PLANE WAVES IN SIMPLE MEDIA

WE EXPAND OUR CONSIDERATIONS SOMEWHAT BY CONSIDERING THE PROPAGATION OF WAVES IN SIMPLE MEDIA, FOR WHICH WE HAVE

$$\bar{D} = \epsilon \bar{E} \qquad \bar{B} = \mu \bar{H} \qquad \text{AND} \qquad \bar{J} = \sigma \bar{E}$$

WE FURTHER SUPPOSE THERE ARE NO EXTERNAL CHARGES OR CURRENTS IMPOSED ON THE MEDIUM — ALTHOUGH CURRENTS MAY FLOW DUE TO THE WAVE FIELD VIA $\bar{J} = \sigma \bar{E}$.

THEN THE MAXWELL EQUATIONS ARE

$$\bar{\nabla} \cdot \bar{E} = 0 \qquad \bar{\nabla} \cdot \bar{B} = 0 \Rightarrow \bar{\nabla} \cdot \bar{H} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} = -\frac{\mu}{c}\frac{\partial \bar{H}}{\partial t} \qquad \bar{\nabla} \times \bar{H} = \frac{4\pi}{c}\bar{J}_{Free} + \frac{1}{c}\frac{\partial \bar{D}}{\partial t} = \frac{4\pi\sigma\bar{E}}{c} + \frac{\epsilon}{c}\frac{\partial \bar{E}}{\partial t}$$

TAKING THE CURL OF THE LAST 2 EQUATIONS, WE FIND

$$\nabla^2 \bar{E} = \frac{\epsilon\mu}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2}\frac{\partial \bar{E}}{\partial t} \quad ; \quad \nabla^2 \bar{H} = \frac{\epsilon\mu}{c^2}\frac{\partial^2 \bar{H}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2}\frac{\partial \bar{H}}{\partial t}$$

FOR SIMPLE DIELECTRICS WITH $\mu = 1$ AND $\sigma = 0$ WE HAVE A WAVE EQUATION AS FOR THE VACUUM, EXCEPT THAT THE WAVE VELOCITY IS

$$v = \frac{c}{\sqrt{\epsilon}}$$

IN OPTICS ONE DEFINES THE INDEX OF REFRACTION OF A DIELECTRIC BY

$$v = \frac{c}{n}$$

$$\therefore \quad \underline{\underline{n = \sqrt{\epsilon}}}$$

SO THE INDEX OF REFRACTION BECOMES A SECONDARY CONCEPT IN MAXWELL'S VIEW.

HOWEVER WE WILL FIND IT NECESSARY TO ENLARGE OUR CONCEPTION OF THE DIELECTRIC CONSTANT. FROM OPTICS WE KNOW THAT THE INDEX OF REFRACTION IS NOT CONSTANT BUT DEPENDS ON THE WAVELENGTH (OR FREQUENCY) OF THE LIGHT. THUS WE MUST EXPECT THAT THE DIELECTRIC CONSTANT VARIES WITH FREQUENCY, AND THAT

$$n \neq \sqrt{\epsilon_{STATIC}}$$

FOR EXAMPLE $n_{WATER} \sim 1.33$ FOR VISIBLE LIGHT, BUT $\epsilon_{WATER} \approx 81$ ACCORDING TO ELECTROSTATIC MEASUREMENTS.

A MODEL OF THE FREQUENCY DEPENDENCE OF THE DIELECTRIC CONSTANT

TO EXPLAIN THE VARIATION OF THE INDEX OF REFRACTION WITH WAVELENGTH WE CONSIDER AN ATOMIC MODEL OF A DIELECTRIC MEDIUM - DUE TO LORENTZ.

ELECTRONS ARE TIED TO FIXED POINTS IN THE MEDIUM BY SPRINGS. THE NATURAL FREQUENCIES OF THE SPRINGS ARE THOSE OF THE SPECTRAL LINES OF THE MEDIUM. EMPIRICALLY THERE SEEMS TO BE SOME MECHANISM FOR CONVERTING KINETIC ENERGY OF THE ELECTRONS INTO OTHER FORMS OF INTERNAL ENERGY. WE SIMULATE THIS BY INCLUDING A VELOCITY DEPENDENT DAMPING TERM.

THE EQUATION OF MOTION OF AN ELECTRON IN THE PRESENCE OF AN ELECTRIC FIELD IS THEN

$$m\left(\ddot{\bar{x}} + \gamma\dot{\bar{x}} + \omega_0^2\bar{x}\right) = e\bar{E}.$$

ω_0 = NATURAL FREQUENCY $\qquad$ γ = DAMPING COEFFICIENT

FOR A PLANE WAVE $\bar{E} = \bar{E}_0\, e^{i(\bar{k}\cdot\bar{r} - \omega t)}$ AT SOME FIXED POINT:

$$\bar{E} = \bar{E}_0\, e^{-i\omega t} \qquad (\text{BECKER ASSUMES } e^{+i\omega t})$$

WE LOOK FOR ELECTRON MOTION OF THE FORM $\bar{x} = \bar{x}_0 e^{-i\omega t}$ ($\bar{x}_0$ COMPLEX)

THEN $\quad m\bar{x}_0\left(-\omega^2 - i\gamma\omega + \omega_0^2\right) = e\bar{E}$

$$\text{SO} \qquad \bar{x} = \frac{e\bar{E}}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

THE DIPOLE MOMENT OF THE ELECTRON DURING THIS MOTION IS $\bar{p} = e\bar{x}$

$$\text{SO} \qquad \bar{p} = \frac{e^2\bar{E}}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} = \alpha\bar{E} \qquad \left(\alpha = \text{MOLECULAR POLARIZABILITY p. 21}\right)$$

FOR LOW DENSITY MATERIALS WE FOUND

$$\epsilon = 1 + 4\pi N\alpha \qquad (N = \#/\text{VOLUME}) \qquad [\text{LORENZ - LORENTZ}]$$

$$\text{SO} \qquad \epsilon = 1 + \frac{4\pi Ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

IN REAL MATERIALS MANY DIFFERENT NATURAL FREQUENCIES ARE PRESENT. IF WE CALL THESE ω_i AND LABEL f_i AS THE FRACTIONAL POPULATION OF SPRING TYPE i ($f_i \equiv$ OSCILLATOR STRENGTH) THEN

$$\epsilon = 1 + \frac{4\pi Ne^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}, \qquad \sum_i f_i = 1.$$

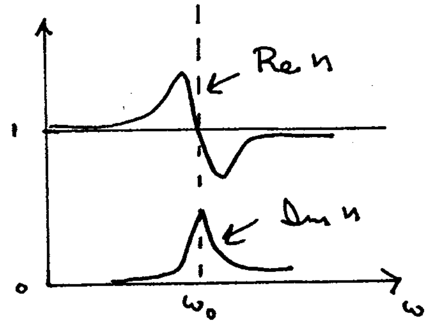
IN LECTURE 11 WE IDENTIFIED THE INDEX OF REFRACTION AS $n = \sqrt{\epsilon}$

IF ϵ IS NOT VERY DIFFERENT FROM 1

$$n = \sqrt{1 + \mathrm{Re}\,\chi + i\,\mathrm{Im}\,\chi} \approx 1 + 2\pi \mathrm{Re}\,\chi + 2\pi i\, \mathrm{Im}\,\chi$$

WHERE $\epsilon = 1 + 4\pi\chi$

AND $$\chi \sim \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$



WHICH LEADS TO $n(\omega)$ AS SKETCHED.

