integral equations. This discussion follows E.T. Copson, Proc. Edin. Math. Soc. 8, 14 (1947).



Let $\sigma(\rho')$ be the unknown charge distribution on a conducting disk of radius a. (For the moment σ = sum of density on top and bottom)

Then certainly $\phi(\rho, z) = \int \frac{\sigma(\rho')\, dS'}{r} = \int_0^a \sigma(\rho')\,\rho'\, d\rho' \int_0^{2\pi} \frac{d\phi}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi + z^2}}$

If the disk has potential V then we can set $z = 0$ above and find

$$V = \int_0^a \sigma(\rho')\,\rho'\, d\rho' \int_0^{2\pi} \frac{d\phi}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi}} \quad \text{for any } 0 < \rho < a$$

This is an <u>integral equation</u> for the unknown $\sigma(\rho')$.

Analytic solutions of such equations are rarer than solutions of differential equations. But integral equations can certainly be solved by a computer....

Mr. Copson knew some tricks (this is not the original solution: Weber 1873)

$$\int_0^{2\pi} \frac{d\phi}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi}} = 4\int_0^{\text{minimum of } \rho \text{ and } \rho'} \frac{dx}{\sqrt{(\rho^2 - x^2)(\rho'^2 - x^2)}}$$

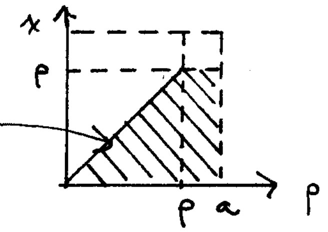
(Proof via complex integration)

PLUGGING THIS INTO OUR EXPRESSION FOR V, AND SPLITTING THE RADIAL INTEGRAL:

$$\frac{V}{4} = \int_0^{\rho} \rho' \sigma(\rho') \int_0^{\rho'} \frac{dx}{\sqrt{(\rho'^2-x^2)(\rho^2-x^2)}}\, d\rho' + \int_{\rho}^{a} \rho' \sigma(\rho') \int_0^{\rho} \frac{dx}{\sqrt{(\rho'^2-x^2)(\rho^2-x^2)}}\, d\rho'$$

$$= \int_0^{\rho} \frac{dx}{\sqrt{\rho^2-x^2}} \int_x^a \frac{\sigma(\rho')\rho' d\rho'}{\sqrt{\rho'^2-x^2}}$$

REGION OF INTEGRATION



IF WE LET $S(x) = \int_x^a \frac{\sigma(\rho')\rho' d\rho'}{\sqrt{\rho'^2-x^2}}$

THEN $\frac{V}{4} = \int_0^{\rho} \frac{S(x)\, dx}{\sqrt{\rho^2-x^2}}$

THE ORIGINAL DOUBLE INTEGRAL EQUATION HAS BEEN CONVERTED INTO TWO SINGLE INTEGRAL EQUATIONS.

NOW WE CAN INQUIRE IN INTEGRAL TABLES. FOR EXAMPLE DWIGHT 320.01 SAYS $\int \frac{dx}{\sqrt{\rho^2-x^2}} = \sin^{-1}\frac{x}{\rho}$ (OR JUST LET $x = \rho\sin\theta$)

SO $S(x) = \frac{V}{2\pi}$ = CONST WORKS

HENCE $\frac{V}{2\pi} = \int_x^a \frac{\sigma(\rho')\rho' d\rho'}{\sqrt{\rho'^2-x^2}} = \frac{1}{2}\int_{x^2}^{a^2} \frac{\sigma(\rho')\, d\rho'^2}{\sqrt{\rho'^2-x^2}}$

THE SOLUTION $\sigma(\rho') = \frac{V}{\pi^2\sqrt{a^2-\rho'^2}}$ CAN BE VERIFIED BY DWIGHT 380.001

THE TOTAL CHARGE IS $Q = \int_0^a 2\pi\sigma(\rho)\rho\, d\rho = \frac{2aV}{\pi}$

SO $\sigma(\rho) = \frac{Q}{2\pi a\sqrt{a^2-\rho^2}}$ OR $\frac{Q}{4\pi a\sqrt{a^2-\rho^2}}$ PER SIDE

THE CAPACITANCE IS $C = \frac{Q}{V} = \frac{2a}{\pi}$

NOTE THAT THE CHARGE DISTRIBUTION $\sigma(\rho)$ HAS THE SAME FORM AS THAT FOUND FOR THE CONDUCTING PROLATE ELLIPSOID! ALL CONDUCTING ELLIPSOIDS, FROM NEEDLES TO SPHERES TO DISKS HAVE THE SAME CHARGE DISTRIBUTION IN TERMS OF THE COORDINATE PERPENDICULAR TO THEIR AXIS OF ROTATION.

Ph 206 Lecture 6

Solution by Conjugate Functions

If $f(z)$ is an analytic function of the complex variable $z = x + iy$, where $f = u + iv$, then the Cauchy–Riemann equations tell us $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\boxed{\begin{array}{l} i\frac{\partial f}{\partial x} = i f' \frac{\partial z}{\partial x} = i f' = i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} = f' \frac{\partial z}{\partial y} = i f' = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \end{array}}$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

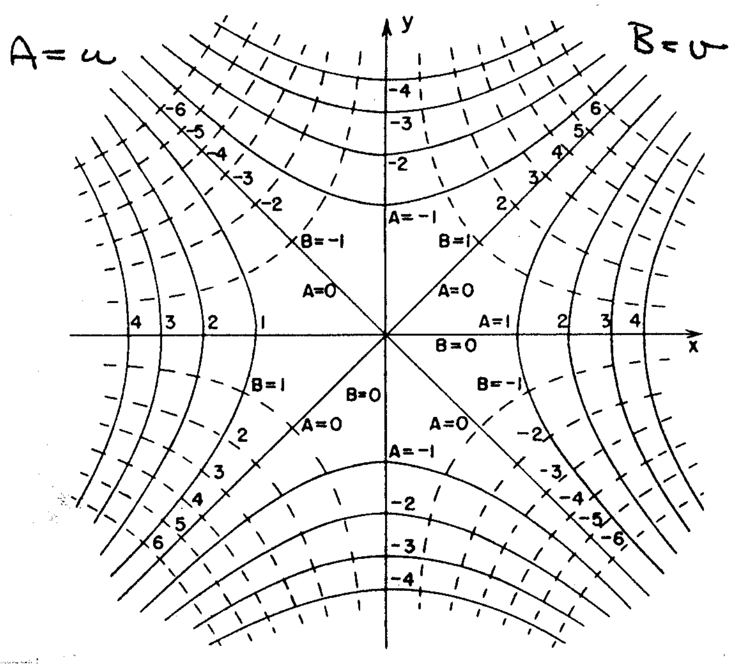
Thus u and v automatically satisfy Laplace's equation in two dimensions. In this sense, any function $f(z)$ is a solution in search of a problem!

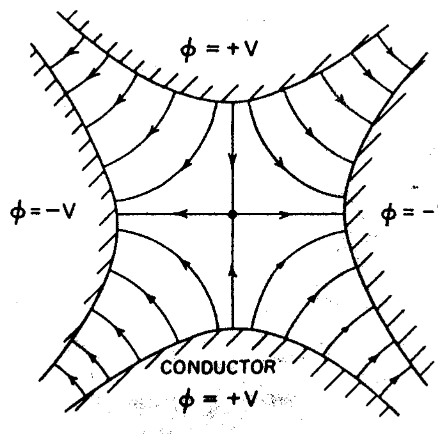
Furthermore, the C-R equations state that lines of constant u are orthogonal to lines of constant v.

Hence if we take u as our potential, v plays the role of electric field lines, and vice versa.

Example $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\therefore \quad u = x^2 - y^2 \qquad v = 2xy$





The field in a quadrupole lens.

Quadrupole magnets are quite common in particle accelerators. The field pattern is the same as shown here.

