Four Expressions for Electromagnetic Field Momentum

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1 Problem

Show that electromagnetic field momentum can be written for static fields in four equivalent forms,

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}^{(\rm C)}}{c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\, d{\rm Vol}, \tag{1}$$

in Gaussian units, where ϱ is the (total) electric charge density, $\mathbf{A}^{(\rm C)} = \int \mathbf{J}\, d{\rm Vol}/cr$ is the magnetic vector potential in the Coulomb gauge,[1] $\mathbf{J}$ is the (total) electric current density, c is the speed of light in vacuum, $\mathbf{E} = \int \varrho\, \hat{\mathbf{r}}\, d{\rm Vol}/r^2$ is the electric field, $\mathbf{B} = \int \mathbf{J}\times\hat{\mathbf{r}}\, d{\rm Vol}/cr^2$ is the magnetic field, and $V^{(\rm C)} = \int \varrho\, d{\rm Vol}/r$ is the electric scalar potential in the Coulomb gauge.[2]

The first form of eq. (1) is due to Maxwell (sec. 57 of [3]), who regarded the vector potential $\mathbf{A}^{(\rm C)}$ at the location of an electric charge q as providing a measure, $q\mathbf{A}^{(\rm C)}/c$, of electromagnetic momentum and an interpretation of Faraday's electrotonic state (secs. 60-61 of [4]). That Faraday associated with some kind of momentum with this state is hinted in sec. 1077 of [5]. The second form is due to J.J. Thomson, who speculated as to electromagnetic mass/momentum (1881) [6], and continued with a concept of momentum stored in the electromagnetic field (1891) [7, 8], with the field-momentum density being the Poynting vector [9] divided by c^2, $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2$, where c is the speed of light in vacuum.[3] The third form was introduced by Furry [11],[4] and the fourth form is due to Aharonov *et al.* [15].

Of these four forms, the second depends only on the electromagnetic fields, and so is the most "Maxwwellian". As such, we consider the volume density of electromagnetic field momentum to be,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,, \tag{2}$$

[1]The Coulomb-gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)} = 0$ also holds for static fields in the Lorenz gauge [1], whose gauge condition is $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm L)} = -\partial V^{(\rm L)}/\partial ct$.

[2]For discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence, see, for example [2].

[3]This relation was also advocated by Poincaré [10].

[4]The density $V^{(\rm C)}\mathbf{J}/c^2$ is the static limit of an expression for the field-momentum density proposed by Livens (1926), bottom of p. 263 of [12]. This density is also the static limit of $\mathbf{S}/c^2$ for several of the alternative forms of the Poynting vector proposed by Slepian (1942) [13].

For a review of alternatives to the Poynting vector, see [14].

and not $\varrho\mathbf{A}^{(\mathrm{C})}/c$, $V^{(\mathrm{C})}\mathbf{J}/c^2$ or $(\mathbf{J}\cdot\mathbf{E})\mathbf{r}/c^2$.[5,6]

Also discuss which of the four forms of eq. (1) lead to valid expressions for electromagnetic-field angular momentum.[7]

2 Solution

2.1 Minkowski's Derivation of the Poynting-Thomson

We follow an argument of Minkowski [26][8] as to field momentum by considering the total force density on electromagnetic media (of unit permittivity and unit permeability),

$$\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} = \mathbf{f} = \varrho\mathbf{E} + \frac{\mathbf{J}}{c}\times\mathbf{B}, \tag{3}$$

where $\mathbf{p}_{\mathrm{mech}}$ is the density of mechanical momentum in the media.[9] Using the Maxwell equations $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\varrho$ and $\boldsymbol{\nabla}\times\mathbf{B} = 4\pi\mathbf{J}/c + (1/c)\partial\mathbf{E}/\partial t$ for the macroscopic fields,[10]

$$\begin{aligned}
\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} &= \frac{\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})}{4\pi} - \mathbf{B}\times\frac{1}{4\pi}\left(\boldsymbol{\nabla}\times\mathbf{B} + \mathbf{B}\times\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= -\frac{\partial}{\partial t}\left(\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\right) + \frac{1}{4\pi}\left[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})\right] \\
&\equiv -\frac{\partial\mathbf{p}_{\mathrm{EM}}}{\partial t} + \boldsymbol{\nabla}\cdot\mathsf{T}_{\mathrm{EM}},
\end{aligned} \tag{4}$$

where,

$$\mathbf{p}_{\mathrm{EM}} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{\mathbf{S}}{c^2}\,, \tag{5}$$

is the density of momentum associated with the electromagnetic field,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \tag{6}$$