THIS IS IN QUALITATIVE AGREEMENT WITH REALITY IF MOST NATURAL FREQUENCIES OF MATERIALS ARE AT FREQUENCIES SLIGHTLY GREATER THAN OPTICAL FREQUECIES. THIS IS LARGELY TRUE....

$n = 1 + 2\pi \mathrm{Re}\,\chi \rightarrow$

$\alpha = 2\pi K \,\mathrm{Im}\,\chi \rightarrow$

THE INTENSITY VARIES AS $e^{-\alpha z}$

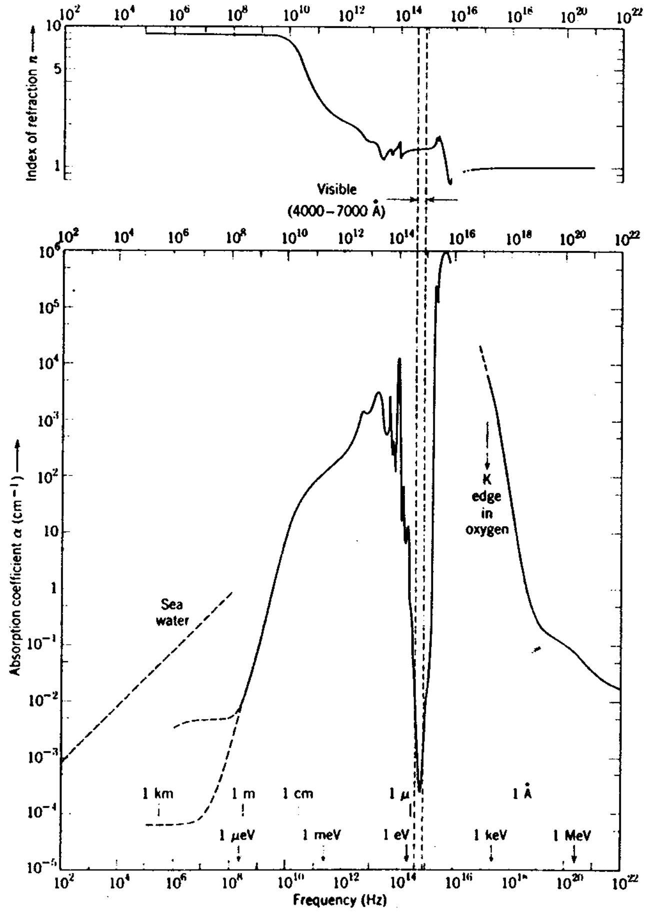


Fig. 7.9 The index of refraction (top) and absorption coefficient (bottom) for liquid water as a function of linear frequency. Also shown as abscissas are an energy scale (arrows) and a wavelength scale (vertical lines). The visible region of the frequency spectrum is indicated by the vertical dashed lines. The absorption coefficient for sea water is indicated by the dashed diagonal line at the left. Note that the scales are logarithmic in both directions.

THE BEHAVIOR OF THE INDEX OF REFRACTION NEAR A NATURAL FREQUENCY IS QUITE STRIKING.

THE INDEX DROPS WITH FREQUENCY NEAR ω_0 — THE RELATIVE BENDING (DISPERSION) OF COLOR BY A PRISM WOULD BE REVERSED — HENCE THE NAME <u>ANOMALOUS</u> <u>DISPERSION</u>.

ALSO MATERIALS ABSORB STRONGLY NEAR NATURAL FREQUENCIES. (ELEMENTS IN THE SUN'S CORONA CAN BE IDENTIFIED BY <u>DARK</u> LINES IN THE SPECTRUM DUE TO ABSORPTION)

THIS IS PREDICTED BY THE PRESENCE OF THE IMAGINARY PART OF n

$$\nabla^2 \bar{E} = \frac{\epsilon}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{AND} \quad \bar{E} = \bar{E}_0\, e^{i(kz-\omega t)} \implies k^2 = \frac{\epsilon\omega^2}{c^2} \quad \text{OR} \quad k = \frac{\omega n}{c}$$

$$\text{SO} \quad \bar{E} = E_0\, e^{-\frac{\mathrm{Im}(n)\omega}{c} z}\, e^{i\frac{\omega}{c}(\mathrm{Re}(n) z - ct)}$$

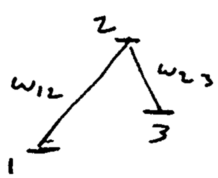
ABSORPTION

NEAR THE NATURAL FREQUENCY THE ELECTRON EXHIBITS RESONANT OSCILLATIONS — THE MOTION IS VERY LARGE, AND MUCH LIKE IN A CONDUCTOR. THE CHARGES MOVE SO AS TO CANCEL THE APPLIED FIELD....

NOTE THAT IF $\mathrm{Im}(n) < 0$ THEN THE WAVE WOULD <u>GROW</u> AS $e^{+|\mathrm{Im}(n)|\frac{\omega}{c}z}$. IT HAS PROVEN POSSIBLE TO PUT DIELECTRICS INTO UNUSUAL INTERNAL STATES IN WHICH THIS IS TRUE, LEADING TO THE <u>LASER</u> PHENOMENON!

IF AN ATOMIC LEVEL WITH FREQUENCY ω_i RELATIVE TO THE GROUND STATE IS "PUMPED" INTO AN INVERTED POPULATION, MUCH OF THE QUANTUM PHYSICS CAN BE APPROXIMATED BY A CLASSICAL MODEL IN WHICH THE OSCILLATOR STRENGTH f_i IS NEGATIVE.

A QUANTUM SITUATION LESS WELL ANTICIPATED BY CLASSICAL PHYSICS IS A 3-LEVEL SYSTEM WITH A "Λ" CONFIGURATION, IN WHICH NO TRANSITION $1 \leftrightarrow 3$ IS POSSIBLE.



THIS CONFIGURATION WAS THE BASIS OF THE RECENT OBSERVATION OF "SLOW LIGHT", HAU ET AL., NATURE <u>397</u>, 594 (1999), IN WHICH THE TRANSITION $2 \leftrightarrow 3$ WAS PUMPED BY A LASER. THIS DOES NOT, HOWEVER, RESULT IN AN INVERTED POPULATION, SINCE LEVEL 2 IS NOT THE GROUND STATE.