EXAMPLE $f(z) = \sqrt{z}$

IF WE WRITE $z = \rho e^{i\theta}$, THEN $\rho = \sqrt{x^2+y^2}$ AND $\tan\theta = \frac{y}{x}$.

$$f = \sqrt{z} = \sqrt{\rho}\, e^{i\theta/2} = \sqrt{\rho}\cos\frac{\theta}{2} + i\sqrt{\rho}\sin\theta/2$$

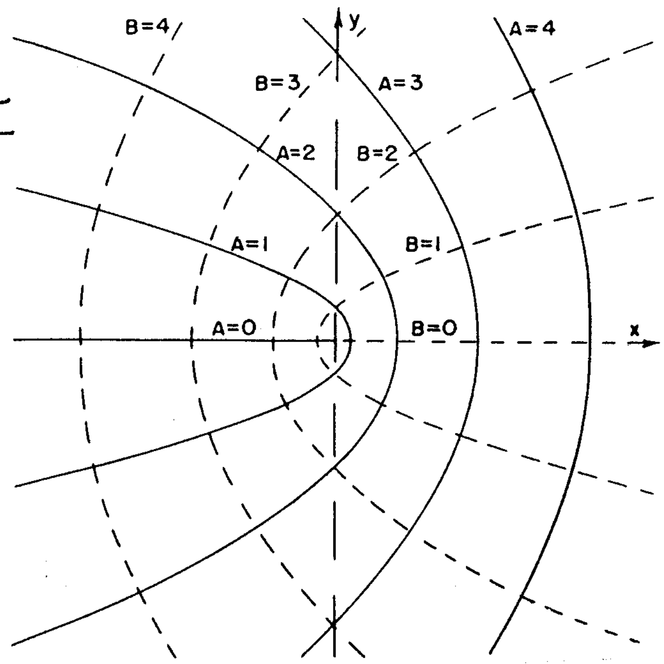
$$f = \sqrt{\rho}\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\rho}\sqrt{\frac{1-\cos\theta}{2}}$$

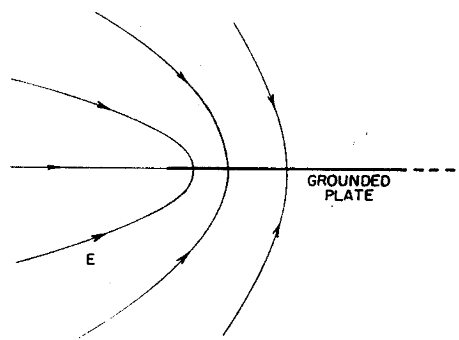
$$\cos\theta = \frac{x}{\rho}$$

$$\text{So } f = \underbrace{\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}}}_{u} + i\underbrace{\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}}_{v}$$

IN PICTURES:

$A = u$
$B = v$





The electric field near the edge of a thin grounded plate.

THE SURFACE $v = 0$ IS THE HALF PLANE $x > 0$, $y = 0$ THAT WE CONSIDERED THE FIELD NEAR SUCH A CONDUCTOR IN LECTURE 5.

WITH $\phi = \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$

THEN THE FIELD IS $E_y = -\frac{\partial\phi}{\partial y}\Big| = \frac{-1}{2\sqrt{2}}\frac{y}{\sqrt{\sqrt{x^2+y^2}-x}}\frac{1}{\sqrt{x^2+y^2}}$

AS $y \to 0$ $\quad E_y \to -\frac{1}{2\sqrt{x}}$ IN AGREEMENT WITH p.50.

EXAMPLE HELMHOLTZ NOTICED AN INTERESTING FORM:

$$z = f + e^{f}$$

WHICH IN PRINCIPLE CAN BE INVERTED TO GIVE $f = f(z)$

BUT WE CAN PROCEED AS IS. WITH $f = u + iv$

$$z = f + e^{f} = u + iv + e^{u+iv} = (\underbrace{u + e^{u}\cos v}_{x}) + i(\underbrace{v + e^{u}\sin v}_{y})$$

NOW WE JUST MAP OUT A FEW CASES OF, SAY, $v =$ CONSTANT AND LOOK TO SEE IF WE RECOGNIZE AN INTERESTING PHYSICAL SITUATION.

$v = 0$: $\begin{cases} x = u + e^{u} \\ y = 0 \end{cases}$. AS u RUNS FROM $-\infty$ TO $+\infty$ x GOES FROM $-\infty$ TO $+\infty$

$\therefore$ THE SURFACE IS THE PLANE $y = 0$

$v = \pi$: $\begin{cases} x = u - e^{u} \\ y = \pi \end{cases}$ NOW x GOES FROM $-\infty$ TO -1 AND BACK TO $-\infty$

$v = -\pi$: $\begin{cases} x = u - e^{u} \\ y = -\pi \end{cases}$ AGAIN x COVERS ONLY $-\infty$ TO -1

WE CAN IDENTIFY THIS AS THE EDGE OF A PARALLEL PLATE CAPACITOR, AND SO WE HAVE A WAY OF MAPPING OUT THE 'EDGE EFFECT' IN DETAIL

SEE FIGURE ON THE NEXT PAGE.

BECAUSE THE FIELD IS STRONGER NEAR THE EDGE THE CHARGE DENSITY IS HIGHER. $\therefore$ THE CAPACITANCE $C = \dfrac{Q}{V}$ IS HIGHER THAN IN AN IDEAL PARALLEL PLATE CAPACITOR. WE ESTIMATE THIS EFFECT AS FOLLOWS:

IF $\sigma(x) =$ SURFACE CHARGE DENSITY, $Q = \int \sigma(x)\, dx$

NOW $\sigma(x) = \dfrac{1}{4\pi} E_y\Big|_{\text{CONDUCTOR}} = -\dfrac{1}{4\pi}\dfrac{\partial\phi}{\partial y} = -\dfrac{1}{4\pi}\dfrac{\partial v}{\partial y} = \dfrac{1}{4\pi}\dfrac{\partial u}{\partial x}$

USING THE C-R EQUATIONS

HENCE $Q(x) = \dfrac{1}{4\pi}\displaystyle\int \frac{\partial u}{\partial x}\, dx = \frac{1}{4\pi}\left(u(x) - u(x_0)\right)$ IS THE CHARGE LOCATED BETWEEN x_0 AND x ON, SAY, THE UPPER PLATE OF THE CAPACITOR.

