

# Ph 406: Elementary Particle Physics

## Problem Set 6

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### 1. Two Body Decay

Consider the decay of the neutral  $\pi$  meson of (total) energy  $E_\pi$  to two photons,  $\pi^0 \rightarrow \gamma\gamma$ .

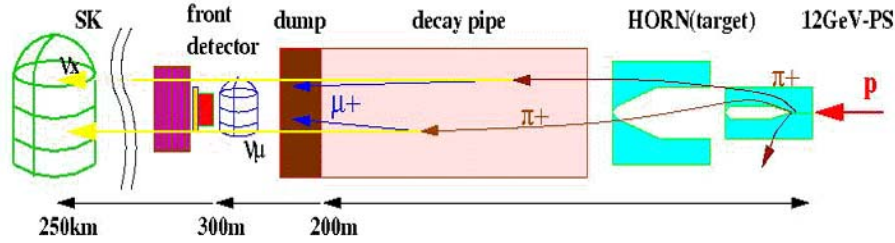
- If the two photons are observed in the laboratory with energies  $E_1$  and  $E_2$  and angle  $\alpha$  between them, what is their invariant mass?
- If the decay of the  $\pi^0$  is isotropic in its rest frame, what is the laboratory distribution  $dN/dE_\gamma$  of the energies of the decay photons?
- What is the minimum opening angle,  $\alpha_{\min}$ , between the two photons in the lab frame?
- What is the distribution  $dN/d\alpha$  of the opening angle between the two photons in the lab frame?
- If the two photons are detected at positions  $x_1$  and  $x_2$  in a plane perpendicular to the direction of the  $\pi^0$  at a distance  $D$ , what is the projected impact point  $x$  of the  $\pi^0$  had it not decayed? You may assume that  $|x_1 - x_2| \ll D$ , which is true for most, but not quite all, decays if  $E_\pi/m_\pi \gg 1$ .
- What is the maximum laboratory angle  $\theta_{\max}$  between the direction of a photon from  $\pi^0$  decay and the direction of the  $\pi^0$ , supposing the photon is observed to have energy  $E_\gamma \gg m_\pi$ ?
- Suppose  $\pi^0$ 's are produced in some scattering process with distribution  $N_\pi(E_\pi, \theta_\pi)$ , where angle  $\theta_\pi$  is measured with respect to the beam direction. That is,  $N_\pi(E_\pi, \theta_\pi) dE_\pi d\Omega_\pi$  is the number of  $\pi^0$ 's in energy interval  $dE_\pi$  centered about energy  $E_\pi$  that point towards solid angle  $d\Omega_\pi$  centered about angles  $(\theta_\pi, \phi_\pi)$ . A detector is placed at angle  $\theta$  to the beam and records the energy spectrum  $N_\gamma(E_\gamma, \theta)$  of the photons that strike it. Show that the  $\pi^0$  spectrum can be related to the photon spectrum by

$$N_\pi(E_\pi, \theta) = -\frac{E_\pi}{2} \frac{dN_\gamma(E_\gamma = E_\pi, \theta)}{dE_\gamma}, \quad (1)$$

if  $E_\pi \gg m_\pi$ .

## 2. Neutrino Beam from Pion Decay

A typical high-energy neutrino beam is made from the decay of  $\pi$  mesons that have been produced in proton interactions on a target, as sketched in the figure below.



Suppose that only positively charged particles are collected by the “horn.” The main source of neutrinos is then the decay  $\pi^+ \rightarrow \mu^+ \nu_\mu$ .

- Give a simple estimate of the relative number of other types of neutrinos than  $\nu_\mu$  in the beam (due to decays in the decay pipe).
- If the decay pions have energy  $E_\pi \gg m_\pi$ , what is the characteristic angle  $\theta_C$  of the decay neutrinos with respect to the direction of the  $\pi^+$ ?
- If a neutrino is produced with energy  $E_\nu \gg m_\pi$ , what is the maximum angle  $\theta_{\max}(E_\nu)$  between it and the direction of its parent pion (which can have any energy)? What is the maximum energy  $E_\nu$  at which a neutrino can be produced in the decay of a pion if it appears at a given angle  $\theta$  with respect to the pion’s direction?