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[5] The first and third forms of eq. (1) involve the electromagnetic potentials, which are not gauge invariant. Note also that even the Coulomb-gauge potentials are not unique for a given set of charges and currents, as use of a gauge function χ which obeys $\nabla^2\chi = 0$ everywhere leads to alternative potentials that satisfy $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. See, for example, sec. IIIC of [16]. Some alternative potentials for an infinite, static solenoid magnet are displayed in sec. 2.1 of [17] and in [18].

[6] We avoid here the famous Abraham-Minkowski debate as to the form of electromagnetic field momentum in media. Some comments by the author on this topic are at [19, 20].

[7] Electromagnetic field angular momentum was first computed for a special case by Darboux [21] and by Poincaré [22], without their realizing the physical significance of the computations [23]. The first explicit mention of electromagnetic field momentum appears to be by J.J. Thomson in [24]. See also [25].

[8] Heaviside gave the form $\mathbf{p}_{\mathrm{EM}} = \mathbf{D}\times\mathbf{B}/4\pi c$ in 1891, p. 108 of [27], and a derivation (1902) essentially that of Minkowski on pp. 146-147 of [28].

[9] Subtle difficulties with the Lorentz force density (3) for permeable media are considered in [29].

[10] Minkowski [26] actually used ϱ_{free} and $\mathbf{J}_{\mathrm{free}}$ rather than the total charge and current densities ϱ and $\mathbf{J}$, as well as the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$.

is the Poynting vector, and,

$$\mathsf{T}_{\mathrm{EM},ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \delta_{ij}\frac{E^2 + B^2}{2}\right] \tag{7}$$

is the symmetric Maxwell stress 3-tensor associated with the electromagnetic fields.[11] To arrive at eq. (7) we note that,

$$\begin{aligned}[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})]_i &= E_i\frac{\partial E_j}{\partial x_j} - E_j\frac{\partial E_j}{\partial x_i} + E_j\frac{\partial E_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j}\left[E_i E_j - \delta_{ij}\frac{\mathbf{E}\cdot\mathbf{E}}{2}\right].\end{aligned} \tag{8}$$

The total electromagnetic field momentum (in the form associated with Poynting, Thomson and Poincaré) follows from eq. (5) as,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})} = \int \mathbf{p}_{\mathrm{EM}}\, d\mathrm{Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol}. \tag{9}$$

2.2 Equivalence of the Faraday-Maxwell and Poynting-Thomson

This section is based on an argument by Vladimir Hnizdo. The result that $\mathbf{P}_{\mathrm{EM_P}} = \mathbf{P}_{\mathrm{EM_M}}$ *was first deduced for a special case by Thomson [24], who gave the first definition of electromagnetic field angular momentum,* $\mathbf{L}_{\mathrm{EM_P}}$*, in the same paper (which is also the origin of the Feynman cylinder paradox [31]). The results of this section may have first been given, via a more compact derivation, by Trammel [32], and independently by Calkin [33] for linear momentum only.*[12]

Static electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be characterized by the time-independent Maxwell equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\varrho, \qquad \boldsymbol{\nabla}\times\mathbf{E} = 0, \tag{10}$$

where ϱ is the electric charge density, and,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \tag{11}$$

which implies that the time-independent current density $\mathbf{J}$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$.

For present purposes we can avoid use of the current density $\mathbf{J}$ and instead consider the vector potential $\mathbf{A}^{(\mathrm{C})}$, which has zero divergence in the Coulomb gauge (and also in the Lorentz gauge for static problems),

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{C})}, \qquad \boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})} = 0. \tag{12}$$

To confirm that the electromagnetic momentum,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})} = \int \frac{\varrho\mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol}, \tag{13}$$

[11] For an extension of this argument if magnetic monopoles existed, see [30].