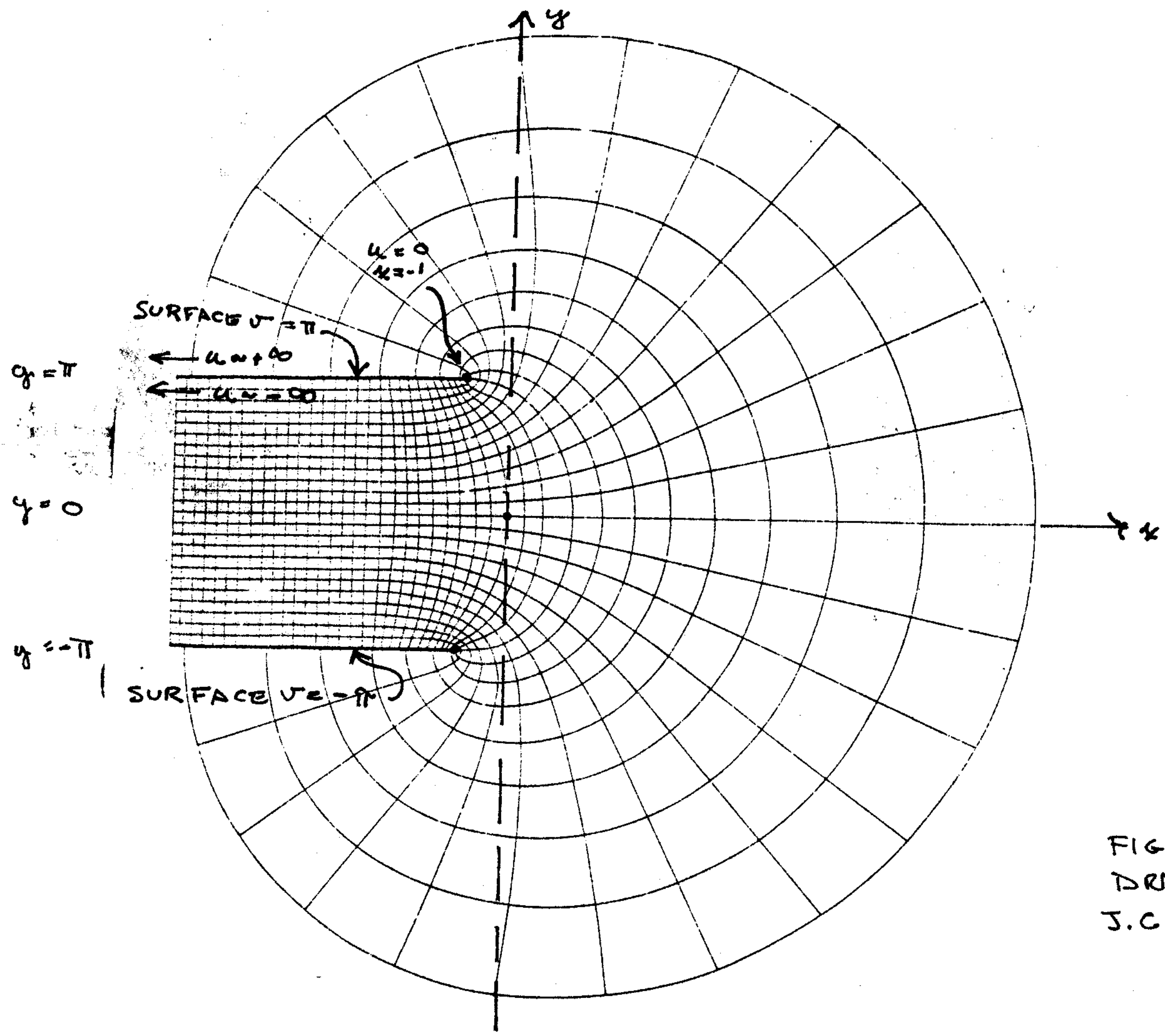


FIGURE DRAWN BY J.C. MAXWELL

Lines of Force between two Plates

ON THE UPPER PLATE, $x = u - e^{u}$. FOR $u < 0$ WE ARE ON THE BOTTOM SIDE OF THE UPPER PLATE AND $x \sim u$ FOR $|u| >> 1$

THE PLATE ENDS AT $x = -1$, WHERE $u = 0$

HENCE $Q(x)\Big|_{\text{BOTTOM}} = \dfrac{|u|}{4\pi} \sim \dfrac{|x|}{4\pi}$ WHILE WE EXPECT

ONLY $Q = \dfrac{|x-1|}{4\pi}$ FOR AN IDEAL CAPACITOR

THUS $\dfrac{\Delta Q}{Q} \sim \dfrac{1}{x}$. (BUT WE ALSO SHOULD INTERPRET x AS THE HALF WIDTH OF THE CAPACITOR)

NOW THE GAP HEIGHT IS π (ACTUALLY THIS IS THE HALF HEIGHT)

SO $\dfrac{\Delta Q}{Q} = \dfrac{1}{\pi} \dfrac{\text{GAP HEIGHT}}{\text{WIDTH}} \Rightarrow \dfrac{\Delta C}{C} = \dfrac{1}{\pi} \dfrac{\text{GAP HEIGHT}}{\text{WIDTH OF CAPACITOR}}$

$\Rightarrow$ A SMALL CORRECTION. WE HAVE IGNORED THE EXTRA CHARGE ON TOP OF THE UPPER PLATE – ANOTHER SMALL CORRECTION.

Ph 206 Lecture 6

Line charge at origin λ = charge per unit length.

Gauss' law $\Rightarrow$ $E_r = \frac{2\lambda}{r}$, and potential $\phi(r) = -\int^r \vec{E}\cdot d\vec{r} = -2\lambda \ln r$.

We seek $f(z)$ such that $\phi = U(z) = \mathrm{Re}\, f(z)$

Now, $z = x+iy = re^{i\theta}$, so $\ln z = \ln r + i\theta$

Hence, $f(z) = -2\lambda \ln z = -2\lambda \ln r - 2i\lambda\theta$.

Then, $V = \mathrm{Im}\, f = -2\lambda\theta$ should describe the field lines,

which are indeed $\theta = $ const, i.e., radial.

Line charge at $(x_0, y_0) = (r_0, \theta_0)$

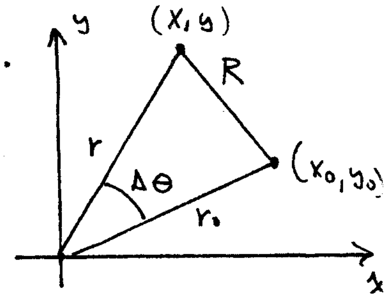
We readily infer that $f(z) = -2\lambda \ln(z - z_0)$ will describe

the potential, where $z_0 = x_0 + iy_0 = r_0 e^{i\theta_0}$.

We verify this by noting that $f = -2\lambda \ln[x - x_0 + i(y-y_0)]$

$$= -2\lambda \ln Re^{i\Delta\theta} = -2\lambda \ln R - 2i\lambda\Delta\theta$$

where $R = \sqrt{(x-x_0)^2 + (y-y_0)^2}$ and $\Delta\theta = \tan^{-1}\left(\frac{y-y_0}{x-x_0}\right)$.



Then $\phi = \mathrm{Re}\, f = -2\lambda \ln R$ as expected.

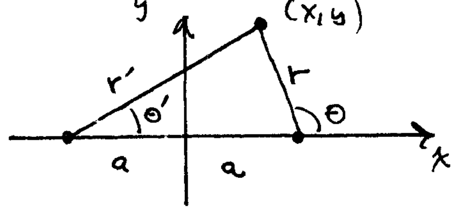
Two line charges λ_1 at (x_1, y_1) & λ_2 and (x_2, y_2)

will be described by $f(z) = -2\lambda_1 \ln(z - z_1) - 2\lambda_2 \ln(z - z_2)$

where $z_j = x_j + iy_j$, according to the superposition principle.

Of particular interest is the example $\lambda_1 = \lambda = -\lambda_2$,

$(x_1, y_1) = (a, 0)$, $(x_2, y_2) = (-a, 0)$.



Then, $f(z) = -2\lambda \ln\left(\frac{z-a}{z+a}\right) = -2\lambda \ln \frac{r}{r'} - 2i(\theta - \theta')$,

where $z - a = re^{i\theta}$ with $r = \sqrt{(x-a)^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x-a}$

$z + a = r'e^{i\theta'}$ with $r' = \sqrt{(x+a)^2 + y^2}$, $\theta' = \tan^{-1}\frac{y}{x+a}$

Ph 206 LECTURE 6

THE EQUIPOTENTIALS ARE DESCRIBED BY $\phi = \mathrm{Re}\, f = -2\lambda \ln \dfrac{\sqrt{(x-a)^2+y^2}}{\sqrt{(x+a)^2+y^2}}$,

OR $e^{-\phi/\lambda} = \dfrac{(x-a)^2+y^2}{(x+a)^2+y^2}$

$$\Rightarrow (x^2+2ax+a^2+y^2)\,e^{-\phi/\lambda} = x^2-2ax+a^2+y^2$$

$$x^2(1-e^{-\phi/\lambda}) - 2ax(1+e^{-\phi/\lambda}) + y^2(1-e^{-\phi/\lambda}) = -a^2(1-e^{-\phi/\lambda})$$

$$x^2 \sinh\frac{\phi}{2\lambda} - 2ax\cosh\frac{\phi}{2\lambda} + y^2\sinh\frac{\phi}{2\lambda} = -a^2\sinh\frac{\phi}{2\lambda}$$

$$x^2 - 2ax\coth\frac{\phi}{2\lambda} + y^2 = -a^2$$

$$\left(x - a\coth\frac{\phi}{2\lambda}\right)^2 + y^2 = -a^2 + a^2\coth^2\frac{\phi}{2\lambda} = a^2\operatorname{csch}^2\frac{\phi}{2\lambda}$$

$\Rightarrow$ THE EQUIPOTENTIALS ARE CIRCLES.

