

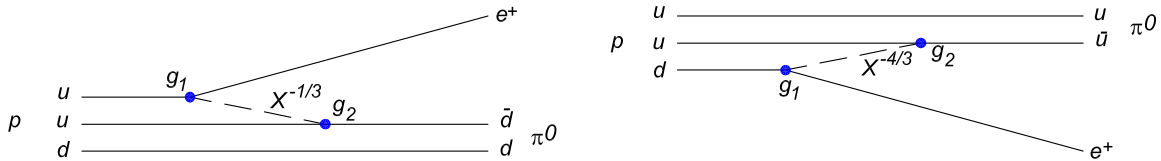
# Ph 406: Elementary Particle Physics

## Problem Set 12

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Due Monday, January 12, 2015 (updated August 5, 2016)

- Proton and Neutron Decay.** The emerging understanding that the strong force becomes weaker at high energies while the electromagnetic force grows stronger, such that  $\alpha_s \approx \alpha_{EM}$  at  $E \approx 10^{15}$  GeV, led<sup>1</sup> to the conjecture of a “grand unification” of strong, electromagnetic (and weak) forces, associated with various new bosons of mass  $\approx 10^{15}$  GeV that couple quarks and leptons. Such bosons (with fractional electric charge  $1/3e$  and  $-4/3e$ , etc.) might mediate baryon-number- and lepton-number-violating transitions, as sketched below, such that a proton could decay via  $p \rightarrow e^+ \pi^0$ .



The coupling constants  $g_1$  and  $g_2$  might have strengths at energy  $\approx 1$  GeV such that  $g_1^2 \approx \alpha_{EM}$  and  $g_2^2 \approx \alpha_s$ .

Make a simple estimate of the proton lifetime based on the above model.

Give a similar simple estimate for (3-body) neutron decay,  $n \rightarrow pe^- \bar{\nu}_e$ , whose observed lifetime is  $\approx 1000$  s.

- The M-S-W Effect.** An interesting phenomenon in the  $K^0-\bar{K}^0$  system is that the difference in the interaction of a  $K^0$  and a  $\bar{K}^0$  with matter permits regeneration of the mass/lifetime eigenstate  $K_S^0$  by a block of matter placed in a beam of Kaons where the  $K_S^0$  component has essentially died out and only  $K_L^0$  remain.<sup>2</sup> Use of two regenerators with variable separation permits measurement of the magnitude of the  $K_S^0-K_L^0$  mass difference,<sup>3,4</sup> and if the two regenerators are placed close to the Kaon source so that the

<sup>1</sup>J.C. Pati and A. Salam, *Lepton number as the fourth “color,”* Phys. Rev. D **10**, 275 (1974), [http://physics.princeton.edu/~mcdonald/examples/EP/pati\\_prd\\_10\\_275\\_74.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/pati_prd_10_275_74.pdf)

H. Georgi and S.L. Glashow, *Unity of All Elementary-Particle Forces,* Phys. Rev. Lett. **32**, 438 (1974), [http://physics.princeton.edu/~mcdonald/examples/EP/georgi\\_prl\\_32\\_438\\_74.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/georgi_prl_32_438_74.pdf).

<sup>2</sup>A. Pais and O. Piccioni, *Note on the Decay and Absorption of the  $\theta^0$ ,* Phys. Rev. **100**, 1487 (1955), [http://physics.princeton.edu/~mcdonald/examples/EP/pais\\_pr\\_100\\_1487\\_55.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/pais_pr_100_1487_55.pdf).

<sup>3</sup>M.L. Good, *Method for Determining the  $K_+^0 - K_-^0$  Mass Difference,* Phys. Rev. **110**, 550 (1958), [http://physics.princeton.edu/~mcdonald/examples/EP/good\\_pr\\_110\\_550\\_58.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/good_pr_110_550_58.pdf).

<sup>4</sup>R.H. Good *et al.*, *Regeneration of Neutral K Mesons and Their Mass Difference,* Phys. Rev. **124**, 1223 (1961), [http://physics.princeton.edu/~mcdonald/examples/EP/good\\_pr\\_124\\_1223\\_61.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/good_pr_124_1223_61.pdf);

V.L. Fitch *et al.*, *Mass Difference of Neutral K Mesons,* Nuovo Cim. **22**, 1160 (1961), [http://physics.princeton.edu/~mcdonald/examples/EP/fitch\\_nc\\_22\\_1160\\_61.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/fitch_nc_22_1160_61.pdf).