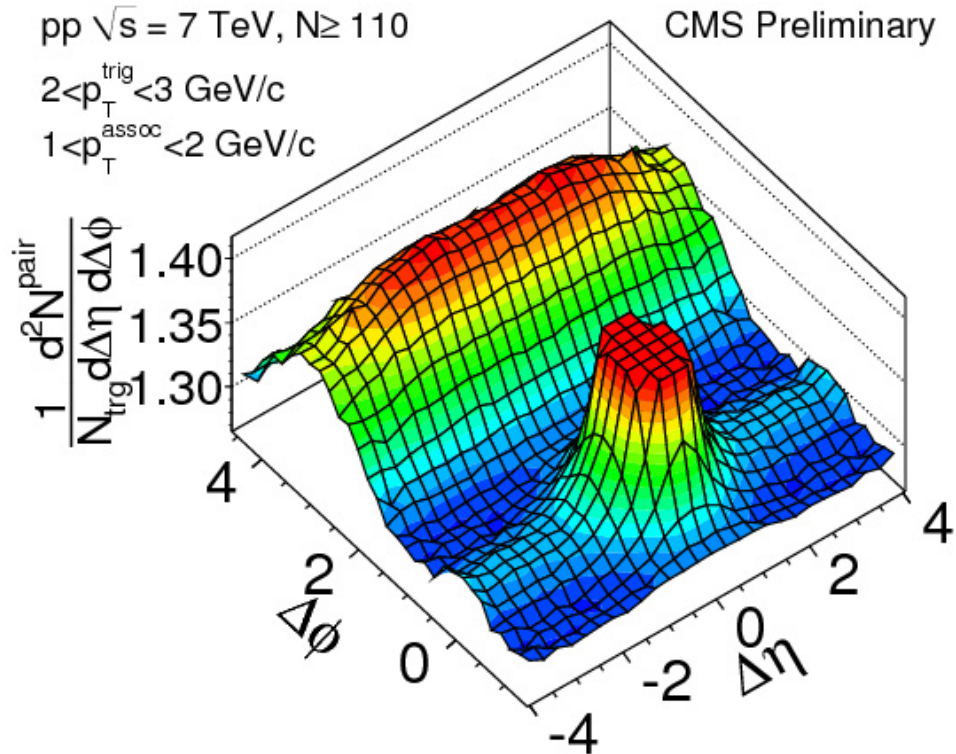
Parts (d) and (f) explore consequences of the existence of these maxima.

- Deduce an analytic expression for the energy-angle spectrum  $d^2N/dE_\nu d\Omega$  for neutrinos produced at angle  $\theta \leq \theta_C$  to the proton beam. You may suppose that  $E_\nu \gg m_\pi$ , that the pions are produced with an energy spectrum  $dN/dE_\pi \propto (E_p - E_\pi)^5$ , where  $E_p$  is the energy of the proton beam, and that the “horn” makes all pion momenta parallel to that of the proton beam.
- At what energy  $E_{\nu,\text{peak}}$  does the neutrino spectrum peak for  $\theta = 0$ ?
- Compare the characteristics of a neutrino beam at  $\theta = 0$  with an off-axis beam at angle  $\theta$  such that  $E_{\nu,\max}(\theta)$  is less than  $E_{\nu,\text{peak}}(\theta = 0)$ .

Facts:  $m_\pi = 139.6 \text{ MeV}/c^2$ ,  $\tau_\pi = 26 \text{ ns}$ ,  $m_\mu = 105.7 \text{ MeV}/c^2$ ,  $\tau_\mu = 2.2 \mu\text{s}$ . In this problem, neutrinos can be taken as massless.

### 3. Pseudorapidity Ridge

An unexpected feature in recent data from high energy  $pp$  collisions is the appearance of a “ridge” along  $\Delta\phi = 0$  in  $\Delta\eta$ - $\Delta\phi$  space in 2-particle correlations in events that contain at least 2 particles at moderately high transverse momentum, where  $\eta = -\ln \tan(\theta/2)$  and  $\phi$  are the pseudorapidity and azimuthal angle of a particle relative to the  $pp$  axis. See, for example, CMS Collaboration, *Observation of long-range, near-side angular correlations in proton-proton collisions at the LHC*, JHEP09, 091 (2010), [http://physics.princeton.edu/~mcdonald/examples/EP/cms\\_jhep09\\_091\\_10.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/cms_jhep09_091_10.pdf).



The peak at  $\Delta\eta = 0 = \Delta\phi$  is due to  $\rho \rightarrow \pi\pi$  decay (although this is also attributed to Bose-Einstein correlations among pions), and the “ridge” at  $\Delta\phi \approx \pi$  is attributed to pairs of particles with transverse momentum opposite to that of the  $\rho \rightarrow \pi\pi$ .

The (open-ended) problem is to explain the “same-side ridge.”

The answer to this is not considered to be clear yet. You may, of course, consult recent literature on this topic.

## Solutions

1. (a) Since a (real) photon has no mass, its energy and momentum are the same:  
 $E_\gamma = P_\gamma$ .

In this part we suppose that photon 1 propagates along the  $+z$  axis, so its energy-momentum 4-vector can be written (in units where  $c = 1$ ) as

$$q_1 = (E, P_x, P_y, P_z) = (E_1, 0, 0, E_1). \quad (2)$$

We can define photon 2 to be moving in the  $x$ - $z$  plane, so its 4-vector is

$$q_2 = (E_2, E_1 \sin \alpha, 0, E_1 \cos \alpha). \quad (3)$$

The invariant mass of the two photons is related by

$$\begin{aligned} m^2 &= (q_1 + q_2)^2 = q_1^2 + q_2^2 + 2q_1 \cdot q_2 = 0 + 0 + 2E_1 E_2 (1 - \cos \alpha) \\ &= 4E_1 E_2 \sin^2 \alpha / 2. \end{aligned} \quad (4)$$

If we had defined the  $\pi^0$  to propagate along the  $+z$  axis, we could still define the decay plane to be the  $x$ - $z$  plane and write

$$q_1 = (E_1, E_1 \sin \theta_1, 0, E_1 \cos \theta_1), \quad q_2 = (E_2, -E_2 \sin \theta_2, 0, E_1 \cos \theta_2), \quad (5)$$

so that

$$m^2 = (q_1 + q_2)^2 = 2E_1 E_2 (1 - \cos(\theta_1 + \theta_2)) = 4E_1 E_2 \sin^2 \alpha / 2, \quad (6)$$

where the opening angle is  $\alpha = \theta_1 + \theta_2$ .