FOR A CLASSICAL MODEL OF THIS, SET $f_{23} > 0$, BUT TAKE $\gamma_{23} < 0$. SEE http://arxiv.org/abs/physics/0007097

REFLECTION AND REFRACTION AT A DIELECTRIC BOUNDARY

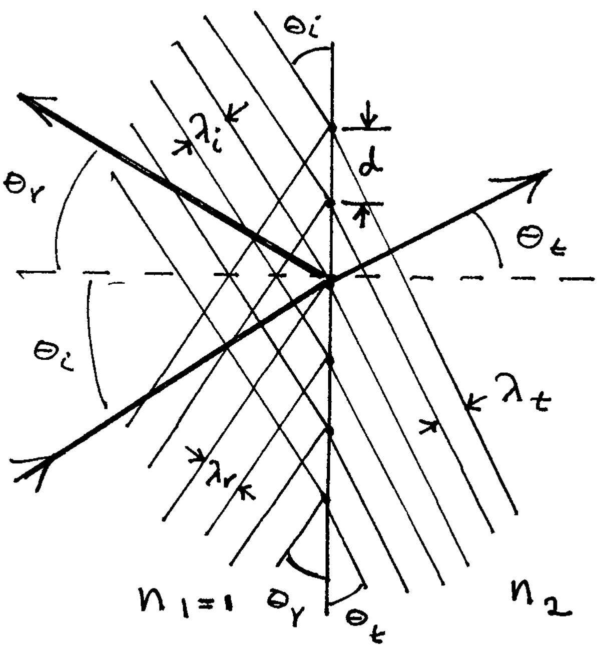
OUR STUDY OF ELECTROMAGNETISM HAS BEEN AN EXTRAPOLATION FROM ELECTROSTATIC AND MAGNETOSTATICS VIA FARADAY'S LAW AND MAXWELL'S DISPLACEMENT CURRENT. WE FIND WAVE PHENOMENA WHICH PROPAGATE WITH THE VELOCITY OF LIGHT, AND SURELY MUST BE LIGHT!

HISTORICALLY STUDIES INTO THE NATURE OF LIGHT WERE SEPARATE FROM THOSE OF ELECTRICITY AND MAGNETISM UNTIL MAXWELL. SOME OF THE MOST DETAILED INVESTIGATIONS ON LIGHT WERE DUE TO FRESNEL (~1825). FOLLOWING A SUGGESTION OF YOUNG THAT LIGHT WAVES ARE TRANSVERSELY, BUT NOT LONGITUDINALLY POLARIZED, HE DEMONSTRATED THAT THE COMPLEX LAWS OF REFLECTION AND REFRACTION FOLLOW - ASSUMING LIGHT TO BE A VIBRATION OF AN ELASTIC SOLID - THE ETHER.

A DETAILED VERIFICATION OF FRESNEL'S LAW FROM MAXWELL'S EQUATIONS REPRESENTED STRONG CONFIRMATION THAT ELECTROMAGNETIC WAVES ARE INDEED LIGHT WAVES.

SOME 'ELEMENTARY' RESULTS

PLANE WAVES OF WAVELENGTH λ_i TRAVEL IN VACUUM (INDEX $n_1 = 1$). THE WAVES ARE INCIDENT ON THE BOUNDARY BETWEEN A SECOND MEDIUM WITH INDEX n_2. THE ANGLE OF INCIDENCE Θ_i IS MEASURED BETWEEN THE NORMAL TO THE BOUNDARY AND THE DIRECTION OF WAVE PROPAGATION



AS THIS WAVE ENTERS MEDIUM 2 IT SHAKES THE ELECTRONS WHICH CREATES NEW FIELDS ⇒ NEW WAVES

THE NEW WAVES ENTER THE VACUUM AND BECOME THE REFLECTED WAVE. IN MEDIUM 2 THE NEW WAVES COMBINE WITH THE ORIGINAL WAVES TO BECOME THE TRANSMITTED WAVE. DUE TO THIS INTERFERENCE BETWEEN THE ORIGINAL AND NEW WAVES THE TRANSMITTED WAVE VELOCITY BECOMES $v_2 = \frac{c}{n_2}$.

FOR A STEADY STATE SITUATION ALL WAVES MUST HAVE THE SAME FREQUENCY.

THEN $\lambda_2 = \frac{2\pi v_2}{\omega} = \frac{2\pi c}{n_2 \omega}$ WHILE $\lambda_1 = \frac{2\pi v_1}{\omega} = \frac{2\pi c}{\omega}$ $\Rightarrow \lambda_2 = \frac{\lambda_1}{n_2}$

AT THE BOUNDARY THE CRESTS OF THE INCIDENT WAVE MUST MATCH WITH THE CRESTS (OR PERHAPS TROUGHS) OF THE REFLECTED AND TRANSMITTED WAVES.

HENCE $$d = \frac{\lambda_1}{\sin\theta_i} = \frac{\lambda_1}{\sin\theta_r} = \frac{\lambda_2}{\sin\theta_t}$$

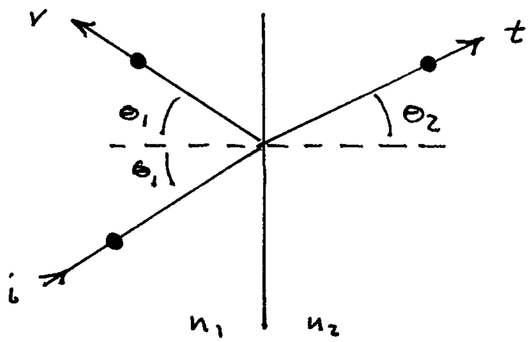
$\Rightarrow \theta_r = \theta_i \Rightarrow$ MIRROR REFLECTION!

AND $\sin\theta_i = n_2 \sin\theta_t \Longleftrightarrow$ <u>SNELL'S LAW</u>

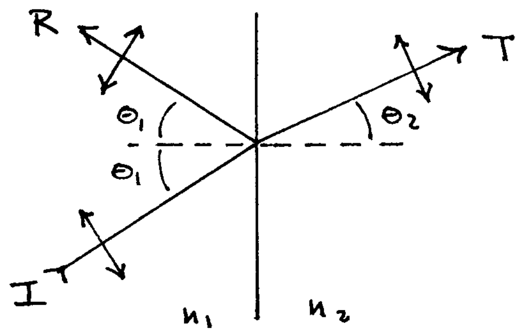
[IF $n_1 \neq 1$, THEN $n_1 \sin\theta_i = n_2 \sin\theta_t$]

WHAT CAN WE SAY ABOUT THE <u>INTENSITIES</u>?

WE MUST DISTINGUISH BETWEEN THE TWO POSSIBLE POLARIZATIONS OF THE INCIDENT WAVE



$\bar{E}$ POLARIZED <u>PERPENDICULAR</u> TO THE PLANE CONTAINING THE NORMALS.

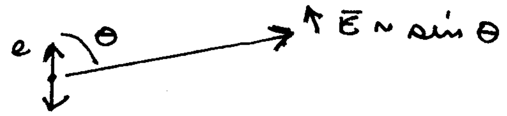


$\bar{E}$ POLARIZED <u>PARALLEL</u> TO THE PLANE CONTAINING THE NORMALS

THE INTENSITY OF THE WAVES IS GIVEN BY THE POYNTING VECTOR $\bar{S} \sim E^2$

LABEL THE SIZES OF THE VARIOUS FIELDS SUCH THAT $S_i = i^2$ $\quad S_T = T^2$ ETC.