THE FIELD LINES ARE GIVEN BY $\psi = \mathrm{Im}\, f = -2\lambda(\theta-\theta')$

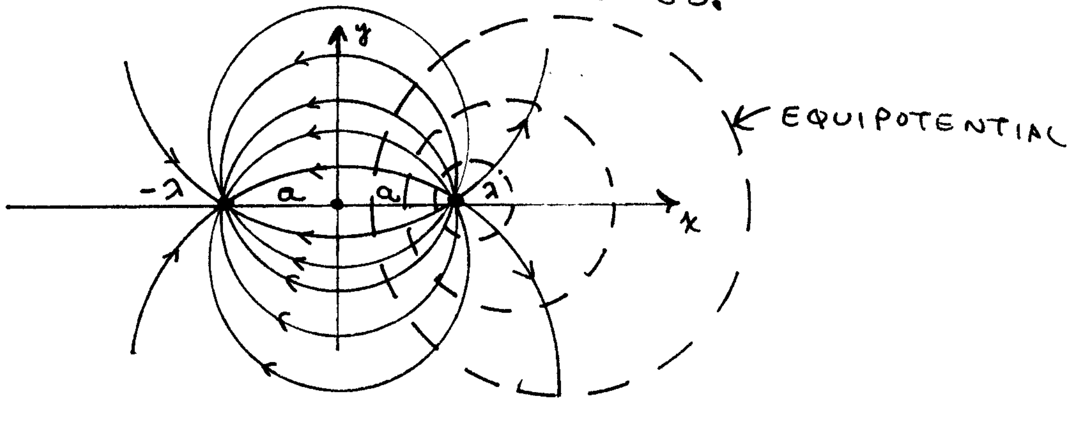
NOW, $\tan(\theta-\theta') = \dfrac{\tan\theta - \tan\theta'}{1+\tan\theta\tan\theta'} = \dfrac{\dfrac{y}{x-a} - \dfrac{y}{x+a}}{1+\dfrac{y^2}{x^2-a^2}} = \dfrac{2ay}{x^2-a^2+y^2} = \tan\left(-\dfrac{\psi}{2\lambda}\right)$

THEN, $x^2-a^2+y^2 = -2ay\cot\dfrac{\psi}{2\lambda}$

$$x^2+y^2+2ay\cot\frac{\psi}{2\lambda} = a^2$$

$$x^2 + \left(y + a\cot\frac{\psi}{2\lambda}\right)^2 = a^2 + a^2\cot^2\frac{\psi}{2\lambda} = a^2\csc^2\frac{\psi}{2\lambda}$$

THE FIELD LINES ARE ALSO CIRCLES! THE POINTS $(x,y) = (\pm a, 0)$ LIE ON THESE CIRCLES, AS IS TO BE EXPECTED.



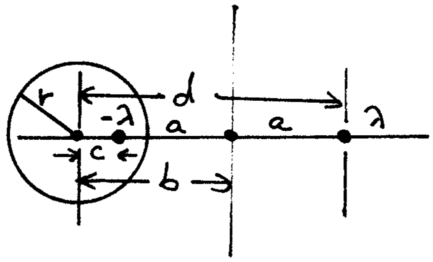
Ph 206 Lecture 6

Line Charge and Conducting Cylinder

Clearly, this case is contained within the preceding example. Let r = radius of the conducting cylinder, and d = distance between the line charge λ and the axis of the cylinder.

A line charge $-\lambda$ at an appropriate distance $2a$ from line charge λ would make the cylinder an equipotential.

This is the <u>image method</u> for 2-dimensions!



From the preceding, we identify $r = \dfrac{a}{\sinh \frac{\phi}{2\lambda}}$ or $\sinh \dfrac{\phi}{2\lambda} = \dfrac{a}{r}$

And the center of the cylinder is at $b = a \coth \dfrac{\phi}{2\lambda} = r \cosh \dfrac{\phi}{2\lambda} = r\sqrt{1 + \dfrac{a^2}{r^2}} = \sqrt{a^2 + r^2}$

That is, $b^2 - a^2 = r^2$

$$(b+a)(b-a) = r^2$$

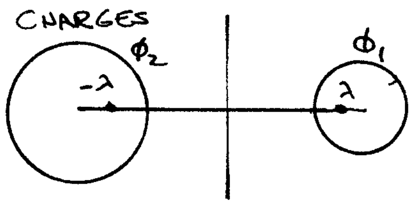
$$d \cdot c = r^2 \qquad \text{where } d = a + b \text{ , and } c = b - a$$

Thus $c = \dfrac{r^2}{d}$ = distance between the center of the cylinder and the image charge $-\lambda$.

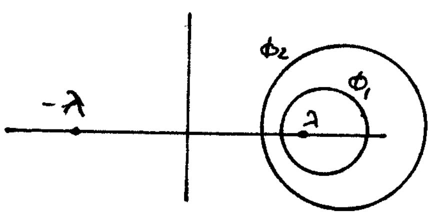
Two Conducting Cylinders

This case is now readily solved using the image method – or the method of complex variables.

The cylinders can be either on the same side or on the opposite sides of the bisector of the line joining the image charges

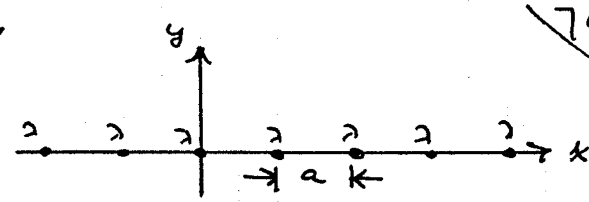


or



The capacitance is $C = \dfrac{\phi_1 - \phi_2}{\lambda}$

A GRID OF WIRES (PROB. 1, SET 3)



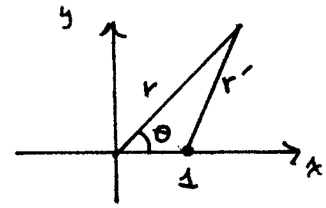
EACH WIRE CARRIES CHARGE λ PER UNIT LENGTH, AND IS DISTANCE a FROM ITS NEIGHBORS.

FROM p. 74a, A POSSIBLE WAY TO FIND THE POTENTIAL IS VIA THE SUM

$$f(z) = -2\lambda \sum_n{}' \ln(z - na)$$

A TRICK TO DEAL WITH THIS IS A SPECIAL CASE OF A "SCHWARZ TRANSFORMATION", NAMELY $z' = e^{-\frac{2\pi i z}{a}}$, WHICH MAPS ALL POINTS $z = na$ ONTO $z' = 1$.

WE FIRST CONSIDER THE CASE OF A WIRE OF CHARGE DENSITY λ AT $(x,y) = (1,0)$, PLUS A WIRE OF CHARGE DENSITY λ' AT $(x,y) = (0,0)$.



FROM p. 74a, THE POTENTIAL IS GIVEN BY THE REAL PART OF

$$f(z) = -2\lambda \ln(z-1) - 2\lambda' \ln z$$

WRITING $z = r e^{i\theta}$ $\quad z - 1 = r' e^{i\theta'}$ WHERE $r'^2 = 1 - 2r\cos\theta + r^2$.

THAT IS, $\phi(x,y) = \mathrm{Re}\,(f(z)) = -2\lambda \ln r' - 2\lambda' \ln r = \lambda \ln(1 - 2r\cos\theta + r^2) - 2\lambda' \ln r$

NOW CONSIDER $f(z')$ WHERE $z' = e^{-\frac{2\pi i z}{a}} = e^{-\frac{2\pi i}{a}(x+iy)} = e^{\frac{2\pi y}{a}} e^{-\frac{2\pi i x}{a}} \equiv r e^{i\theta}$.