- (b) In this part we suppose the  $\pi^0$  propagates along the  $+z$  axis, and we define  $\theta^*$  as the angle of photon 1 to the  $z$  axis in the rest frame of the  $\pi^0$ .

The decay is isotropic in the rest frame, so the distribution is flat as a function of  $\cos \theta^*$ . We write

$$\frac{dN}{d \cos \theta^*} = \frac{1}{2}, \quad (7)$$

normalized to unity over the interval  $-1 \leq \cos \theta^* \leq 1$ . The desired distribution of photon energies can be related to this via

$$\frac{dN}{dE_\gamma} = \frac{dN}{d \cos \theta^*} \frac{d \cos \theta^*}{dE_\gamma} = \frac{1}{2} \frac{d \cos \theta^*}{dE_\gamma}. \quad (8)$$

To relate  $E_\gamma$  to  $\cos \theta^*$ , we examine the transformation between the lab frame and the rest frame of the  $\pi^0$ , for which the boost is described by the Lorentz factors  $\gamma = E_\pi / m_\pi$  and  $\beta = v_\pi / c = P_\pi c / E_\pi$  (although we use units where  $c = 1$ ).

Of course, the energy and momentum of the photons in the  $\pi^0$  rest frame is  $E_\gamma^* = P_\gamma^* = m_\pi / 2$ . The 4-vector of photon  $i$ , for  $1 = 1, 2$ , in the  $\pi^0$  rest frame is therefore,

$$(E_i^*, P_{i,x}^*, P_{i,y}^*, P_{i,z}^*) = \frac{m_\pi}{2} (1, \pm \sin \theta^*, 0, \pm \cos \theta^*). \quad (9)$$

Then, the lab-frame energy of photon  $i$  is given by

$$E_i = \gamma E_i^* + \gamma \beta P_{i,z}^* = \gamma \frac{m_\pi}{2} (1 \pm \beta \cos \theta^*). \quad (10)$$

Thus,

$$\frac{dE_\gamma}{d \cos \theta^*} = \gamma \beta \frac{m_\pi}{2} = \frac{P_\pi}{2}, \quad (11)$$

and the energy distribution follows from eq. (8) as

$$\frac{dN}{dE_\gamma} = \frac{1}{P_\pi}. \quad (12)$$

The distribution is flat, with limiting values of  $(E_\pi \pm P_\pi)/2$ , according to eq. (10).

- (c) Since the two decay products have equal mass (zero), the minimum decay angle in the lab occurs at either  $\cos \theta^* = 1$  or 0. If  $\cos \theta^* = 1$ , one of the photons goes forward, and the other goes backwards. Since the mass of the photon is zero, its backwards velocity is  $c$ , and the boost of the pion to the lab frame cannot overcome this. The opening angle between the two photons is then  $\pi$ , a maximum rather than a minimum. (If the decay products have mass, it is possible that the velocity of the backward going particles is less than that of the parent, and both particles go forward in the lab, with minimum opening angle of zero.)

We conclude that the minimum opening angle  $\alpha_{\min}$  occurs for the symmetric decay,  $\cos \theta^* = 0$ . In this case, the transverse momentum of the photons is  $m_\pi/2$ , both in lab frame and the  $\pi^0$  rest frame. In the lab frame, the photons' total momentum equals their total energy, which is just  $E_\pi/2$  for the symmetric decay. Hence,

$$\sin \frac{\alpha_{\min}}{2} = \frac{m_\pi}{E_\pi} = \frac{1}{\gamma}. \quad (13)$$

- (d) The distribution of decays in opening angle  $\alpha$  can be written as

$$\frac{dN}{d\alpha} = \frac{dN}{d \cos \theta^*} \frac{d \cos \theta^*}{d\alpha} = \frac{1}{2} \frac{d \cos \theta^*}{d\alpha}, \quad (14)$$

recalling eq. (7).