TO PROCEED BY OUR 'SIMPLE' ARGUMENT WE NEED ONE FACT NOT YET PROVEN: IF ELECTRONS OSCILLATE ALONG SOME DIRECTION THEY CREATE WAVE FIELDS WITH ELECTRIC FIELD $\bar{E}$ PROPORTIONAL TO THE PROJECTION OF THE OSCILLATION ONTO THE WAVE DIRECTION



WE WILL DEMONSTRATE THIS IN LECTURE 15

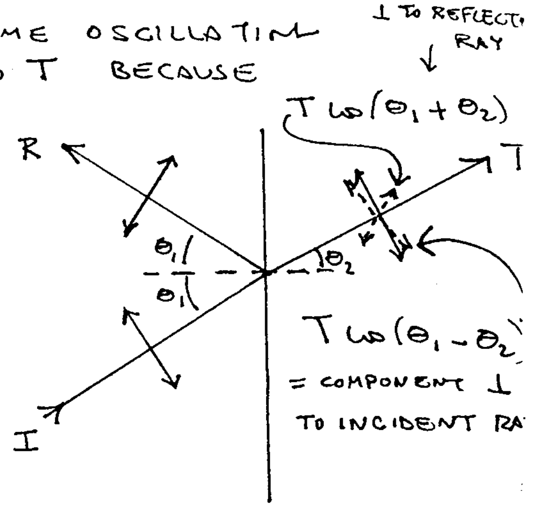
THUS IF THE INCIDENT WAVE IS POLARIZED $\perp$ TO THE SCATTERING PLANE, SO WILL BE THE REFLECTED AND TRANSMITTED WAVES, AS SHOWN ABOVE.

NOTE THAT IF $\theta_1 + \theta_2 = 90°$ THEN THE REFLECTED WAVE R (∥ POL.) WILL VANISH, SINCE IT IS ALONG THE DIRECTION OF OSCILLATION OF THE CHARGES IN MEDIUM 2, WHICH CAUSE THE WAVE (BREWSTER ANGLE EFFECT)

NOW WAVES r AND t ARE CAUSED BY THE SAME OSCILLATING CHARGES, SO WE EXPECT

$$\frac{r}{t} = \alpha = \text{GEOMETRICAL FACTOR.}$$

LIKEWISE R AND T ARE CAUSED BY THE SAME OSCILLATING CHARGES, BUT R IS REDUCED RELATIVE TO T BECAUSE OF OUR FACT ABOUT RADIATED WAVES



HENCE $$\frac{R}{T\cos(\theta_1+\theta_2)} = \alpha = \frac{r}{t}$$

MEANWHILE IN MEDIUM 2, THE INCIDENT WAVE IS CANCELLED BY THE RADIATION OF THE OSCILLATING CHARGES. IF WE SAY THAT

$$\frac{t}{i} = -\beta = \text{GEOMETRICAL FACTOR}$$

THEN $$\frac{T\cos(\theta_1-\theta_2)}{I} = -\beta$$ SINCE ONLY PART OF T CONTRIBUTES TO THE CANCELLATION OF I IN CASE OF PARALLEL POLARIZATION.

FOR COMPARISON, WE DEFINE $i = I = 1$

THEN $t = T\cos(\theta_1-\theta_2)$

AND $$\frac{R}{r} = \frac{T\cos(\theta_1+\theta_2)}{t} = \frac{\cos(\theta_1+\theta_2)}{\cos(\theta_1-\theta_2)}$$

ALSO, TOTAL ENERGY MUST BE CONSERVED:

$$i^2 = 1 = r^2 + t^2 \quad \text{AND} \quad I^2 = 1 = R^2 + T^2$$

THUS $$\frac{t^2}{T^2} = \cos^2(\theta_1-\theta_2) = \frac{1-r^2}{1-R^2}$$

OR $$1 - r^2 = \cos^2(\theta_1-\theta_2)\left(1 - r^2\frac{\cos^2(\theta_1+\theta_2)}{\cos^2(\theta_1-\theta_2)}\right)$$

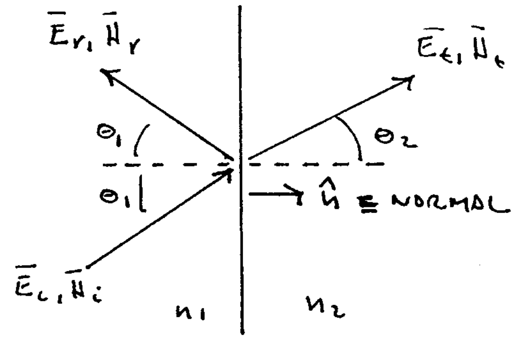
SO $$r^2 = \frac{1-\cos^2(\theta_1-\theta_2)}{1-\cos^2(\theta_1+\theta_2)} = \frac{\sin^2(\theta_1-\theta_2)}{\sin^2(\theta_1+\theta_2)} \Rightarrow t^2 = 1 - r^2 = \frac{\sin 2\theta_1 \sin 2\theta_2}{\sin^2(\theta_1+\theta_2)}$$

$$R^2 = \frac{\tan^2(\theta_1-\theta_2)}{\tan^2(\theta_1+\theta_2)}; \quad T = \frac{t}{\cos(\theta_1-\theta_2)} \Rightarrow T^2 = 1 - R^2 = \frac{\sin 2\theta_1 \sin 2\theta_2}{\sin^2(\theta_1+\theta_2)\cos^2(\theta_1-\theta_2)}$$

THESE LAST RESULTS ARE CALLED <u>FRESNEL'S EQUATIONS</u> (FOR INTENSITIES).

Fresnel's Equations as a Boundary Value Problem

The above results can also be derived without explicit mention of the motion of charges, by noting the boundary conditions on $\bar{E}$ and $\bar{H}$. (Helmholtz)



Each of the 3 plane waves is of the form $\bar{E} = \bar{E}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)}$

$i\bar{k}\times\bar{H} = -\frac{i\omega}{c}\bar{D} \Rightarrow \bar{H} = \sqrt{\epsilon}\,\hat{k}\times\bar{E} = n\,\hat{k}\times\bar{E}$

(for media with $\mu = 1$) noting $k = \sqrt{\epsilon}\frac{\omega}{c}$ and $\bar{D} = \epsilon\bar{E}$.

At the boundary there are no free charges, so $\bar{\nabla}\cdot\bar{D} = 0$, $\bar{\nabla}\cdot\bar{B} = 0$

$\Rightarrow D_\perp, B_\perp$ continuous

We use $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$ and $\bar{\nabla}\times\bar{H} = \frac{4\pi}{c}\bar{J}_{free} + \frac{1}{c}\frac{\partial\bar{D}}{\partial t}$ with a narrow loop straddling the boundary. For a loop of vanishing surface area, we see that

Loop →

$E_\parallel$, $H_\parallel$ are continuous across the boundary.

With $\hat{n}$ = normal to the boundary, we have ($\mu = 1$)

$$D_\perp \Rightarrow \quad n_1^2(\bar{E}_{0i} + \bar{E}_{0r})\cdot\hat{n} = n_2^2\,\bar{E}_{0t}\cdot\hat{n}$$

$$B_\perp \Rightarrow \quad n_1(\hat{k}_i\times\bar{E}_{0i} + \hat{k}_r\times\bar{E}_{0r})\cdot\hat{n} = n_2(\hat{k}_t\times\bar{E}_{0t})\cdot\hat{n}$$

$$E_\parallel \Rightarrow \quad (\bar{E}_{0i} + \bar{E}_{0r})\times\hat{n} = \bar{E}_{0t}\times\hat{n}$$

$$H_\parallel \Rightarrow \quad n_1(\hat{k}_i\times\bar{E}_{0i} + \hat{k}_r\times\bar{E}_{0r})\times\hat{n} = n_2(\hat{k}_t\times\bar{E}_{0t})\times\hat{n}$$

which is surely enough relations.