THEN $r = e^{\frac{2\pi y}{a}}$ AND $\theta = -\frac{2\pi x}{a}$.

WE GET THE POTENTIAL

$$\phi(x,y) = \mathrm{Re}\,(f(z')) = -\lambda \ln\left(1 - 2e^{\frac{2\pi y}{a}} \cos\frac{2\pi x}{a} + e^{\frac{4\pi y}{a}}\right) - 2\lambda' \ln e^{\frac{2\pi y}{a}}$$

$$= -\lambda \ln\left[e^{\frac{2\pi y}{a}}\left(e^{\frac{2\pi y}{a}} + e^{-\frac{2\pi y}{a}} - 2\cos\frac{2\pi x}{a}\right)\right] - 2\lambda'\left(\frac{2\pi y}{a}\right)$$

$$= -\lambda \ln\left[2\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right)\right] - \frac{2\pi y}{a}(\lambda + 2\lambda')$$

IF WE TAKE $\lambda' = -\lambda/2$, THEN $\phi(x,y)$ IS SYMMETRIC IN y, AND IS THE DESIRED POTENTIAL OF THE GRID:

$$\phi(x,y) = -\lambda \ln\left[2\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right)\right]$$

[IF $\lambda' \neq -\lambda/2$, WE GET THE POTENTIAL DUE TO THE GRID, PLUS A UNIFORM ELECTRIC FIELD IN THE y DIRECTION.]

5.291. Bessel's Equation and Bessel Functions.—In 5.29 (2), we set

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2 \tag{1}$$

and

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \tag{2}$$

Then R satisfies the equation

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (k^2\rho^2 - n^2)R = 0$$

If we set $v = k\rho$, this equation takes the form

$$\frac{d^2R}{dv^2} + \frac{1}{v}\frac{dR}{dv} + \left(1 - \frac{n^2}{v^2}\right)R = 0 \tag{3}$$

Complete solutions of 1 and 2 are

$$\Phi = Ae^{jn\phi} + Be^{-jn\phi} = A' \cos n\phi + B' \sin n\phi \tag{4}$$
$$Z = Ce^{kz} + De^{-kz} = C' \cosh kz + D' \sinh kz \tag{5}$$

A solution of (3), which is known as Bessel's equation, is called a Bessel function of the nth order. The cylindrical harmonic may now be written

$$V = R_n(k\rho)\Phi(n\phi)Z(kz) \tag{6}$$

except when $k = 0$ when, from 4.01 (5) and 4.07 (2), solutions are

$$V_0 = (M\rho^n + N\rho^{-n})(Cz + D)(A \cos n\phi + B \sin n\phi) \tag{7}$$
$$V_0 = [M \cos (n \ln \rho) + N \sin (n \ln \rho)](Cz + D)(A \cosh n\phi + B \sinh n\phi) \tag{8}$$

When k and n are both zero, we have

$$V_{00} = (M \ln \rho + N)(Cz + D)(A\phi + B) \tag{9}$$

5.292. Modified Bessel Equation and Functions.—A somewhat different equation is obtained by substituting jk for k in 5.291 (2). We then get in place of 5.291 (3)

$$\frac{d^2R^0}{dv^2} + \frac{1}{v}\frac{dR^0}{dv} - \left(1 + \frac{n^2}{v^2}\right)R^0 = 0 \tag{1}$$

which is called the modified Bessel equation and whose solutions are called modified Bessel functions. In place of 5.291 (5), we now have

$$Z = Ce^{jk_1z} + De^{-jk_1z} = C' \cos k_1z + D' \sin k_1z \tag{2}$$

The cylindrical harmonic is now written

$$V = R_n^0(k_1\rho)e^{\pm jn\phi}e^{\pm jk_1z} \tag{3}$$

Solutions of this type will be considered in Arts. 5.32 to 5.36.

5.293. Solution of Bessel's Equation.—Let $R_n = \Sigma a_s v^{n+s}$ in 5.291 (3), and we obtain

$$v^n\Sigma[s(2n + s)a_sv^{s-2} + a_sv^s] = 0 \tag{1}$$

Whence,

$$a_s = -[s(2n + s)]^{-1}a_{s-2} \tag{2}$$

If a_0 is finite, then a_{-2}, a_{-4}, etc. are zero. Let a_0 be $[2^n\Gamma(n + 1)]^{-1}$, which reduces to $(2^nn!)^{-1}$ when n is an integer, then we have, by repeated use of (2), writing $J_n(v)$ for $R_n(v)$,

$$J_n(v) = \frac{v^n}{2^n\Gamma(n + 1)}\left[1 - \frac{v^2}{2^2(n + 1)} + \frac{v^4}{2^42!(n + 1)(n + 2)} - \cdots\right]$$
$$= \sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}v)^{n+2r}}{r!\Gamma(n + r + 1)} \tag{3}$$

The function $J_n(v)$ is called a Bessel function of the nth order of the first kind. Evidently $J_n(0) = 0$ when $n \neq 0$, and it will be shown in 5.303 that $J_n(\infty) = 0$ also.

Since 5.291 (3) is a second-order differential equation, there must be a second solution. Where n is not an integer, this is $J_{-n}(v)$; but when n is an integer, $J_{-n}(v)$ and $J_n(v)$ are not independent solutions. To show this, write $-n$ for n in (3) then, since $[\Gamma(-n)]^{-1}$ is zero, the series for $J_{-n}(v)$ and $(-1)^nJ_n(v)$ are identical. When n is not integral, the formula

$$Y_\nu(v) = \frac{J_\nu(v) \cos \nu\pi - J_{-\nu}(v)}{\sin \nu\pi} \tag{4}$$

defines a second solution. When ν is integral, this becomes 0/0. To find its value, write $-\nu$ for n in (3), split the 0 to ∞ r-summation into one from n to ∞ and one from 0 to $n - 1$, write $n + s$ for r in the former, and $\pi^{-1}\Gamma(\nu - r) \sin \nu\pi$ for $(-1)^r/\Gamma[1 - (\nu - r)]$ by *Dw* 850.3 in the latter. When $J_{-\nu}(v)$ in (4) is replaced by this result and $\cos \nu\pi$ by $(-1)^n$, it becomes

$$\frac{(-1)^n}{\sin \nu\pi}\left[J_\nu(v) - \sum_{s=0}^{\infty}\frac{(-1)^s(\frac{1}{2}v)^{\nu+2s}}{(n+s)!\Gamma(n-\nu+s+1)}\right] - \sum_{r=0}^{n-1}\frac{(\frac{1}{2}v)^{2r-\nu}\Gamma(\nu - r)}{\pi r!}$$

The bracket is zero when $\nu = n$ from (3) so the first term still has the form 0/0. To evaluate this, differentiate each factor with respect to ν and write n for ν. By referring to page 19 of Jahnke and Emde we find

$$\frac{d}{dz}\left[\frac{1}{\Gamma(z)}\right] = \frac{-1}{\Gamma(z)}\frac{d \ln \Gamma(z)}{dz} = \frac{-1}{\Gamma(z)}\left(-C + \sum_{m=1}^{z}\frac{1}{m}\right)$$

where $C = .5772157$ is Euler's number. With this formula and *Dw* 563.3 or *Pc* 828, the 0/0 form is evaluated and writing $\ln \alpha$ for $C - \ln 2$ gives

$$Y_n(v) = \frac{2}{\pi}J_n(v)\ln(\alpha v) - \sum_{r=0}^{n-1}\frac{(\frac{1}{2}v)^{2r-n}(n-r-1)!}{\pi r!}$$
$$- \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}v)^{n+2r}}{\pi r!(n+r)!}\left(\sum_{m=1}^{r}\frac{1}{m} + \sum_{m=1}^{n+r}\frac{1}{m}\right) \quad (5)$$

A complete solution of Bessel's equation when n is an integer is

$$R_n(v) = A'J_n(v) + B'Y_n(v) \quad (6)$$

We note that $Y_n(0)$ is infinite and shall see in 5.303 that $Y_n(\infty)$ is zero. There are many notations for the function defined by (5). Watson and the British Association Tables use $Y_n(v)$, Jahnke and Emde, Schelkunoff, and Stratton use $N_n(v)$, and Gray, Mathews, and MacRobert use $\bar{Y}_n(v)$.