One way to relate  $\alpha = \theta_1 + \theta_2$  and  $\cos \theta^*$  is to combine eqs. (4) and (10) in the form

$$\sin^2 \alpha/2 = \frac{m_\pi^2}{4E_1 E_2} = \frac{1}{\gamma^2 (1 - \beta^2 \cos^2 \theta^*)}, \quad (15)$$

or

$$\cos \theta^* = \frac{1}{\beta} \sqrt{1 - \frac{1}{\gamma^2 \sin^2 \alpha/2}} = \frac{\sqrt{\gamma^2 \sin^2 \alpha/2 - 1}}{\gamma \beta \sin \alpha/2}. \quad (16)$$

Taking the derivative, we use eq. (14) to find

$$\frac{dN}{d\alpha} = \frac{1}{4\gamma\beta} \frac{\cos \alpha/2}{\sin^2 \alpha/2} \frac{1}{\sqrt{\gamma^2 \sin^2 \alpha/2 - 1}}. \quad (17)$$

This distribution is peaked at  $\alpha_{\min}$  where  $\sin \alpha_{\min}/2 = 1/\gamma$ , and vanishes at  $\alpha_{\max} = \pi$ .

A subtle issue is revealed on integration of eq. (17), letting  $x = \gamma \sin \alpha/2$ , so that

$$\int_{\alpha_{\min}}^{\pi} \frac{dN}{d\alpha} d\alpha = \frac{1}{2\beta} \int_1^{\gamma} \frac{dx}{x^2 \sqrt{x^2 - 1}} = \frac{1}{2\beta} \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \frac{1}{2}, \quad (18)$$

using Dwight 282.01. The integral is only 1/2, rather than 1, because as the decay angle  $\theta^*$  in the pion rest frame varies from 0 to  $\pi$ , the lab-frame opening angle varies from  $\alpha_{\text{rm}}$  at  $\theta^* = 0$  up to  $\pi$  (for  $\theta^* = \pi/2$ ) and then back down to  $\alpha_{\min}$  at  $\theta^* = \pi$ . That is,  $\theta^*$  is a double-valued function of  $\alpha$ , so integration (once) over  $\alpha$  includes only half of the total decays.

If it is desired that the distribution  $dN/d\alpha$  be normalized to unity, eq. (17) should be multiplied by 2.

- (e) The transverse momenta of the two decay photons (with respect to the lab direction of the  $\pi^0$ ) are equal and opposite. When the observed separation of the two photons obeys  $|x_1 - x_2| \ll D$ , the angles of the photons with respect to the direction of the  $\pi^0$  are small, and the transverse momenta can be written as

$$P_i \frac{x_i - x}{D} = E_i \frac{x_i - x}{D}, \quad (19)$$

Hence,

$$E_1(x_1 - x) = E_2(x - x_2), \quad (20)$$

and the virtual impact point of the  $\pi^0$  is

$$x = \frac{x_1 E_1 + x_2 E_2}{E_1 + E_2} = \frac{x_1 E_1 + x_2 E_2}{E_{\pi}}. \quad (21)$$

- (f) The transverse momentum of a decay photon with respect to the direction of the  $\pi^0$  is

$$P_{\perp} = P_{\gamma} \sin \theta = E_{\gamma} \sin \theta, \quad (22)$$

where  $\theta$  is the angle between the direction of the photon and the  $\pi^0$ . This quantity is invariant with respect to the boost to the rest frame of the  $\pi^0$ , so

$$P_{\perp} = P_{\perp}^* = P_{\gamma}^* \sin \theta^* = \frac{m_{\pi}}{2} \sin \theta^*. \quad (23)$$

Comparing eqs. (22) and (23) we see that

$$\sin \theta = \frac{m_{\pi}}{2E_{\gamma}} \sin \theta^*. \quad (24)$$

So long as  $\theta \leq \pi/2$ , we find that

$$\sin \theta_{\max} = \frac{m_{\pi}}{2E_{\gamma}}, \quad (25)$$

and for  $E_\gamma \gg m_\pi$ ,

$$\theta_{\max} \approx \frac{m_\pi}{2E_\gamma}. \quad (26)$$

However, there are cases when  $\theta > \pi/2$ , for which  $P_{\parallel} = P_\gamma \cos \theta < 0$ . Recalling the boost formalism of part (b),

$$P_{\parallel} = \gamma_\pi(P_{\parallel}^* + \beta_\pi E^*) = \frac{\gamma_\pi m_\pi}{2}(\cos \theta^* + \beta_\pi), \quad (27)$$

we see that  $P_{\parallel} = 0$  and  $\theta = \pi/2$  when  $\cos \theta^* = -\beta_\pi$ . In this case,

$$E_\gamma = P_{\perp} = \frac{m_\pi}{2}\sqrt{1 - \beta_\pi^2} = \frac{m_\pi}{2\gamma_\pi} = \frac{m_\pi^2}{2E_\pi} < \frac{m_\pi}{2}, \quad (28)$$

since  $E_\pi \geq m_\pi$ . Thus, the result (25) holds for  $E_\gamma > m_\pi/2$ .

- (g) We will use information about the photon spectrum for energies  $E_\gamma \gg m_\pi$ , so the maximum angle between the photon and its parent  $\pi^0$  is negligibly small, according to the result of part (f). Then, the probability that a photon hits a detector of a fixed solid angle is the same as the probability that its parent  $\pi^0$  would have hit the detector, had the  $\pi^0$  not decayed. That is, we can ignore any possible complication due to solid angle transformation between the  $\pi^0$  and the photon.