$K_S^0$  component is still significant at the first regenerator, the sign of the mass difference can be determined.<sup>5</sup>

In 1978, Wolfenstein noted<sup>6</sup> that an electron (anti)neutrino can interact with matter via diagrams with either  $W$  or  $Z^0$  exchange, while  $\nu_\mu$  and  $\nu_\tau$  can only interact via  $Z^0$  exchange, which leads to “matter effects” for neutrinos that are equivalent to regeneration in the  $K^0$ - $\bar{K}^0$  system.

In this problem you should deduce some consequences of these matter effects in a two-neutrino scenario, supposing that an electron or muon (anti)neutrino is produced at time  $t = 0$  with a definite momentum,  $m_W \gg P_\nu \gg m_i$ , such that the neutrino-mass eigenstates  $\nu_1$  and  $\nu_2$  have energies  $E_i \approx P + m_i^2/2P \approx P + m_i^2/2E$  for  $i = 1, 2$ .<sup>7</sup> Note that

$$E \equiv \frac{E_1 + E_2}{2}, \quad E_{1,2} \approx E \pm \frac{\Delta m^2}{4E}, \quad \text{with} \quad \Delta m^2 = \Delta m_{12}^2 \equiv m_1^2 - m_2^2. \quad (1)$$

The goal of this exercise is to display how the oscillations of a beam of initial muon neutrinos that propagates through uniform matter are sensitive to the sign of  $\Delta m_{12}^2$ .

The task is to identify the mass eigenstates of neutrinos propagating inside matter. An approximate analysis of the neutrino wavefunction,  $|\psi(t)\rangle = a_e(t)|\nu_e\rangle + a_\mu(t)|\nu_\mu\rangle = b_1(t)|\nu_1\rangle + b_2(t)|\nu_2\rangle$ , is that it obeys the Schrödinger equation  $i\psi = H\psi = (H_0 + H')\psi$ , where the Hamiltonian (a  $2 \times 2$  matrix)  $H_0$  applies for propagation in vacuum, and  $H'$  describes the additional effect of propagation through matter. Of course, the amplitudes in the flavor basis and the mass basis are related by

$$\begin{pmatrix} a_e \\ a_\mu \end{pmatrix} = U \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad (2)$$

where  $\theta = \theta_{12}$  is the neutrino-mixing angle.

**Problem:** Give the forms of the vacuum Hamiltonian  $H_0$  in the mass basis and in the flavor basis for a neutrino of momentum  $P$ . Show that in the flavor basis  $H_0$  is the average energy  $E$  times the unit matrix  $\mathbf{I}$  plus  $\Delta m^2/4E$  times a symmetric matrix that is a function of  $2\theta$ . Note a relation between the mixing angle and all four of the

<sup>5</sup>J.V. Jovanovich *et al.*, *Experiment on the Sign and Magnitude of the  $K_L^0$ - $K_S^0$  Mass Difference*, Phys. Rev. Lett. **124**, 1223 (1961), [http://physics.princeton.edu/~mcdonald/examples/EP/jovanovich\\_pr1\\_17\\_1075\\_66.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/jovanovich_pr1_17_1075_66.pdf); W.A.W. Melhop *et al.*, *Interference between Neutral Kaons and Their Mass Difference*, Phys. Rev. **172**, 1613 (1968), [http://physics.princeton.edu/~mcdonald/examples/EP/melhop\\_pr\\_172\\_1613\\_68.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/melhop_pr_172_1613_68.pdf).

<sup>6</sup>L. Wolfenstein, *Neutrino oscillations in matter*, Phys. Rev. D **17**, 2369 (1978), [http://physics.princeton.edu/~mcdonald/examples/neutrinos/wolfenstein\\_prd\\_17\\_2369\\_78.pdf](http://physics.princeton.edu/~mcdonald/examples/neutrinos/wolfenstein_prd_17_2369_78.pdf); *Neutrino oscillations and stellar collapse*, Phys. Rev. D **20**, 2634 (1979), [http://physics.princeton.edu/~mcdonald/examples/neutrinos/wolfenstein\\_prd\\_20\\_2634\\_79.pdf](http://physics.princeton.edu/~mcdonald/examples/neutrinos/wolfenstein_prd_20_2634_79.pdf).