But we must also have $e^{i(\bar{k}\cdot\bar{r}-\omega t)}$ the same for all waves at the boundary,

$\Rightarrow \bar{k}_i\cdot\bar{r} = \bar{k}_r\cdot\bar{r} = \bar{k}_t\cdot\bar{r} \Rightarrow n_i\sin\theta_i = n_i\sin\theta_r = n_t\sin\theta_t$ using $k = n\omega/c$. (Snell)

We leave it to the problem set to show that this leads to

$$\frac{E_{0r}}{E_{0i}} = -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}\ ; \quad \frac{E_{0t}}{E_{0i}} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)}$$

for $\bar{E}$ polarized perp. to the plane of $\hat{k}_i, \hat{k}_r$ & $\hat{k}_t$

and

$$\frac{E_{0r}}{E_{0i}} = \frac{-\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}\ ; \quad \frac{E_{0t}}{E_{0i}} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}$$

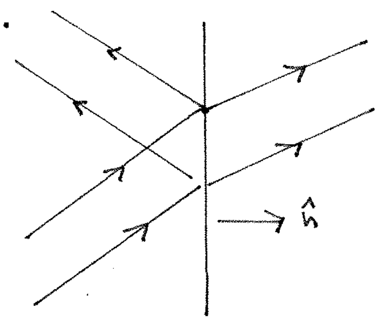
for $\bar{E}$ polarized parallel to the plane.

THE INTENSITIES OF THE WAVES ARE DESCRIBED BY THE POYNTING VECTOR

$$\bar{S} = \frac{c}{4\pi}\,\bar{E}\times\bar{H}$$ = ENERGY FLOWING ACROSS UNIT AREA PER SECOND

$$\langle \bar{S} \rangle = \frac{c}{4\pi}\cdot\frac{1}{2}\,\mathrm{Re}(\bar{E}\times\bar{H}^*) = \frac{c}{8\pi}\,\bar{E}_0 \times (n\hat{k}\times E_0) = \frac{nc}{8\pi}E_0^2\,\hat{k}$$

AT THE BOUNDARY WE EXPECT CONSERVATION OF ENERGY: AS MUCH ENERGY FLOWS INTO A GIVEN AREA AS FLOWS OUT.



NOTE THAT THE AREA TRANSVERSE TO THE WAVE WHICH FLOWS INTO UNIT AREA ON THE BOUNDARY IS ONLY $\cos\Theta$.

SO IN TERMS OF THE POYNTING VECTOR

$$\bar{S}_i\cdot\hat{n} = \bar{S}_r\cdot\hat{n} + \bar{S}_t\cdot\hat{n}$$

EXPRESSES CONSERVATION OF ENERGY

THIS MOTIVATES THE DEFINITION OF THE COEFFICIENTS OF REFLECTED AND TRANSMITTED INTENSITIES

$$\mathcal{R} \equiv \left|\frac{\bar{S}_r\cdot\hat{n}}{\bar{S}_i\cdot\hat{n}}\right| = \frac{E_{0r}^2}{E_{0i}^2} \quad \text{AND} \quad \mathcal{T} \equiv \left|\frac{\bar{S}_t\cdot\hat{n}}{\bar{S}_i\cdot\hat{n}}\right| = \frac{n_2\cos\theta_2\,E_{0t}^2}{n_1\cos\theta_1\,E_{0i}^2}$$

WHICH LEAD TO THE FRESNEL EQUATIONS FOUND ON p 142

(WITH A CHANGE OF NOTATION: COEF $\mathcal{R}$ = r^2 OR R^2 OF p142...)

NOTE $\mathcal{R} + \mathcal{T} = 1$ ALWAYS

[SOME PEOPLE DEFINE $\mathcal{R} = \frac{|\bar{S}_r|}{|\bar{S}_i|}$ AND $\mathcal{T} = \frac{|\bar{S}_t|}{|\bar{S}_i|}$ WHICH LEADS TO SLIGHTLY DIFFERENT EXPRESSIONS FOR $\mathcal{T}$ BUT NOT $\mathcal{R}$]

SOME GRAPHS OF $\mathcal{R}$ AND $\mathcal{T}$ ARE APPENDED TO THE NOTES.

AT NORMAL INCIDENCE, WE FIND $\mathcal{R} = \dfrac{\left(\frac{n_2}{n_1}-1\right)^2}{\left(\frac{n_2}{n_1}+1\right)^2}$; $\mathcal{T} = \dfrac{4\,n_2/n_1}{\left(\frac{n_2}{n_1}+1\right)^2}$

[$\sin(\theta_1-\theta_2) \rightarrow \sin\theta_1 - \sin\theta_2 = \left(\frac{n_2}{n_1}-1\right)\sin\theta_2$ VIA SNELL, ETC]

AN ADDITIONAL DETAIL CAN BE NOTED BY REFERRING TO THE EXPRESSIONS FOR E_{0r}/E_{0i} ON p 143.

UNDER MANY CIRCUMSTANCES $E_{0r}/E_{0i} < 0$. WE INTERPRET THIS AS A 180° PHASE CHANGE OF THE REFLECTED WAVE.

PH 206 LECTURE 12

BY SNELL'S LAW $n_1 \sin\theta_1 = n_2 \sin\theta_2$

SO IF $n_1 < n_2$ THEN $\theta_1 > \theta_2$

ALSO NOTE THAT IF $\theta_1 + \theta_2 > 90°$ $\tan(\theta_1+\theta_2) < 0$ ADDING FURTHER COMPLICATION. AT $\theta_1 = \theta_{BREWSTER}$, $\theta_1 + \theta_2 = 90°$

WE MAY NOW SUMMARIZE:

TRANSMITTED WAVE : NO PHASE CHANGE

REFLECTED WAVE

1) $\bar{E} \perp$ TO PLANE

$n_1 < n_2 \Rightarrow 180°$ CHANGE

$n_2 < n_1 \Rightarrow$ NO CHANGE

2) $\bar{E} \parallel$ TO PLANE

$n_1 < n_2 \begin{cases} \theta_1 < \theta_B \Rightarrow 180° \text{ CHANGE} \\ \theta_1 > \theta_B \Rightarrow \text{NO CHANGE} \end{cases}$

$n_2 < n_1 \begin{cases} \theta_1 < \theta_B \Rightarrow \text{NO CHANGE} \\ \theta_1 > \theta_B \Rightarrow 180° \text{ CHANGE} \end{cases}$

TOTAL INTERNAL REFLECTION

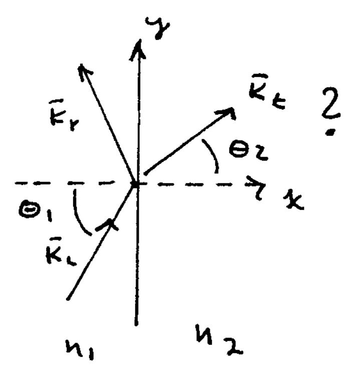
AN INTERESTING CASE REMAINS TO BE DISCUSSED. SUPPOSE $n_1 > n_2$. THEN SNELL'S LAW SAYS

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 \quad \text{WHICH IS} > 1 \text{ IF } \sin\theta_1 > \frac{n_2}{n_1}$$

WE CONCLUDE THAT THERE CAN BE NO TRANSMITTED WAVE IN THE SENSE OF OUR PREVIOUS ANALYSIS. THIS EFFECT IS CALLED TOTAL INTERNAL REFLECTION - IN THAT $n_1 > n_2$ OCCURS FOR WAVES ORIGINATING INSIDE A DENSE MEDIUM.