[12] The results of this section are also discussed for the case of axial symmetry in [34], without awareness of the prior work of [24, 32] or that axial symmetry is not required.

is equal to,

$$\mathbf{P}_{\rm EM}^{\rm (P)} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}, \qquad (14)$$

and that the electromagnetic angular momentum,

$$\mathbf{L}_{\rm EM}^{\rm (M)} = \int \mathbf{r}\times\mathbf{P}_{\rm EM_M}\ d{\rm Vol} = \int \mathbf{r}\times\frac{e\mathbf{A}^{\rm (C)}}{c}\ d{\rm Vol}, \qquad (15)$$

is equal to,

$$\mathbf{L}_{\rm EM}^{\rm (P)} = \int \frac{\mathbf{r}\times(\mathbf{E}\times\mathbf{B})}{4\pi c}\, d{\rm Vol}, \qquad (16)$$

we show that $\mathbf{E}\times\mathbf{B}$ is equal to $\varrho\mathbf{A}^{\rm (C)}$ plus the divergence of a vector field, and that $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ is equal to $\mathbf{r}\times\varrho\mathbf{A}^{\rm (C)}$ plus the divergence of another vector field. Then, we transform the volume integrals of the auxiliary vector field into surface integral using Gauss' law, and if the auxiliary field fall off sufficiently quickly with distance the equivalence of the various forms of electromagnetic momenta is established.[13]

In addition to well-known vector-calculus relations, it is useful to define the operation,[14]

$$\boldsymbol{\nabla}\cdot\mathbf{Ab} \equiv (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{b} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{b} = (\boldsymbol{\nabla}\cdot b_x\mathbf{A})\,\hat{\mathbf{x}} + (\boldsymbol{\nabla}\cdot b_y\mathbf{A})\,\hat{\mathbf{y}} + (\boldsymbol{\nabla}\cdot b_z\mathbf{A})\,\hat{\mathbf{z}}. \qquad (17)$$

Then, for quasistatic $\mathbf{E}$, *i.e.*, $\boldsymbol{\nabla}\times\mathbf{E}\approx 0$, and any vector potential $\mathbf{A}$ such that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$,

$$\begin{aligned}\mathbf{E}\times\mathbf{B} &= \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{E} - (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A} - \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{E}) \\ &= (\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A} + \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E}) - [(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{E}] - (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}. \qquad (18)\end{aligned}$$

For a Coulomb-gauge potential, we can subtract a term $(\boldsymbol{\nabla}\cdot\mathbf{A}^{\rm (C)})\mathbf{E} = 0$ to obtain,

$$\begin{aligned}\mathbf{E}\times\mathbf{B} &= 4\pi\varrho\mathbf{A}^{\rm (C)} + \boldsymbol{\nabla}(\mathbf{A}^{\rm (C)}\cdot\mathbf{E}) - \boldsymbol{\nabla}\cdot\mathbf{E}\mathbf{A}^{\rm (C)} - [(\boldsymbol{\nabla}\cdot\mathbf{A}^{\rm (C)})\mathbf{E} + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}^{\rm (C)}] \\ &= 4\pi\varrho\mathbf{A}^{\rm (C)} + \boldsymbol{\nabla}(\mathbf{A}^{\rm (C)}\cdot\mathbf{E}) - \boldsymbol{\nabla}\cdot\mathbf{E}\mathbf{A}^{\rm (C)} - \boldsymbol{\nabla}\cdot\mathbf{A}^{\rm (C)}\mathbf{E}, \qquad (19)\end{aligned}$$

so that, using various versions of Gauss' theorem,

$$\begin{aligned}\mathbf{P}_{\rm EM}^{\rm (P)} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \\ &= \int \frac{\varrho\mathbf{A}^{\rm (C)}}{c}\ d{\rm Vol} + \oint(\mathbf{A}^{\rm (C)}\cdot\mathbf{E})\ d\mathbf{Area} - \oint \mathbf{E}(\mathbf{A}^{\rm (C)}\cdot d\mathbf{Area}) - \oint \mathbf{A}^{\rm (C)}(\mathbf{E}\cdot d\mathbf{Area}) \\ &= \int \frac{\varrho\mathbf{A}^{\rm (C)}}{c}\ d{\rm Vol} = \mathbf{P}_{\rm EM}^{\rm (M)}. \qquad (20)\end{aligned}$$

The surface integrals in eq. (20) are negligible when the charges and currents that create the electric field $\mathbf{E}$ and the vector potential $\mathbf{A}^{\rm (C)}$ lie within a finite volume that is small compared to the volume of integration, and when radiation can be neglected.