Setting $M = 1/v$ and $N = 1 - (n/v)^2$ in 5.111 (6) reduces it to 5.291 (3) so that, if $J_n(v)$ is the known solution, 5.111 (7) becomes

$$Y_n(v) = J_n(v)\{A + B\int[vJ_n^2(v)]^{-1}\,dv\} \quad (7)$$

Differentiating this and omitting the argument give

$$\frac{d}{dv}\left(\frac{Y_n}{J_n}\right) = \frac{Y'_n}{J_n} - \frac{J'_nY_n}{J_n^2} = \frac{B}{vJ_n^2} \quad (8)$$

From the recurrence formulas, B is independent of both n and v and so may be evaluated for integral n's by taking the simple case where n is zero and v is very small. Then only the ln v term in (5) is important, so it and its derivative may be used for $Y_n(v)$ and $Y'_n(v)$ in (8). The logarithm terms cancel and B comes out $2/\pi$ so (8) may be rearranged in the form

$$Y'_n(v)J_n(v) - J'_n(v)Y_n(v) = \frac{2}{\pi v} \quad (9)$$

For cylindrical electromagnetic waves we need the "Hankel" functions which combine with $e^{j\omega t}$ to give traveling waves and are defined by

$$H_n^{(1)}(v) = J_n(v) + jY_n(v), \qquad H_n^{(2)}(v) = J_n(v) - jY_n(v) \quad (10)$$

5.294. Recurrence Formulas for Bessel Functions.—If we multiply 5.293 (3) by v^n and differentiate, we obtain

$$\frac{d[v^nJ_n(v)]}{dv} = \frac{v^{2n-1}}{2^{n-1}\Gamma(n)}\left[1 - \frac{v^2}{2^2n} + \frac{v^4}{2^4 2!n(n+1)} - \cdots\right]$$

Factoring out v^n on the right side and comparing with 5.293 (3), we see that

$$\frac{d[v^nJ_n(v)]}{dv} = v^nJ_{n-1}(v)$$

Differentiate left side, rearrange, divide out v^n, and indicate differentiation with respect to v by a prime, and we have

$$J'_n = J_{n-1} - \frac{n}{v}J_n \quad (1)$$

If we use the same procedure multiplying by v^{-n} instead of by v^n, we obtain

$$J'_n = -J_{n+1} + \frac{n}{v}J_n = \frac{1}{2}(J_{n-1} - J_{n+1}) \quad (2)$$

Subtracting (1) from (2), we have

$$\frac{2n}{v}J_n = J_{n-1} + J_{n+1} \quad (3)$$

Writing $(-1)^nJ_{-n}(v)$ for $J_n(v)$ in (1), (2), and (3) gives the recurrence formulas for $J_{-n}(v)$. Differentiation of 5.293 (4) and substitution for $J'_\nu(v)$ from (1) and for $J'_{-\nu}(v)$ from a similar formula give, as $\nu \to n$, so that cos $\nu\pi$ may be replaced by $-\cos(\nu-1)\pi$ and sin $\nu\pi$ by $-\sin(\nu-1)\pi$,

$$\frac{J'_\nu\cos\nu\pi - J'_{-\nu}}{\sin\nu\pi} \xrightarrow[\nu\to n]{} \frac{J_{\nu-1}\cos(\nu-1)\pi - J_{-\nu+1}}{\sin(\nu-1)\pi} - \frac{\nu(J_\nu\cos\nu\pi - J_{-\nu})}{v\sin\nu\pi}$$

A similar process can be applied to (2) so that

$$Y'_n = Y_{n-1} - \frac{n}{v}Y_n \quad (4)$$

$$Y'_n = -Y_{n+1} + \frac{n}{v}Y_n \quad (5)$$

Subtraction of these equations gives

$$\frac{2n}{v}Y_n = Y_{n-1} + Y_{n+1} \quad (6)$$

Two useful integral formulas are got by integrating the equation from which (1) was derived and the analogous equation associated with (4).

$$\int v^nJ_{n-1}(v)\,dv = v^nJ_n(v) \quad (7)$$
$$\int v^nY_{n-1}(v)\,dv = v^nY_n(v) \quad (8)$$

A similar integration of (2) and (5) gives

$$\int v^{-n}J_{n+1}(v)\,dv = -v^{-n}J_n(v) \quad (9)$$
$$\int v^{-n}Y_{n+1}(v)\,dv = -v^{-n}Y_n(v) \quad (10)$$

5.295. Values of Bessel Functions at Infinity.—In potential problems involving $\rho = \infty$, one must know how $J_n(k\rho)$ and $Y_n(k\rho)$ behave there. To find this limiting value, we use a trick often employed by Sommerfeld. In 5.291 (3), as $v \to \infty$ let us, as a first approximation, drop the terms containing v^{-1} and v^{-2}. This gives the approximate differential equation

$$\frac{d^2R}{dv^2} + R = 0 \quad (1)$$

The solution of this equation is

$$R = R'e^{\pm jv} \tag{2}$$

We now insert this trial solution in Bessel's equation 5.291 (3) and consider R' to vary so slowly with v that d^2R'/dv^2, $v^{-1}\,dR'/dv$, and $v^{-2}R'$ can be neglected compared with dR'/dv and $v^{-1}R'$, and we obtain

$$2\frac{dR'}{dv} + \frac{R'}{v} = 0 \qquad \text{or} \qquad R' = Cv^{-\frac{1}{2}}$$

so that, from (2) our asymptotic solution becomes

$$R = Cv^{-\frac{1}{2}}e^{\pm jv} \tag{3}$$

We now see that the largest term neglected was of the order $v^{-\frac{3}{2}}$. $J_n(v)$ and $Y_n(v)$ must be real linear combinations of the two solutions given by taking the plus or minus sign so that they must be of the form

$$J_n(v) \xrightarrow[v\to\infty]{} Av^{-\frac{1}{2}} \cos (v + \alpha) \tag{4}$$

$$Y_n(v) \xrightarrow[v\to\infty]{} Bv^{-\frac{1}{2}} \cos (v + \beta) \tag{5}$$

To find how A and α depend on n, put (4) into 5.294 (1) and 5.294 (2) which give, as $v \to \infty$, $J'_n = J_{n-1}$ and $J'_n = -J_{n+1}$, respectively. This gives the relation $\alpha_{n\pm1} = \alpha_n \mp \frac{1}{2}\pi$ which is satisfied by $\alpha_n = -\frac{1}{2}n\pi + \gamma$ and shows that A does not depend on n. Because n need not be an integer, we may write $n = \frac{1}{2}$ in (4) and compare it with 5.31 (2) which shows that $\gamma = -\frac{1}{4}\pi$ and gives

$$J_n(v) \xrightarrow[v\to\infty]{} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} \cos\left(v - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) \tag{6}$$

where terms in $v^{-\frac{1}{2}m}$ have been neglected if $m \geqslant 3$ and n is real.