BUT DO WE REALLY EXPECT ABSOLUTELY NO FIELDS INSIDE MEDIUM 2 IN THIS CASE?

WE LOOK FOR A TRANSMITTED WAVE $\bar{E}_t = \bar{E}_{0t}\, e^{i(\bar{k}_t\cdot\bar{r} - \omega t)}$



WE WOULD EXPECT $\bar{k}_t = n_2 \frac{\omega}{c}\hat{k}_t$

AND SO $\bar{k}_t\cdot\bar{r} = \frac{n_2\omega}{c}\left(x\cos\theta_2 + y\sin\theta_2\right)$

WITH $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 \Rightarrow \cos\theta_2 = \sqrt{1-\sin^2\theta_2}$

$$= \frac{1}{n_2}\sqrt{n_2^2 - n_1^2\sin^2\theta_1}$$

$$= \frac{i}{n_2}\sqrt{n_1^2\sin^2\theta_1 - n_2^2} \quad \text{IF } \theta_1 \text{ BIG ENOUGH!}$$

Thus $\bar{k}_t \cdot \bar{r} = \frac{\omega}{c}\left(i\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\, x + n_1 \sin\theta_1\, y\right)$

And $\bar{E}_t = \bar{E}_{0t}\, e^{-\frac{\omega}{c}\sqrt{\quad}\, x}\, e^{i\omega\left(\frac{n_1 \sin\theta_1 y}{c} - t\right)}$

which is a wave which runs parallel to the boundary in medium 2 with velocity $v = \frac{c}{n_1 \sin\theta_1}$, but it barely penetrates due to the damping $e^{-\frac{\omega}{c}\sqrt{\quad}\, x}$.

[Can also find $\bar{H}_t$ to show that $\bar{S} \parallel \hat{y}$ .....]

The phase change of the reflected wave is quite complicated in the case of total internal reflection.

If we return to our expressions for E_{0r}/E_{0i} on p 143 and replace θ_2 in favor of θ_1 and n_1, n_2 via Snell's law:

$$\frac{E_{0r}}{E_{0i}} = \frac{\cos\theta_1 - i\sqrt{\sin^2\theta_1 - (n_2/n_1)^2}}{\cos\theta_1 + i\sqrt{\quad}}$$ for $\bar{E}$ perp to plane

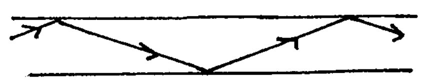
and $$\frac{E_{0r}}{E_{0i}} = \frac{\cos\theta_1 - i\left(\frac{n_1}{n_2}\right)^2\sqrt{\sin^2\theta_1 - (n_2/n_1)^2}}{\cos\theta_1 + i\left(\frac{n_1}{n_2}\right)^2\sqrt{\quad}}$$ for $\bar{E}$ parallel to the plane

In both cases $\left|\frac{E_{0r}}{E_{0i}}\right| = 1 \Rightarrow$ energy totally reflected.

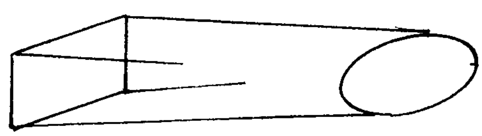
But the complex phase change is different in the two cases.

On the problem set you can consider how to use these facts to convert linearly polarized light into circularly polarized light by a double internal reflection....

Example Light Pipes. Suppose we have light inside a cylindrical or rectangular dielectric tube.



If the light is totally internally reflected at one surface, it will be also at the other. The light is trapped! — and it propagates without losses due to reflections. This is much better than bouncing light off metal surfaces where Joule heating losses always occur and consume a few % per bounce.



If the shape of the tube changes slowly compared to the distance between bounces, the condition for total internal reflection can be maintained. Thus we can 'pipe' light from one aperture to another, and even around bends of large radius ....

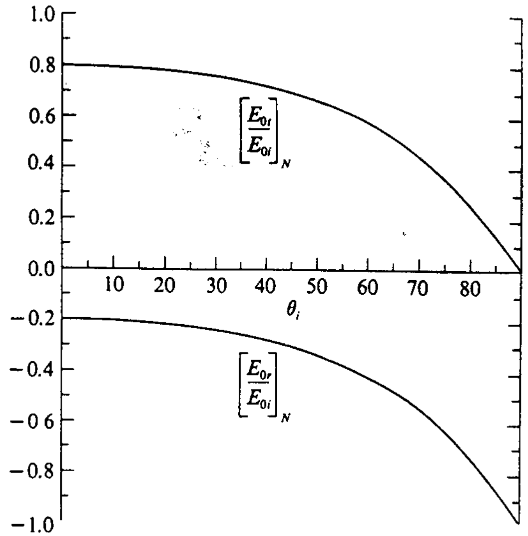


Figure 12-7. The ratios $(E_{0r}/E_{0i})_N$ and $(E_{0t}/_{0i})_N$ as functions of the angle of incidence θ_i for $n_1/n_2 = 1/1.5$. This corresponds to light incident in air on a glass with $n = 1.5$. The wave is polarized with its $\boldsymbol{E}$ vector *normal* to the plane of incidence.

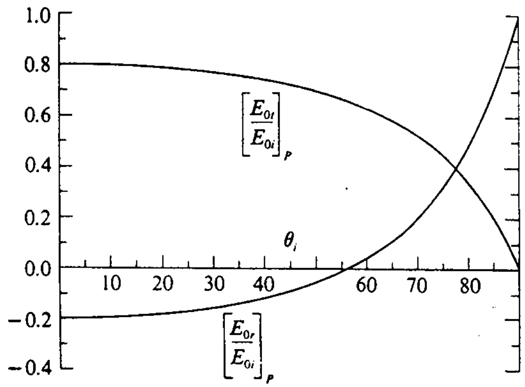


Figure 12-8. The ratios $(E_{0r}/E_{0i})_P$ and $(E_{0t}/_{0i})_P$ as functions of the angle of incidence θ_i for $n_1/n_2 = 1/1.5$. This corresponds to light incident in air on a glass with $n = 1.5$. The wave is polarized with its $\boldsymbol{E}$ vector *parallel* to the plane of incidence.