<sup>7</sup>For aspects of a more detailed description in which the neutrino is produced in an entangled state such that energy is conserved during neutrino oscillations, see K.T. McDonald, *Do Neutrino Oscillations Conserve Energy?* (July 21, 2013), [http://physics.princeton.edu/~mcdonald/examples/neutrino\\_osc.pdf](http://physics.princeton.edu/~mcdonald/examples/neutrino_osc.pdf).

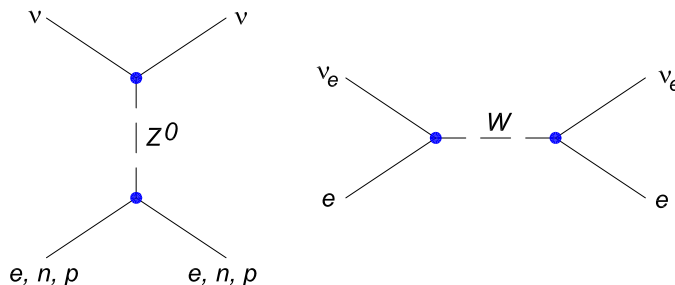
elements of the matrix  $H_0$ , which relation can be used to identify the effective mixing angle  $\theta'$  for the full Hamiltonian  $H$  once you have found this.

Next, we need to add the effect of neutrino interactions with matter. One way of thinking about this is that the matter interactions scatter neutrinos out of the beam, such that the beam is attenuated, which effect seems equivalent to neutrino decay. However, another view is that the matrix elements of the Hamiltonian (in, say, the flavor basis) represent the transition amplitude that a neutrino of one flavor becomes a neutrino of another (or the same) flavor. Since a matter interaction cannot (directly) change the flavor of a neutrino, we can/should consider the amplitude that after a matter interaction the neutrino of a given flavor remains one of the same flavor.

These two viewpoints are related by the so-called **optical theorem**<sup>8</sup> that the total scattering cross section is proportional to the imaginary part of the forward elastic scattering cross section.

A technical detail that has been omitted in this course is that the propagators in Feynman diagrams include a factor  $i$ , such that the amplitudes that we have discussed for, say, elastic scattering are actually imaginary, as to be consistent with the optical theorem.

Then, the matrix  $H'$  in the Hamiltonian  $H_0 + H'$  that characterizes the matter effects (in the flavor basis) has  $H'_{e\mu} = H'_{\mu e} = 0$ , and  $H'_{ee}$  and  $H'_{\mu\mu}$  as the amplitudes from the Feynman diagrams for forward elastic scattering of the neutrinos off the electrons, neutrons, and protons in matter.



Consider low energies, where the term  $q^2$  in the propagators is negligible compared to  $m_{W,Z}^2$ , and note that lefthanded protons and neutrons form a weak doublet with weak isospin  $I_3 = 1/2$  and weak hypercharge  $Y = 1$ , while righthanded protons have weak  $I_3 = 0$ , weak  $Y = 2$  and righthanded neutrons have weak  $I_3 = 0$ , weak  $Y = 0$ .

The forward scattering amplitudes are then proportional to the sum of amplitudes for scattering by  $e$ ,  $p$  and  $n$  (since in forward scattering one can't tell while particle did the scattering; the forward scattering amplitude is **coherent**), where each subamplitude is the product of the number density  $N/\text{cm}^3$  of scatterers, the vertex factors, and the (simplified) propagator. Recall (p. 388, Lecture 22 of the Notes) that introducing the

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<sup>8</sup>A historical survey is given in R.G. Newton, *Optical theorem and beyond*, Am. J. Phys. **44**, 639 (1976), [http://physics.princeton.edu/~mcdonald/examples/EM/newton\\_ajp\\_44\\_639\\_76.pdf](http://physics.princeton.edu/~mcdonald/examples/EM/newton_ajp_44_639_76.pdf). For discussion of the optical theorem in classical electromagnetism, see sec. 10.11 of J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999).

coupling factors  $C_L$  and  $C_R$  permits summing over left- and righthanded states  $j$  in the amplitudes, so the vertex factors for  $Zjj$  coupling are proportional to the relevant sum  $C_L + C_R$ .