To get $Y_n(v)$ substitute (6), with ν and $-\nu$ for n, in 5.293 (4). The result gives 0/0 when ν is integral, but replacement of numerator and denominator by their derivatives with respect to ν gives, when $\nu = n$,

$$Y_n(v) \xrightarrow[v\to\infty]{} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} \sin\left(v - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) \tag{7}$$

Thus both Bessel functions vanish at infinity. From (6), (7), and 5.293 (10) we find that for the Hankel functions

$$H_n^{(1)}(v) \xrightarrow[v\to\infty]{} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{j(v-\frac{1}{2}n\pi-\frac{1}{4}\pi)}, \qquad H_n^{(2)}(v) \xrightarrow[v\to\infty]{} \left(\frac{2}{\pi v}\right)^{\frac{1}{2}} e^{-j(v-\frac{1}{2}n\pi-\frac{1}{4}\pi)} \tag{8}$$

5.296. Integrals of Bessel Functions.—In 5.261, we made an expansion in spherical harmonics satisfy the condition $V = 0$ on the cone $\theta = \alpha$ by choosing only such orders n of the harmonics $\Theta_n^m(\cos\theta)$ as made $\Theta_n^m(\cos\alpha) = 0$. To determine the coefficients in this expansion, it was first necessary, in 5.26, to evaluate the integral of the product of two such

harmonics over the range of θ from 0 to α. In the same way, if we are to get an expansion in Bessel functions that meets the condition $V = 0$ or $E = 0$ on the cylinder $\rho = a$, we must evaluate the integral of the product of $R_n(k_p\rho)$ and $R_n(k_q\rho)$ over this range, where k_p and k_q are chosen to meet the boundary conditions.

Let $u = R_n(k_p\rho)$ and $v = R_n(k_q\rho)$ be two solutions of Bessel's equation. Then from 5.291

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \left(k_p^2 - \frac{n^2}{\rho^2}\right)u = 0$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dv}{d\rho}\right) + \left(k_q^2 - \frac{n^2}{\rho^2}\right)v = 0$$

Multiply the first by ρv and the second by ρu subtract, and integrate, and we have

$$(k_p^2 - k_q^2)\int_0^a \rho uv\, d\rho = -\int_0^a\left[v\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) - u\frac{d}{d\rho}\left(\rho\frac{dv}{d\rho}\right)\right]d\rho$$

Integrate each term on the right side by parts, and the integrals cancel, leaving

$$(k_p^2 - k_q^2)\int_0^a \rho uv\, d\rho = -\left(\rho v\frac{du}{d\rho} - \rho u\frac{dv}{d\rho}\right)_0^a$$
$$= -a[k_pR_n(k_qa)R'_n(k_pa) - k_qR_n(k_pa)R'_n(k_qa)]$$

This is zero if

$$R_n(k_pa) = R_n(k_qa) = 0 \tag{1}$$

or if

$$R'_n(k_pa) = R'_n(k_qa) = 0 \tag{2}$$

or if

$$k_paR'_n(k_pa) + BR_n(k_pa) = k_qaR'_n(k_qa) + BR_n(k_qa) = 0 \tag{3}$$

Thus if $k_p \neq k_q$, we have the result

$$\int_0^a \rho R_n(k_p\rho)R_n(k_q\rho)\, d\rho = 0 \tag{4}$$

If $R_n(k_pa) = R_n(k_qa) = R_n(k_pb) = R_n(k_qb) = 0$ we have, since

$$\int_a^b f(x)\,dx = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx$$

the result

$$\int_a^b \rho R_n(k_p\rho)R_n(k_q\rho)\, d\rho = 0 \tag{5}$$

To evaluate the integral when $k_p = k_q$, multiply Bessel's equation [5.291 (3)] through by $v^2(dR_n/dv)\,dv$ giving

$$v^2\frac{dR_n}{dv}d\left(\frac{dR_n}{dv}\right) + v\left(\frac{dR_n}{dv}\right)^2 dv + v^2R_n\frac{dR_n}{dv}\,dv - n^2R_n\frac{dR_n}{dv}\,dv = 0$$

Integrate from 0 to a using integration by parts on the first and third terms. We thus find the following expression to be zero;

$$\left|\frac{v^2}{2}\left(\frac{dR_n}{dv}\right)^2\right|_0^a - \int_0^a v\left(\frac{dR_n}{dv}\right)^2 dv + \int_0^a v\left(\frac{dR_n}{dv}\right)^2 dv + \left|\frac{v^2}{2}R_n^2\right|_0^a - \int_0^a vR_n^2\,dv - \left|\frac{n^2R_n^2}{2}\right|_0^a$$

Canceling the second and third terms and solving for the fifth give

$$\int_0^a v[R_n(v)]^2\,dv = \frac{1}{2}\left|v^2\left[\frac{dR_n(v)}{dv}\right]^2 + (v^2 - n^2)[R_n(v)]^2\right|_0^a \qquad (6)$$

Substituting for the derivative from 5.294 (2) gives

$$\int_0^a v[R_n(v)]^2\,dv = \tfrac{1}{2}a^2\{[R_n(a)]^2 + [R_{n+1}(a)]^2\} - naR_n(a)R_{n+1}(a)$$
$$= \tfrac{1}{2}a^2\{[R_n(a)]^2 + [R_{n-1}(a)]^2\} - naR_n(a)R_{n-1}(a) \qquad (7)$$

In dealing with the vector potential we shall have occasion to make use of the orthogonal properties of the vector function defined by

$$\mathbf{R}_n(k_p\rho) = \boldsymbol{\rho}R'_n(k_p\rho) \pm \boldsymbol{\phi}\frac{n}{k_p\rho}R_n(k_p\rho) \qquad (8)$$

The integral of the scalar product of two such functions from 0 to a is

$$\int_0^a \mathbf{R}_n(k_p\rho)\cdot\mathbf{R}_n(k_q\rho)\rho\,d\rho$$
$$= \int_0^a\left[R'_n(k_p\rho)R'_n(k_q\rho) + \frac{n^2}{k_pk_q\rho^2}R_n(k_p\rho)R_n(k_q\rho)\right]\rho\,d\rho \qquad (9)$$

With the aid of (1), (2), (3), (4), (5), (6) of Art. 5.294 we may write this as the sum of two integrals of the form of (4). Thus

$$\tfrac{1}{2}\int_0^a R_{n+1}(k_p\rho)R_{n+1}(k_q\rho)\rho\,d\rho + \tfrac{1}{2}\int_0^a R_{n-1}(k_p\rho)R_{n-1}(k_q\rho)\rho\,d\rho \qquad (10)$$

Evaluate each integral by the formula for $\int\rho uv\,d\rho$ already given and add the results, then eliminate the derivatives by 5.294 (1) and 5.294 (2), cancel terms not involving the nth order, and combine the resultant $n+1$ and $n-1$ orders by the same formulas. These operations give, if $v = k_pa$ and $v' = k_qa$,

$$\int_0^a \mathbf{R}_n(k_p\rho)\cdot\mathbf{R}_n(k_q\rho)\rho\,d\rho = (k_p^2 - k_q^2)^{-1}[vR_n(v')R'_n(v) - v'R_n(v)R'_n(v')] \qquad (11)$$

Thus, if $k_p \neq k_q$, the integral vanishes under conditions (1), (2), or (3). But, if $k_p = k_q$, evaluation of each integral in (10) by (7) gives for their sum

$$\tfrac{1}{2}[a^2 - (n/k_p)^2][R_n(k_pa)]^2 + \tfrac{1}{2}[aR'_n(k_pa)]^2 + (a/k_p)R'_n(k_pa)R_n(k_pa) \qquad (12)$$

Thus a surface vector function of ρ and ϕ, one component of which vanishes at $\rho = a$, may be written as a sum of terms of the form

$$\mathbf{R}_n(k_p\rho)\sin(n\phi + \delta_n)$$

5.297. Expansion in Series of Bessel Functions.—Consider a function $f(v)$ which satisfies the conditions for expansion into a Fourier series in the range from $v = 0$ to $v = a$ and which fulfills one of the following boundary conditions:

(a) $f(a) = 0$. This case arises, if $f(a)$ is a potential function, when the boundary is at zero potential.

(b) $f'(a) = 0$. This case occurs when the boundary is a line of force.

(c) $af'(a) + Bf(a) = 0$. This case reduces to (a) if $B = \infty$ and to (b) if $B = 0$. An example of its use is given in 11.08.