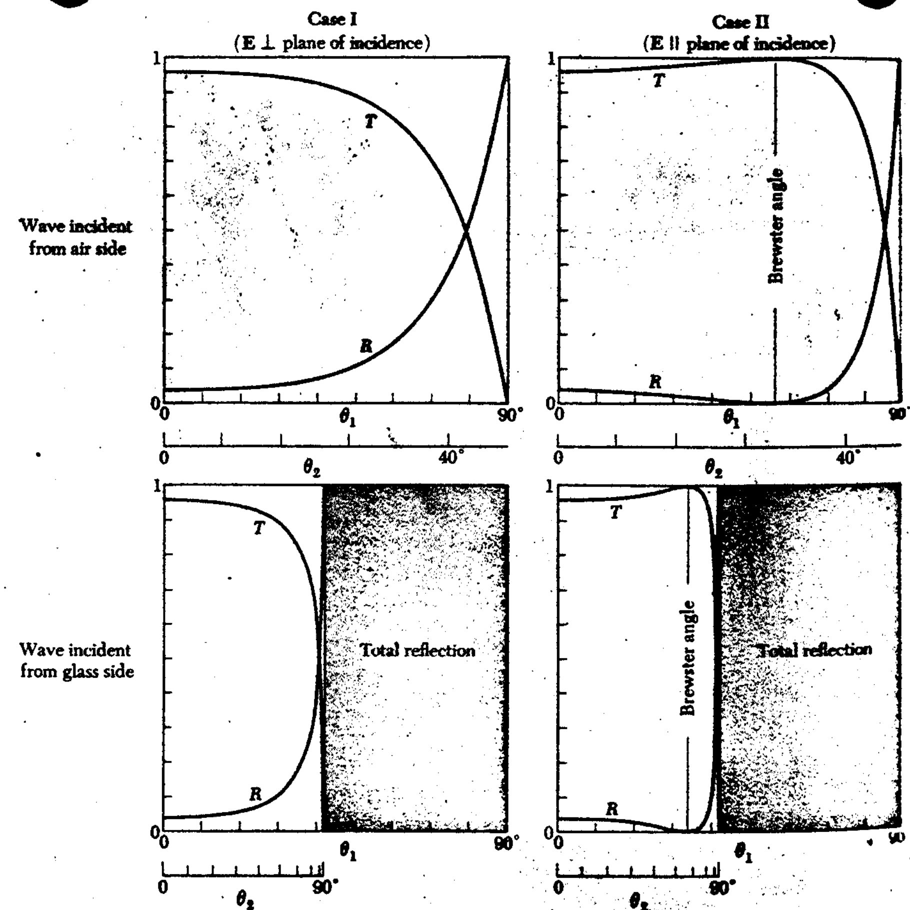


Fig. 8.6.4 Power reflection and transmission coefficients at an air-glass interface; $(\kappa_e)_{glass}/(\kappa_e)_{air} = 2.25$.

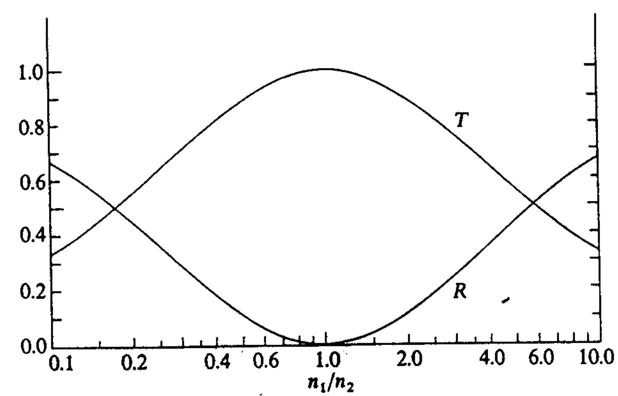


Figure 12-12. The coefficient of reflection R and the coefficient of transmission T at normal incidence as functions of the ratio n_1/n_2.

Ph 206 Lecture 12

Phase Velocity and Group Velocity (Becker p 264-265)

(This section is a review of material covered in Lecture 23, Ph 205).

We have seen how plane waves of a definite frequency ω can propagate in a dielectric medium according to

$$E_0 e^{i(kz-\omega t)} \quad \text{for example.}$$

A given value of the field, say $E_0 e^{i\phi}$ where $\phi = kz - \omega t$ = phase, propagates with velocity $v = \frac{\omega}{k}$

This is the so-called phase velocity. We could relate it to the index of refraction: $v_p = \frac{c}{n}$ (or better, $\frac{c}{Re(n)}$)

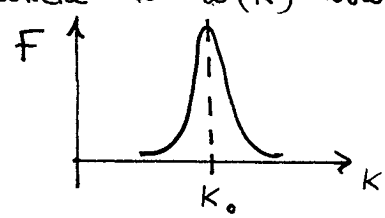
For frequencies slightly above a natural frequency of a medium, $Re(n) < 1 \Rightarrow v_p > c$!

This is no real violation of the theory of relativity, as a wave of pure frequency extends over all space and time, and so is a somewhat fictitious entity. In particular such a wave cannot be used to send a message.

A "wave" containing a message (such as the information as to when the wave was created) must involve some 'modulation' of the pure frequency - as in radio signal transmission.

A modulated wave can be thought of as the superposition of waves of several different pure frequencies - in the sense of the Fourier analysis

$$f(z,t) = \int F(k)\, e^{i(kz-\omega t)}\, dk \qquad \text{where } \omega = \omega(k) \text{ now,}$$



Consider a case where $F(k)$ is non-zero only near a 'central' frequency $\omega_0 = \omega(k_0)$

The 'spectral function' $F(k)$ can be related to the wave $f(z,t)$ by the usual Fourier inversion, at $t=0$.

$$F(k) = \frac{1}{2\pi}\int f(z,0)\, e^{-ikz}\, dz.$$

BECAUSE THE SPECTRAL FUNCTION F IS NARROW, IT IS A GOOD APPROXIMATION TO KEEP ONLY THE FIRST TWO TERMS IN A TAYLOR EXPANSION OF ω ABOUT K_0:

$$\omega \approx \omega_0 + \frac{d\omega}{dk}\Big|_0 (k-k_0)$$

$$\text{so } f(z,t) = \underbrace{e^{-i\left(\omega_0 - k_0 \frac{d\omega}{dk}\big|_0\right)t}}_{\text{PHASE FACTOR}} \underbrace{\int F(k)\, e^{ik\left(z - \frac{d\omega}{dk}\big|_0 t\right)} dk}_{f\left(z - \frac{d\omega}{dk}\big|_0 t,\ 0\right)}$$

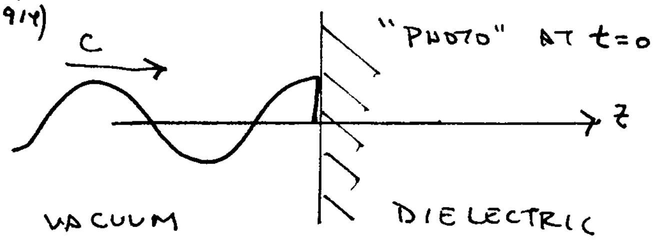
THUS, TO WITHIN A PHASE FACTOR OF MODULUS 1, THE WAVE $f(z,t)$ IS THE SAME AS THE WAVE $f(z,0)$, BUT DISPLACED BY AMOUNT $\frac{d\omega}{dk}\big|_0 t$. HENCE THE 'MESSAGE' CONTAINED IN THE SHAPE $f(z,0)$ PROPAGATES WITH VELOCITY $\underline{\underline{\frac{d\omega}{dK}}}$

THIS IS CALLED THE <u>GROUP VELOCITY</u> = $\underline{v_g} = \frac{d\omega}{dK}$ (RAYLEIGH) OR SOMETIMES THE SIGNAL VELOCITY.