[13]The case of a charged particle together with an infinite magnetic solenoid is delicate in this regard, as discussed in [35, 17]. Here, $\mathbf{P}_{\rm EM}^{\rm (P)} = \mathbf{P}_{\rm EM}^{\rm (M)}$, but $\mathbf{L}_{\rm EM}^{\rm (M)}$ rather than $\mathbf{L}_{\rm EM}^{\rm (P)}$ yields reasonable physical results.

[14]A variant of the argument without use of operation (17) is given in Appendix A of [17].

We now evaluate eq. (16) by taking the cross product of eq. (18) with $\mathbf{r}$ and further transforming the various terms. Thus,

$$\mathbf{r} \times \boldsymbol{\nabla}(\mathbf{A}^{(\mathrm{C})} \cdot \mathbf{E}) = -\boldsymbol{\nabla} \times (\mathbf{A}^{(\mathrm{C})} \cdot \mathbf{E})\, \mathbf{r} + \boldsymbol{\nabla}(\mathbf{A}^{(\mathrm{C})} \cdot \mathbf{E}) \boldsymbol{\nabla} \times \mathbf{r} = -\boldsymbol{\nabla} \times (\mathbf{A}^{(\mathrm{C})} \cdot \mathbf{E})\, \mathbf{r}. \tag{21}$$

Now,

$$\begin{aligned} [\mathbf{r} \times \boldsymbol{\nabla} \cdot \mathbf{E}\mathbf{A}^{(\mathrm{C})}]_x &= y \boldsymbol{\nabla} \cdot (A_z^{(\mathrm{C})} \mathbf{E}) - z \boldsymbol{\nabla} \cdot (A_y^{(\mathrm{C})} \mathbf{E}) \\ &= \boldsymbol{\nabla} \cdot (y A_z^{(\mathrm{C})} \mathbf{E}) - A_z^{(\mathrm{C})} \mathbf{E} \cdot \boldsymbol{\nabla} y - \boldsymbol{\nabla} \cdot (z A_y^{(\mathrm{C})} \mathbf{E}) + A_y^{(\mathrm{C})} \mathbf{E} \cdot \boldsymbol{\nabla} z \\ &= A_y^{(\mathrm{C})} E_z - A_z^{(\mathrm{C})} E_y + \boldsymbol{\nabla} \cdot (y A_z^{(\mathrm{C})} \mathbf{E}) - \boldsymbol{\nabla} \cdot (z A_y^{(\mathrm{C})} \mathbf{E}) \\ &= [\mathbf{A}^{(\mathrm{C})} \times \mathbf{E}]_x + \boldsymbol{\nabla} \cdot ([\mathbf{r} \times \mathbf{A}^{(\mathrm{C})}]_x \mathbf{E}), \end{aligned} \tag{22}$$

so,

$$[\mathbf{r} \times \boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{C})} \mathbf{E}]_x = [\mathbf{E} \times \mathbf{A}^{(\mathrm{C})}]_x + \boldsymbol{\nabla} \cdot ([\mathbf{r} \times \mathbf{E}]_x \mathbf{A}^{(\mathrm{C})}), \tag{23}$$

and,

$$[\mathbf{r} \times \boldsymbol{\nabla} \cdot \mathbf{E}\mathbf{A}^{(\mathrm{C})}]_x + [\mathbf{r} \times \boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{C})} \mathbf{E}]_x = \boldsymbol{\nabla} \cdot ([\mathbf{r} \times \mathbf{A}^{(\mathrm{C})}]_x \mathbf{E}) + \boldsymbol{\nabla} \cdot ([\mathbf{r} \times \mathbf{E}]_x \mathbf{A}^{(\mathrm{C})}). \tag{24}$$