The function $f(v)$ may be expanded in the form

$$f(v) = \sum_{r=1}^{\infty} A_rJ_n(\mu_rv) \qquad (1)$$

where the values of μ_r are chosen so that in case (a) $J_n(\mu_ra) = 0$, in case (b) $J'_n(\mu_ra) = 0$, and in case (c) $\mu_raJ'_n(\mu_ra) + BJ_n(\mu_ra) = 0$. To determine A_r, multiply both sides of (1) by $vJ_n(\mu_sv)$, and integrate from $v = 0$ to $v = a$. By 5.296 (4), all terms on the right vanish except $A_s\int v[J_n(\mu_sv)]^2\,dv$ so that

$$A_s = \frac{\int_0^a vf(v)J_n(\mu_sv)\,dv}{\int_0^a v[J_n(\mu_sv)]^2\,dv} \qquad (2)$$

We can evaluate the lower integral by 5.296 (7) giving

$$\int_0^a v[J_n(\mu_sv)]^2\,dv = \mu_s^{-2}\int_0^{\mu_sa} x[J_n(x)]^2\,dx$$
$$= \frac{1}{2}a^2\{[J_n(\mu_sa)]^2 + [J_{n\pm1}(\mu_sa)]^2\} - \frac{na}{\mu_s}J_n(\mu_sa)J_{n\pm1}(\mu_sa) \qquad (3)$$

In case (a), substituting (3) in (2) gives

$$A_s = \frac{2}{[aJ_{n\pm1}(\mu_sa)]^2}\int_0^a vf(v)J_n(\mu_sv)\,dv \qquad (4)$$

In case (b), substituting (3) in (2) gives

$$A_s = \frac{2}{(a^2 - n^2\mu_s^{-2})[J_n(\mu_sa)]^2}\int_0^a vf(v)J_n(\mu_sv)\,dv \qquad (5)$$

In case (c), substituting (3) in (2) gives

$$A_s = \frac{2}{[a^2 + (B^2 - n^2)/\mu_s^2][J_n(\mu_sa)]^2}\int_0^a vf(v)J_n(\mu_sv)\,dv \qquad (6)$$

5.298. Green's Function for Cylinder. Inverse Distance.—We shall calculate by the principles of the last few articles the potential when a point charge q is placed at the point $z = 0$, $\rho = b$, $\phi = \phi_0$, inside an earthed conducting cylinder. By a point charge, we mean one whose denominator of the sum by $e^{\mu_r L}$, and putting over a common denominator give

$$\frac{2 \sinh \mu_r(L-c) \sinh \mu_r z}{\sinh \mu_r L}$$

Substituting this value in 5.298 (4) gives, when $z < c$, the potential

$$V = \frac{q}{\pi\epsilon a^2}\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(2-\delta_s^0)\frac{\sinh \mu_r(L-c)\sinh \mu_r z}{\sinh \mu_r L}\frac{J_s(\mu_r b)J_s(\mu_r \rho)}{\mu_r[J_{s+1}(\mu_r a)]^2}\cos s(\phi-\phi_0) \quad (1)$$

When $z > c$, substitute $L - z$ for z and $L - c$ for c. If the charge is on the axis of the cylindrical box, drop the summation with respect to s in (1), retaining only the $s = 0$ term.

If, in addition, the planes $\phi = 0$ and $\phi = \phi_1$, where $0 < \phi_0 < \phi_1$, were at zero potential, and if $\phi_1 = \pi/n$, where n is an integer, we could get the Green's function by superimposing, according to the principle of images (see 5.07), $2n$ solutions of the type of (1).

5.30. Bessel Functions of Zero Order.—In the important case where we are dealing with fields symmetrical about the z-axis, the potential is independent of ϕ so that Bessel's equation becomes, from 5.291 (3),

$$\frac{d^2R}{dv^2} + \frac{1}{v}\frac{dR}{dv} + R = 0 \quad (1)$$

and the solution [5.293 (3)] becomes

$$J_0(v) = 1 - \frac{v^2}{2^2} + \frac{v^4}{2^4(2!)^2} - \frac{v^6}{2^6(3!)^2} + \cdots \quad (2)$$

This series is evidently convergent for all values of v. As with $J_n(v)$, $J_0'(\infty) = 0$ but $J_0(0) = 1$. Equation 5.293 (5) becomes, when $n = 0$,

$$Y_0(v) = \frac{2}{\pi}\left[J_0(v) \ln \alpha v + \frac{v^2}{2^2} - \frac{v^4(1+\frac{1}{2})}{2^4(2!)^2} + \frac{v^6(1+\frac{1}{2}+\frac{1}{3})}{2^6(3!)^2} - \cdots\right] \quad (3)$$

where $\ln \alpha$ is -0.11593.

5.301. Roots and Numerical Values of Bessel Functions of Zero Order.—If we plot the values of $J_0(v)$ and $Y_0(v)$ given by 5.30 (2) and (3), we get the curves shown in Figs. 5.301*a* and 5.301*b*. We see that they oscillate up and down across the v-axis. It can be shown that both $J_0(v)$ and $Y_0(v)$ have an infinite number of real positive roots. The same is true of $J_n(v)$ and $Y_n(v)$. As we have seen in finding the Green's function for a cylinder, the existence of these roots is very useful as it makes it possible to choose an infinite number of values of k which will make $J_n(k\rho)$ or $Y_n(k\rho)$ zero for any specified value of ρ. Many excellent tables exist which give numerical values, graphs, roots, etc., of Bessel functions. Care should be taken in observing the notation used as this

varies widely with different authors. Asymptotic expansions provide an easy method of evaluating Bessel functions of large argument.

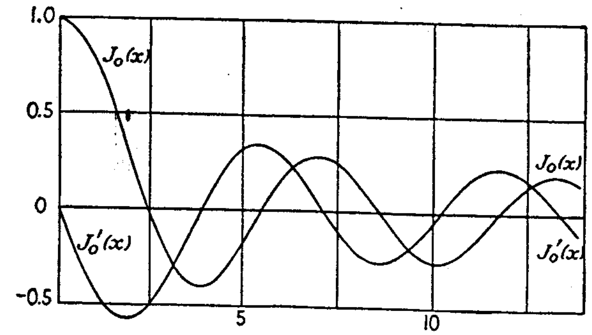


FIG. 5.301*a*.

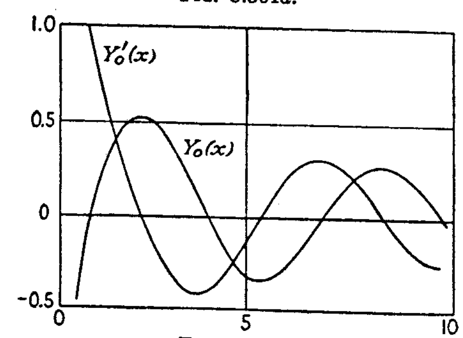


FIG. 5.301*b*.

5.302. Derivatives and Integrals of Bessel Functions of Zero Order. Putting $n = 0$ in 5.294 (2) and (5), we have

$$J_0'(v) = -J_1(v) \quad \text{and} \quad Y_0'(v) = -Y_1(v) \quad (1)$$

From 5.294 (7) and (8), we have

$$\int_0^v vJ_0(v)\,dv = vJ_1(v) \quad \text{and} \quad \int_0^v vY_0(v)\,dv = vY_1(v) + 2\pi^{-1} \quad (2)$$

There are several definite integrals involving $J_0(v)$ which will be useful. From 5.30 (2), using *Dw* 854.1 or *Pc* 483, we have

$$J_0(v) = \sum_{n=0}^{\infty}\frac{(-1)^n v^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty}(-1)^n\frac{v^{2n}}{(2n)!}\frac{(2n)!}{2^{2n}(n!)^2}$$

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n 2v^{2n}}{(2n)!}\frac{\pi}{2}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}$$

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{v^{2n}}{(2n)!}\int_0^{\pi}\cos^{2n}t\,dt$$