WE FURTHER ILLUSTRATE THE IDEA OF SIGNAL VELOCITY WITH A SIMPLE BUT NON-TRIVIAL EXAMPLE DUE TO SOMMERFELD & BRILLOUIN (1914)

<u>EXAMPLE</u> WAVE FRONT INCIDENT ON A DIELECTRIC MEDIUM.
(HAMILTON, 1839, SOMMERFELD & BRILLOUIN, 1914)

A WAVE OF 'PURE' FREQUENCY ω_0, BUT WITH A SHARPLY DEFINED WAVEFRONT ARRIVES AT THE DIELECTRIC BOUNDARY ($z=0$) AT TIME $t=0$.



IT PROVES TO BE MORE CONVENIENT TO DO THE FOURIER ANALYSIS IN FREQUENCY, RATHER THAN WAVE NUMBER AS ABOVE.

AT $z=0$, THE <u>INCIDENT</u> WAVE HAS THE FORM

$$f(t) = \begin{cases} 0 & t<0 \\ e^{-i\omega_0 t} & t>0 \end{cases}$$

WE CAN FOURIER ANALYSE THIS: $f(t) = \int F(\omega)\, e^{-i\omega t}\, d\omega$

SINCE $f(t)$ IS NOT $e^{-i\omega_0 t}$ FOREVER, $F(\omega)$ IS NOT JUST $\delta(\omega-\omega_0)$

FORMALLY $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt = \frac{1}{2\pi}\int_0^{\infty} e^{i(\omega-\omega_0)t}\, dt$

AND $F(\omega)$ IS SIGNIFICANTLY DIFFERENT FROM ZERO ONLY FOR $\omega \sim \omega_0$

WE EXPECT THE <u>TRANSMITTED</u> WAVE MAY BE FOURIER ANALYSED AS $f(z,t) = \int G(\omega)\, e^{i(kz-\omega t)}\, d\omega$ WITH $\omega = \omega(k)$ KNOWN FROM PROPERTIES OF THE MEDIUM: $v_p = \frac{\omega}{k} = \frac{c}{n} \Rightarrow \omega = \frac{ck}{n}$

IF THERE WERE NO REFLECTION AT THE BOUNDARY WE COULD JUST IDENTIFY $G(\omega)$ WITH $F(\omega)$ OF THE INCIDENT WAVE. THE REFLECTION COEFICIENT IS SOMEWHAT FREQUENCY DEPENDENT SINCE THE INDEX n VARIES WITH FREQUENCY (P 144)

BUT WE CAN SAFELY SURMISE $G(\omega) = T(\omega)\, F(\omega)$ WHERE $T(\omega)$ VARIES SLOWLY.

$$f(z,t)\Big|_{z>0} = \int_{-\infty}^{\infty} d\omega\, T(\omega)\, e^{i(kz-\omega t)} \underbrace{\int_0^{\infty} \frac{e^{i(\omega-\omega_0)t'}}{2\pi}\, dt'}_{F(\omega)}$$

SINCE $F(\omega) \neq 0$ ONLY FOR $\omega \sim \omega_0$ WE CAN EVALUATE THIS INTEGRAL APPROXIMATELY BY EXPANDING $T(\omega)\, e^{i(kz-\omega t)}$ ABOUT ω_0

$T(\omega) \sim T(\omega_0)$ SINCE IT VARIES SLOWLY. WE LEARN MORE BY GOING TO 2ND ORDER:

$$k \sim k_0 + \frac{dk}{d\omega}\Big|_0 (\omega-\omega_0) + \frac{1}{2}\frac{d^2k}{d\omega^2}\Big|_0 (\omega-\omega_0)^2 = k_0 + \frac{\omega-\omega_0}{v_g} - \frac{v_g'}{2v_g^2}(\omega-\omega_0)^2$$

USING $\frac{dk}{d\omega}\Big|_0 = 1\Big/\frac{d\omega}{dk}\Big|_0 = \frac{1}{v_g}$ AND $\frac{d^2k}{d\omega^2}\Big|_0 = -\frac{\frac{d^2\omega}{dk^2}\Big|_0}{(d\omega/dk|_0)^2} \equiv -\frac{v_g'}{v_g^2}$

DEFINING $s = \omega - \omega_0$ WE HAVE

$$f(z,t)\Big|_{z>0} \approx \frac{T(\omega_0)}{2\pi} \underbrace{e^{i(k_0 z - \omega_0 t)}}_{\text{CARRIER WAVE}} \underbrace{\int_0^{\infty} dt' \int_{-\infty}^{\infty} ds\, e^{is\left(t'-t+\frac{z}{v_g}\right) - \frac{i s^2 z v_g'}{2v_g^2}}}_{\text{MODULATION FACTOR}}$$

THE 'CARRIER WAVE' $e^{ik_0z-\omega_0 t}$ HAS PHASE VELOCITY $v_p = \frac{\omega_0}{k_0}$ WHICH MIGHT BE GREATER THAN C. BUT THE 'MESSAGE' THAT A SHARP WAVE FRONT HIT THE BOUNDARY AT $t=0$ IS CONTAINED IN THE 'MODULATION FACTOR'.

Ph 206 Lecture 12

We can evaluate the integrals in the modulation factor, M:

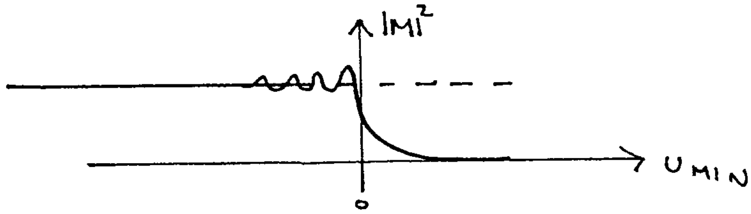
Complete the square in the expression for the exponent –

$$M = \int_0^{\infty} dt' \, e^{\frac{i\left[z+v_g(t'-t)\right]^2}{2 z v_g'}} \underbrace{\int_{-\infty}^{\infty} ds \, e^{-\frac{i z v_g'}{2 v_g^2}\left(s - \frac{z+v_g(t'-t)}{z v_g'/v_g}\right)^2}}_{\sim \sqrt{\frac{2 v_g^2}{z v_g'}} \times \text{constant}}$$

Finally define $u = \dfrac{z + v_g(t'-t)}{\sqrt{2 z v_g'}}$ so $du = \dfrac{v_g}{\sqrt{2 z v_g'}} \, dt'$

and $M = \text{constant} \cdot \displaystyle\int_{u_{min}}^{\infty} e^{i u^2} du$ with $u_{min} = \dfrac{z - v_g t}{\sqrt{2 z v_g'}}$

The remaining integral is a so-called Fresnel Integral and can be found in tables to have the form



The wave intensity, which varies as $|M|^2$, is non-zero primarily for $u_{min} < 0 \Rightarrow z < v_g t$

Thus the wave front propagates inside the dielectric at the group velocity not the phase velocity !

Further, the 'wave-front' becomes less sharp for large z, due to the dispersion of waves of different frequencies inside the dielectric.

In this analysis we emphasized frequencies $\omega \sim \omega_0$ but very high frequencies are present to a small extent. As on p 138, as $\omega \to \infty$, $\epsilon \to 1 \Rightarrow v_{phase} \to c$. It turns out that the high frequency waves add up to small transient pulses which propagate with velocity c inside the dielectric. These are the so-called precursors, which have been observed experimentally. Even in a metal in which the main wave dies out, the precursors get thru. We still say that v_g rather than c is the signal velocity, as the precursors alone don't constitute a clear signal...