Hence,

$$\begin{aligned} \mathbf{L}_{\mathrm{EM}}^{(\mathrm{P})} &= \int \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c}\, d\mathrm{Vol} = \int \mathbf{r} \times \frac{\varrho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} \\ &\quad - \oint (\mathbf{A}^{(\mathrm{C})} \cdot \mathbf{E})\, d\mathbf{Area} \times \mathbf{r} - \oint \mathbf{r} \times \mathbf{E}(\mathbf{A}^{(\mathrm{C})} \cdot d\mathbf{Area}) - \oint \mathbf{r} \times \mathbf{A}^{(\mathrm{C})}(\mathbf{E} \cdot d\mathbf{Area}) \\ &= \int \mathbf{r} \times \frac{\varrho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} = \mathbf{L}_{\mathrm{EM}}^{(\mathrm{M})}. \end{aligned} \tag{25}$$

The arguments of the surface integrals in eq. (25) vanish less quickly with distance than those in eq. (20), so in some cases[15] with sources of infinite extent, we may find that $\mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})} = \mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})}$, but $\mathbf{L}_{\mathrm{EM}}^{(\mathrm{P})} \neq \mathbf{L}_{\mathrm{EM}}^{(\mathrm{M})}$.

The equivalence of $\mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})}$ *and* $\mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})}$ *extends to nonstatic systems in which the currents do not satisfy* $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$, *so long as the velocities of all charges are low and radiation can be neglected [36, 37].*

2.2.1 Alternative Coulomb-Gauge Potentials

It was argued above that the electromagnetic field momentum of a static system is correctly computed via both forms $\mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})}$ and $\mathbf{P}_{\mathrm{EM}}^{()}$, and that the electromagnetic field angular momentum can by computed via both forms $\mathbf{L}_{\mathrm{EM}}^{(\mathrm{P})}$ and $\mathbf{L}_{\mathrm{EM}}^{(\mathrm{M})}$, using any Coulomb-gauge vector potential $\mathbf{A}^{(\mathrm{C})}$, so long as it falls off sufficiently quickly at large distances.

However, it seems unlikely that if a Coulomb-gauge vector potential different from $\mathbf{A}^{(\mathrm{C})}(\mathbf{r}) = \int \mathbf{J}(\mathbf{r}')\, d\mathrm{Vol}'/c\, |\mathbf{r} - \mathbf{r}'|$, were used in the Maxwell forms that the result would be the same. This would be consistent with the argument in sec. 2.2 above if the alternative vector potential did not go to zero at large distances.

[15] See, for example, [17, 35].

We recall (see, for example, sec. IIIC of [16]) that an alternative Coulomb-gauge vector potential would have the form $\mathbf{A}^{(C)} + \boldsymbol{\nabla}\chi$ where the *restricted* gauge function χ obeys Laplace's equation $\nabla^2\chi = 0$ everywhere. Now, the "trivial" case of $\chi = 0$ has $\boldsymbol{\nabla}\chi = 0$ everywhere, which includes large distances. Then, by the uniqueness theorem for solutions to Laplace's equation with Neumann boundary conditions (see, for example sec. 1.9 of [38]), this is the only solution to Laplace's equation for which $\boldsymbol{\nabla}\chi = 0$ everywhere at large distances. Hence, all possible alternative, Coulomb-gauge vector potentials do not go to zero everywhere at large distances, and so none of these would lead to a correct computation of the field momentum (or angular momentum) using the Maxwell forms.[16,17]

2.3 Equivalence of the Faraday-Maxwell and Furry Forms

In (quasi)static situations (where $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$) the potentials can be related to the charge and current densities by,

$$\nabla^2 V^{(C)} = 4\pi\varrho, \qquad \nabla^2\mathbf{A}^{(C)} = \frac{4\pi\mathbf{J}}{c}\,. \tag{26}$$

Hence, we can start from the Faraday-Maxwell form of the electromagnetic momentum and write,