According to eq. (12), the number  $N_\gamma(E_\gamma)$  of photons that appear in energy interval  $dE_\gamma$  due to the decay of a single  $\pi^0$  is

$$N_\gamma = \frac{2}{P_\pi} \approx \frac{2}{E_\pi}, \quad (29)$$

where the factor of 2 occurs because two photon are produced in each decay, and the approximation holds when  $E_\pi \gg m_\pi$  so that it certainly applies when  $E_\gamma \gg m_\pi$ .

If  $\pi^0$ 's are produced with an energy spectrum  $N_\pi(E_\pi, \theta_\pi)$ , then the energy spectrum of the decay photons observed in a detector centered on  $\theta_\pi$  is related by

$$N_\gamma(E_\gamma, \theta_\gamma = \theta_\pi) = \int_{E_\gamma}^{\infty} \frac{2}{E_\pi} N_\pi(E_\pi, \theta_\pi) dE_\pi. \quad (30)$$

Taking the derivative, we find

$$N_\pi(E_\pi, \theta_\pi) = -\frac{E_\pi}{2} \frac{dN_\gamma(E_\gamma = E_\pi, \theta_\gamma = \theta_\pi)}{dE_\gamma}. \quad (31)$$

A more detailed discussion of this problem has been given by R.M. Sternheimer, *Energy Distribution of  $\gamma$  Rays from  $\pi^0$  Decay*, Phys. Rev. **99**, 277 (1955),

[http://physics.princeton.edu/~mcdonald/examples/detectors/sternheimer\\_pr\\_99\\_277\\_55.pdf](http://physics.princeton.edu/~mcdonald/examples/detectors/sternheimer_pr_99_277_55.pdf)

For a discussion of the slightly more complicated case of  $\pi^\pm$  decay, see

<http://physics.princeton.edu/~mcdonald/examples/offaxisbeam.pdf>

2. In this solution we use units where  $c = 1$ .

- (a) Besides the  $\nu_\mu$  from the decay  $\pi^+ \rightarrow \mu^+ \nu_\mu$ , the beam will also contain  $\bar{\nu}_\nu$  and  $\nu_e$  from the subsequent decay  $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$ . Both of these decays occur (primarily) in the “decay pipe” shown in the figure. As both the pions and muons of relevance are relativistic in this problem, they both have about the same amount of time to decay before they are absorbed in the “dump.” Hence, the ratio of number of muon decays to pion decays is roughly the inverse of the ratio of their lifetimes, *i.e.*, about 0.01. Our simple estimate is therefore,

$$\frac{N_{\nu_e}}{N_{\nu_\mu}} = \frac{N_{\bar{\nu}_\mu}}{N_{\nu_\mu}} \approx 0.01. \quad (32)$$

*Experts may note that an additional source of  $\nu_e$  is the decay  $\pi^+ \rightarrow e^+ \nu_e$  at the level of  $10^{-4}$ . Also,  $K^+$  mesons will be produced by the primary proton interaction at a rate about 10% that of  $\pi^+$ . About 65% of  $K^+$  decays are to  $\mu^+ \nu_\mu$ , which add to the main  $\nu_\mu$  beam, but about 5% of the decays are to  $\pi^+ \pi^0 \nu_e$ , which increases the  $\nu_e$  component of the neutrino beam by about  $0.1 \times 0.05 = 0.005$ .*

- (b) Parts (b)-(f) of this problem are based on the kinematics of charged-pion decay, which are closely related to kinematic features of neutral-piondecay,  $\pi^0 \rightarrow \gamma\gamma$  (Prob. 1).

Experts may guess that the characteristic angle of the decay neutrinos with respect to the parent pion is  $\theta_C = 1/\gamma_\pi = m_\pi/E_\pi$ . The details of the derivation are needed in part (c).

We consider the decay  $\pi \rightarrow \mu\nu$  in the rest frame of the pion (in which quantities will be labeled with the superscript  $\star$ ) and transform the results to the lab frame. Energy-momentum conservation can be written as the 4-vector relation,

$$\pi = \mu + \nu, \quad (33)$$

where the squares of the 4-vectors are the particle masses,  $\pi^2 = m_\pi^2$ ,  $\mu^2 = m_\mu^2$  and  $\nu^2 = 0$ . As we are not concerned with details of the muon, it is convenient to rewrite eq. (33) as

$$\mu = \pi - \nu, \quad (34)$$

and square this to find

$$m_\mu^2 = m_\pi^2 - 2(\pi \cdot \nu). \quad (35)$$

In the rest frame of the pion, its 4-vector can be written

$$\pi = (m_\pi, 0, 0, 0). \quad (36)$$

Taking the  $z$  axis to be the direction of the pion in the lab frame, the 4-vector of the (massless) neutrino in the pion rest frame can be written as

$$\nu = (E_\nu^\star, E_\nu^\star \sin \theta^\star, 0, E_\nu^\star \cos \theta^\star), \quad (37)$$



since the energy and momentum of a massless particle are equal. The 4-vector product  $(\pi \cdot \nu) = \pi_0 \nu_0 - \pi_i \nu_i$  is therefore

$$(\pi \cdot \nu) = m_\pi E_\nu^*. \quad (38)$$

Hence, from eq. (35) the energy of the neutrino in the pion rest frame is

$$E_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.8 \text{ MeV}, \quad (39)$$

using the stated facts.