**Problem:** Deduce the form of the (diagonal) transition Hamiltonian  $H'$ , and give an expression for the effective mixing angle  $\theta'$  for  $\nu_e$  and  $\nu_\mu$  that propagate through uniform-density matter. Note that this angle depends on the sign of  $\Delta m^2$ , which could thereby be measured from observation of oscillations of neutrinos that propagate through the Earth from a source to a remote detector, both on the surface. Estimate the characteristic distance scale (oscillation length  $L_{\text{matter}}$ ) over which matter effects become prominent, noting that this length  $\propto 1/\text{energy} \approx 1/H'$ .

Your result should indicate that the effective mixing angle  $\theta'$  goes to zero in very high-density matter (a kind of “quantum watchdog” effect<sup>9</sup>), and that there exists a “resonance” condition for the matter density at which  $\theta' = \pm 45^\circ$ . For example, with  $E = 3$  MeV as representative of solar or reactor (anti)neutrinos, and the empirical results that  $\Delta m_{12}^2 \approx 8 \times 10^{-5}$  eV<sup>2</sup>,  $\sin^2 2\theta_{12} = 0.86$ ,  $\cos 2\theta_{12} \approx 0.4$ , the “resonance” occurs (I believe) for density similar to that of gold. These effects complicate the analysis of oscillations of neutrinos produced in stars, as pointed out by Mikheyev and Smirnov.<sup>10</sup>

3. **The See-Saw Mechanism.** An appealing qualitative “explanation” of why the observed neutrinos have very small mass is that each has a partner in some grand-unified theory, and in principle there could be oscillations between these two states governed by a mass matrix (Hamiltonian) of the form

$$\begin{pmatrix} m_1 & m_{12} \\ m_{12} & m_2 \end{pmatrix}, \quad (3)$$

where the “ideal” mass  $m_1$  of the ordinary neutrino could be 0 while that of the grand-unified partner is  $m_2$ , and  $m_{12}$  describes the coupling between the two “ideal” states. Deduce the “physical” mass eigenvalues  $m_\nu$  and  $m'_\nu$  in the approximation that  $m_1 \ll m_{12} \ll m_2$ , and that the coupling  $m_{12}$  has some effect on  $m_\nu$ . What is the resulting  $m_\nu$  for  $m_2 \approx 10^{15}$  GeV, the grand-unified energy scale, and  $m_{12} \approx m_{\text{Higgs}}$  as representative of the electroweak energy scale.

The “discovery” of the see-saw mechanism is attributed to various people. The earliest paper I have found with some form of it is T.P. Cheng, *Hierarchy of lepton masses in a vectorlike theory with Majorana particles*, Phys. Rev. D **14**, 1367 (1976),

[http://physics.princeton.edu/~mcdonald/examples/EP/cheng\\_prd\\_14\\_1367\\_76.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/cheng_prd_14_1367_76.pdf).

The next occurrence of it seems to be in P. Minkowski,  $\mu \rightarrow e\gamma$  at a Rate of One out

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<sup>9</sup>K. Kraus, *Measuring Processes in Quantum Mechanics I. Continuous Observation and the Watchdog Effect*, Found. Phys. **11**, 549 (1981), [http://physics.princeton.edu/~mcdonald/examples/QM/kraus\\_fp\\_11\\_549\\_81.pdf](http://physics.princeton.edu/~mcdonald/examples/QM/kraus_fp_11_549_81.pdf).

<sup>10</sup>S.P. Mikheyev and A.Y. Smirnov, *Resonant Amplification of  $\nu$  Oscillations in Matter and Solar-Neutrino Spectroscopy*, Nuovo Cim. **9C**, 17 (1986), [http://physics.princeton.edu/~mcdonald/examples/neutrinos/mikheyev\\_nc\\_9c\\_17\\_86.pdf](http://physics.princeton.edu/~mcdonald/examples/neutrinos/mikheyev_nc_9c_17_86.pdf).

*of  $10^9$  Muon Decays?* Phys. Lett. **67B**, 421 (1977),  
[http://physics.princeton.edu/~mcdonald/examples/EP/minkowski\\_pl\\_67b\\_421\\_77.pdf](http://physics.princeton.edu/~mcdonald/examples/EP/minkowski_pl_67b_421_77.pdf).  
The term “see-saw” seems not to have been used in formal papers prior to 1986.