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{(M)} &= \int \frac{\varrho\mathbf{A}^{(C)}}{c}\,d{\rm Vol} = \int \frac{\mathbf{A}^{(C)}\nabla^2 V^{(C)}}{4\pi c}\,d{\rm Vol} \\
&= \int \frac{V^{(C)}\nabla^2\mathbf{A}^{(C)}}{4\pi c}\,d{\rm Vol} + \oint [\mathbf{A}^{(C)}(d\mathbf{Area}\cdot\boldsymbol{\nabla})V^{(C)} - V^{(C)}(d\mathbf{Area}\cdot\boldsymbol{\nabla})\mathbf{A}^{(C)}] \\
&= \int \frac{V^{(C)}\mathbf{J}}{c^2}\,d{\rm Vol} = \mathbf{P}_{\rm EM}^{(F)},
\end{aligned} \tag{27}$$

using Green's identity that for any two well-behaved scalar fields ϕ and ψ,

$$\int_{\rm V} (\phi\nabla^2\psi - \psi\nabla^2\phi)\;d{\rm Vol} = \oint_{\rm S} (\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi)\cdot d\mathbf{Area}, \tag{28}$$

where the surface element $d\mathbf{Area}$ is directly away from the closed surface S that bounds volume V. The integral in eq. (27) over the surface at infinity vanishes for bounded charge and current densities, whose corresponding potentials fall off as $1/r$, and whose gradients fall off as $1/r^2$, at large r.

Another derivation of $\mathbf{P}_{\rm EM}^{(F)}$ notes that since the magnetic field $\mathbf{B}$ is always of order $1/c$ (or higher), we can calculate the electromagnetic momentum $\mathbf{P}_{\rm EM}^{(M)}$ to order $1/c^2$ using approximations to the electric field at zeroth order, *i.e.*, $\mathbf{E} \approx -\boldsymbol{\nabla}V^{(C)}$ (the Coulomb electric

[16] Thanks to D. Griffiths for this argument.

[17] For example, the gauge functions $\chi = \pm Bxy/2$ can be used to transform the axially symmetric vector potential $Br\sin\theta\,\hat{\boldsymbol{\phi}}/2$ within a sphere of uniform magnetization parallel to $\hat{\mathbf{z}}$ to the "Landau" forms $By\,\hat{\mathbf{x}}$ and $-Bx\,\hat{\mathbf{y}}$ within the sphere, the symmetric vector potential goes to zero at infinity while the "Landau" forms do not.

field), and to the magnetic field at order $1/c$, *i.e.*, $\boldsymbol{\nabla} \times \mathbf{B} \approx 4\pi\mathbf{J}/c$ with the neglect of the displacement current. Then,

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (P)} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} \approx -\int \frac{\boldsymbol{\nabla} V^{\rm (C)}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} \\
&= \int \frac{V^{\rm (C)}\boldsymbol{\nabla}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} - \int \frac{\boldsymbol{\nabla}\times V^{\rm (C)}\mathbf{B}}{4\pi c}\,d\mathrm{Vol} \\
&= \int \frac{V^{\rm (C)}\mathbf{J}}{c^2}\,d\mathrm{Vol} - \oint \frac{d\mathbf{Area}\times V^{\rm (C)}\mathbf{B}}{4\pi c} = \int \frac{V^{\rm (C)}\mathbf{J}}{c^2}\,d\mathrm{Vol} = \mathbf{P}_{\rm EM}^{\rm (F)}, \qquad (29)
\end{aligned}$$

whenever the charges and currents are contained within a finite volume.

However, the Furry density $V^{\rm (C)}\mathbf{J}/c^2$ does not lead to a satisfactory computation of field angular momentum,

$$\mathbf{L}_{\rm EM}^{\rm (F)} = \int \mathbf{r} \times \frac{V^{\rm (C)}\mathbf{J}}{c^2}\,d\mathrm{Vol} \neq \mathbf{L}_{\rm EM}^{\rm (P)}. \qquad (30)$$