We can now transform the neutrino 4-vector (37) to the lab frame, using the Lorentz boost  $\gamma_\pi = E_\pi/m_\pi$ ,

$$\begin{aligned} \nu &= (E_\nu, E_\nu \sin \theta, 0, E_\nu \cos \theta) \\ &= (\gamma_\pi E_\nu^*(1 + \beta_\pi \cos \theta^*), E_\nu^* \sin \theta^*, 0, \gamma_\pi E_\nu^*(\beta_\pi + \cos \theta^*)). \end{aligned} \quad (40)$$

The pion has spin zero, so the decay is isotropic in the pion rest frame. A relation for the angle  $\theta$  between the neutrino and its parent pion can be obtained from the 1 and 3 components of eq. (40),

$$\tan \theta = \frac{E_\nu^* \sin \theta^*}{\gamma_\pi E_\nu^*(\beta_\pi + \cos \theta^*)}. \quad (41)$$

The characteristic angle of the decay in the lab frame is usefully associated with decays at  $\theta^* = 90^\circ$  in the pion rest frame. Thus,

$$\tan \theta_C = \frac{1}{\gamma_\pi \beta_\pi}. \quad (42)$$

When  $E_\pi \gg m_\pi$  then  $\gamma_\pi \gg 1$ ,  $\beta_\pi \approx 1$ , and

$$\theta_C \approx \frac{1}{\gamma_\pi} = \frac{m_\pi}{E_\pi} \ll 1. \quad (43)$$

- (c) We now consider the lab angle (41) between the neutrino and its parent pion with emphasis on the neutrino energy rather than the pion energy. If  $E_\nu \gg m_\pi$ , then  $E_\pi \gg m_\pi$  also, so  $\gamma_\pi \gg 1$  and  $\beta_\pi \approx 1$ . Then, we can write

$$\tan \theta \approx \frac{E_\nu^* \sin \theta^*}{\gamma_\pi E_\nu^*(1 + \cos \theta^*)} \approx \frac{E_\nu^* \sin \theta^*}{E_\nu}, \quad (44)$$

using the time component of eq. (40). Since  $\sin \theta^*$  cannot exceed unity, we see that there is a maximum lab angle  $\theta$  relative to the direction of the pion at which a neutrino of energy  $E_\nu$  can appear, namely

$$\theta_{\max} \approx \frac{E_\nu^*}{E_\nu} \approx \frac{30 \text{ MeV}}{E_\nu}, \quad (45)$$

which is small for  $m_\pi \ll E_\nu$ .

If instead, the angle  $\theta$  is given, eq. (44) also tells us that

$$E_\nu \approx \frac{E_\nu^* \sin \theta^*}{\tan \theta} \leq \frac{E_\nu^*}{\tan \theta}. \quad (46)$$

- (d) We desire the neutrino spectrum in terms of the laboratory quantities  $E_\nu$ ,  $\theta$  and  $\phi$ . We expect that the spectrum is uniform in the azimuthal angle  $\phi$ . We are given the energy spectrum  $dN/dE_\pi \propto (E_p - E_\pi)^5$  of the parent pions, and we have deduced that the spectrum is isotropic in the pion rest frame, *i.e.*, flat in  $\cos \theta^*$ . Hence, we seek the transformation

$$\frac{d^2 N}{dE_\nu d\Omega} \propto \frac{d^2 N}{dE_\nu d\cos \theta} = \frac{d^2 N}{dE_\pi d\cos \theta^*} J(E_\pi, \cos \theta^*; E_\nu, \cos \theta) \propto (E_p - E_\pi)^5 J, \quad (47)$$

where the Jacobian is given by

$$J(E_\pi, \cos \theta^*; E_\nu, \cos \theta) = \begin{vmatrix} \frac{\partial E_\pi}{\partial E_\nu} & \frac{\partial \cos \theta^*}{\partial E_\nu} \\ \frac{\partial E_\pi}{\partial \cos \theta} & \frac{\partial \cos \theta^*}{\partial \cos \theta} \end{vmatrix}. \quad (48)$$

The “exact” form of the Jacobian is somewhat lengthy, so we will simplify to the extent we can by noting that when  $E_\nu \gg m_\pi$ , the parent pion has  $E_\pi \gg m_\pi$  also, and so  $\beta_\pi \approx 1$ . Also, part (c) tells us that  $\theta$  is very small for any value of  $\theta^*$ .

We already have relation (44) between  $E_\nu$ ,  $\tan \theta$  and  $\sin \theta^*$ , so we can write

$$\cos \theta^* = \sqrt{1 - \sin^2 \theta^*} \approx \sqrt{1 - \frac{E_\nu^2}{E_\nu^{*2}} \tan^2 \theta} = \sqrt{1 - \frac{E_\nu^2}{E_\nu^{*2}} \left( \frac{1}{\cos^2 \theta} - 1 \right)}. \quad (49)$$

Thus,

$$\frac{\partial \cos \theta^*}{\partial E_\nu} \approx -\frac{\frac{E_\nu}{E_\nu^{*2}} \tan^2 \theta}{\sqrt{1 - \frac{E_\nu^2}{E_\nu^{*2}} \tan^2 \theta}} \approx -\frac{E_\nu \theta^2}{E_\nu^{*2} \cos \theta^*}, \quad (50)$$

for small  $\theta$ , and

$$\frac{\partial \cos \theta^*}{\partial \cos \theta} \approx \frac{\frac{E_\nu^2}{E_\nu^{*2} \cos^3 \theta}}{\sqrt{1 - \frac{E_\nu^2}{E_\nu^{*2}} \tan^2 \theta}} \approx \frac{E_\nu^2}{E_\nu^{*2} \cos \theta^*}. \quad (51)$$

We can also use time components of eq. (40) to write

$$\gamma_\pi = \frac{E_\pi}{m_\pi} = \frac{E_\nu}{E_\nu^*(1 + \beta_\pi \cos \theta^*)} \approx \frac{E_\nu}{E_\nu^*(1 + \cos \theta^*)} \quad (52)$$

Hence,

$$\frac{\partial E_\pi}{\partial E_\nu} \approx \frac{m_\pi}{E_\nu^*(1 + \cos \theta^*)} - \frac{m_\pi E_\nu}{E_\nu^*(1 + \cos \theta^*)^2} \frac{\partial \cos \theta^*}{\partial E_\nu} \approx \frac{E_\pi}{E_\nu} + \frac{E_\pi^2 \theta^2}{m_\pi E_\nu^* \cos \theta^*}, \quad (53)$$

and

$$\frac{\partial E_\pi}{\partial \cos \theta} \approx -\frac{m_\pi E_\nu}{E_\nu^*(1 + \cos \theta^*)^2} \frac{\partial \cos \theta^*}{\partial \cos \theta} \approx -\frac{E_\pi^2 E_\nu}{m_\pi E_\nu^* \cos \theta^*}. \quad (54)$$

The Jacobian (48) is therefore

$$J \approx \left| \begin{array}{cc} \frac{E_\pi}{E_\nu} + \frac{E_\pi^2 \theta^2}{m_\pi E_\nu^* \cos \theta^*} & -\frac{E_\nu \theta^2}{E_\nu^{*2} \cos \theta^*} \\ -\frac{E_\pi^2 E_\nu}{m_\pi E_\nu^* \cos \theta^*} & \frac{E_\nu^2}{E_\nu^{*2} \cos \theta^*} \end{array} \right| = \frac{E_\pi E_\nu}{E_\nu^{*2} \cos \theta^*}, \quad (55)$$

and hence the neutrino spectrum can be written from eq. (47) as

$$\frac{d^2 N}{dE_\nu d \cos \theta} \propto (E_p - E_\pi)^5 \frac{E_\pi E_\nu}{\cos \theta^*}. \quad (56)$$

Because the factor  $\cos \theta^*$  in the denominator of the Jacobian can go to zero, it is possible that the neutrino flux is higher for nonzero values of the lab angle  $\theta$ .

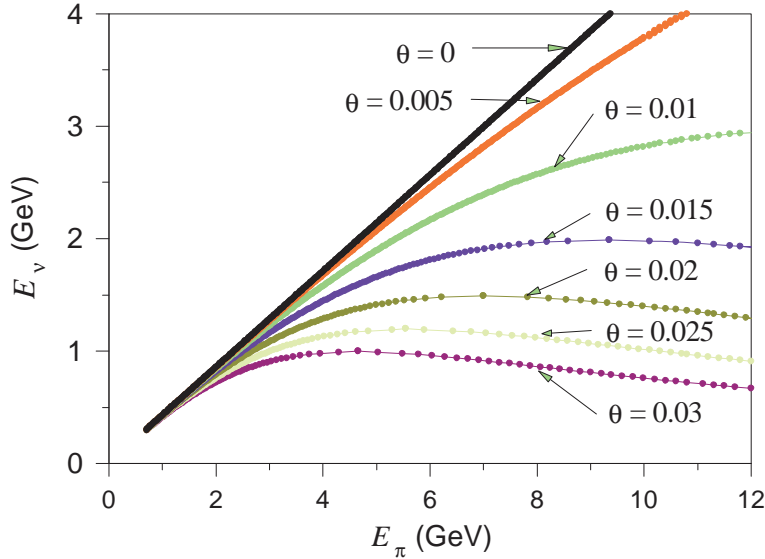
- (e) On the axis,  $\theta = 0$ ,  $\theta^* = 0$ , and  $E_\pi = m_\pi E_\nu / 2E_\nu^* \approx 2E_\nu$  according to eq. (52). In this case, the neutrino spectrum (56) is

$$\frac{d^2 N(\theta = 0)}{dE_\nu d \cos \theta} \propto \left( E_p - \frac{m_\pi E_\nu}{2E_\nu^*} \right)^5 E_\nu^2. \quad (57)$$

The peak of the spectrum occurs at

$$E_{\nu, \text{peak}} = \frac{4E_\nu^*}{7m_\pi} E_p \approx \frac{E_p}{8}. \quad (58)$$

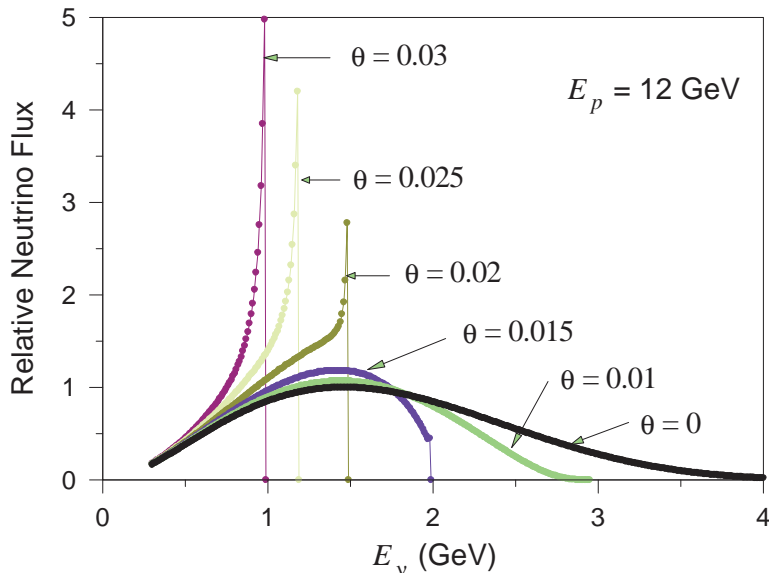
- (f) For an off-axis neutrino beam (at a nonzero value of angle  $\theta$ ) we must evaluate the spectrum (56) using relations (49) and (52). This is readily done numerically. For example, a plot of the pion energy  $E_\pi$  needed to produce a neutrino of energy  $E_\nu$  at various angles  $\theta$  is shown below.



As expected from part (c), we see that for a given angle  $\theta$ , there is a maximum possible neutrino energy, and as the neutrino energy approaches this value, a large range of pion energies contributes to a small range of neutrino energies. This will

result in an enhancement of the neutrino spectrum. If we desire the enhancement at a particular neutrino energy, we should look for the neutrinos close to the angle  $\theta_{\max}$  given in eq. (45), which is independent of the proton/pion energy.

A numerical evaluation of the neutrino spectrum (56) for several values of angle  $\theta$  with respect to the proton/pion beam is shown below.



We see that the spectrum of neutrinos at a nonzero angle is peaked at a lower energy, and is narrower, than that at zero degrees, due to the existence of a maximum possible neutrino energy (46) in decays at a given angle to the direction of the parent pion. This effect is especially prominent when  $E_{\nu, \max}(\theta) \approx (30 \text{ MeV})/\theta$  is less than  $E_{\nu, \text{peak}}(\theta = 0)$ , as then there is a substantial rate of higher energy pions all of which decay into a narrow band of neutrino energies at this angle.

The spectral narrowing of an off-axis neutrino beam remains in more complete calculations<sup>1</sup> that include the nonzero transverse momenta of the pions before and after passing through the “horn,” although the spectrum will not have such hard edges, and the favored angle-energy combination is  $\theta \approx (50 \text{ MeV})/E_{\nu}$ .

In sum, the existence of a maximum energy for neutrinos that decay at a given angle to their parent pions implies that many different pion energies contribute to the this neutrino energy, which enhances the neutrino spectrum at this angle-energy combination,  $\theta \approx (30\text{-}50 \text{ MeV})/E_{\nu}$ .

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<sup>1</sup>D. Beavis *et al.*, *Long Baseline Neutrino Oscillation Experiment, E889, Physics Design Report*, BNL 52459 (April 1995), Chap. 3, <http://physics.princeton.edu/~mcdonald/nufact/e889/chapter3a.pdf>, Y. Itow *et al.*, *The JHF-Kamioka neutrino project*, hep-ex/0106019, <http://arxiv.org/abs/hep-ex/0106019>.

3. The pseudorapidity “ridge” does not yet have a standard explanation. I am intrigued by the comments of J.D. Bjorken, S.J. Brodsky and A.S. Goldhaber, *Possible multiparticle ridge-like correlations in very high multiplicity protonproton collisions*, Phys. Lett. **B726**, 344 (1013), [http://physics.princeton.edu/~mcdonald/examples/EP/bjorken\\_pl\\_b726\\_344\\_13.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/bjorken_pl_b726_344_13.pdf). A somewhat related suggestion is made by C.-Y. Wong, *Momentum-kick model description of the ridge in  $\Delta\phi$ - $\Delta\eta$  correlations in  $pp$  collisions at 7 TeV*, Phys. Rev. C **84**, 024901 (2011), [http://physics.princeton.edu/~mcdonald/examples/EP/wong\\_prc\\_84\\_024901\\_11.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/wong_prc_84_024901_11.pdf), which has references to 37 other suggestions.