To see this, we note that for stationary systems where $\mathbf{E} = -\boldsymbol{\nabla} V^{\rm (C)}$, *i.e.*, $E_i = -\partial_i V^{\rm (C)}$, and $\boldsymbol{\nabla}\times\mathbf{B} = 4\pi\mathbf{J}/c$,

$$\begin{aligned}
[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_i &= \epsilon_{ijk} r_j \epsilon_{klm}(-\partial_l V^{\rm (C)})B_m = -\epsilon_{ijk}\epsilon_{klm} r_j[\partial_l(V^{\rm (C)}B_m) - V^{\rm (C)}\partial_l B_m] \\
&= -\epsilon_{ijk}\epsilon_{klm}[\partial_l(r_j V^{\rm (C)}B_m) - V^{\rm (C)}B_m\partial_l r_j] + \epsilon_{ijk}V^{\rm (C)} r_j(\boldsymbol{\nabla}\times\mathbf{B})_k \\
&\to \epsilon_{ijk}\epsilon_{klm}V^{\rm (C)}B_m\delta_{jl} + \frac{4\pi}{c}\epsilon_{ijk}V^{\rm (C)}r_j J_k = -\epsilon_{ijk}\epsilon_{mjk}V^{\rm (C)}B_m + \frac{4\pi V^{\rm (C)}}{c}(\mathbf{r}\times\mathbf{J})_i \\
&= -\delta_{ij}^{mj}V^{\rm (C)}B_m + \frac{4\pi}{c}(\mathbf{r}\times V^{\rm (C)}\mathbf{J})_i = -2\delta_{im}V^{\rm (C)}B_m + \frac{4\pi}{c}(\mathbf{r}\times V\mathbf{J})_i \\
&= -2V^{\rm (C)}B_i + \frac{4\pi}{c}(\mathbf{r}\times V^{\rm (C)}\mathbf{J})_i, \qquad (31)
\end{aligned}$$

where in the third line we drop the full-differential term $\partial_l(r_j V^{\rm (C)}B_m)$, anticipating that $\int \partial_l(r_j V^{\rm (C)}B_m)\,d\mathrm{Vol} \to \oint r_j V^{\rm (C)}B_m\,d\mathrm{Area} \to 0$ for fields that fall off sufficiently quickly at large distances. Then, for bounded, stationary systems, where $\mathbf{L}_{\rm P} = \mathbf{L}_{\rm M}$,

$$\mathbf{L}_{\rm EM}^{\rm (P)} = \int \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} = -\int \frac{V^{\rm (C)}\mathbf{B}}{2\pi c}\,d\mathrm{Vol} + \int \mathbf{r}\times\frac{V^{\rm (C)}\mathbf{J}}{c^2}\,d\mathrm{Vol} = \mathbf{L}_{\rm EM}^{\rm (F)} - \int \frac{V^{\rm (C)}\mathbf{B}}{2\pi c}\,d\mathrm{Vol}. \qquad (32)$$

That is, the Poynting and Furry forms of field angular momentum are not equal, in general.[18]

2.4 Equivalence of the Aharonov and Furry Forms

For a stationary system, $\mathbf{E} = -\boldsymbol{\nabla} V^{\rm (C)}$ and $\boldsymbol{\nabla}\cdot\mathbf{J} = \partial^j J_j = 0$, so,

$$\begin{aligned}
\mathbf{J}\cdot\mathbf{E} &= -J_j\,\partial^j V^{\rm (C)} = -\partial^j(J_j V^{\rm (C)}) + V^{\rm (C)}\partial^j J_j = -\partial^j(J_j V^{\rm (C)}), \qquad (33) \\
\mathbf{J}\cdot\mathbf{E}\,r_i &= -r_i J_j\,\partial^j V^{\rm (C)} = -\partial^j(r_i J_j V^{\rm (C)}) + V^{\rm (C)}\partial^j(r_i J_j) \\
&= -\partial^j(r_i J_j V^{\rm (C)}) + V^{\rm (C)}J_j\,\partial^j r_i + V^{\rm (C)} r_i\,\partial^j J_j \\
&= -\partial^j(r_i J_j V^{\rm (C)}) + V^{\rm (C)}J_i, \qquad (34)
\end{aligned}$$

[18]For stationary fields that fall off sufficiently quickly at large distance, $\int V^{\rm (C)}\mathbf{B}\,d\mathrm{Vol} = \int \mathbf{E}\times\mathbf{A}^{\rm (C)}\,d\mathrm{Vol}$.

and,

$$P_{\mathrm{EM},i}^{(\mathrm{A})} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, r_i \, d\mathrm{Vol} = \int \frac{V^{(\mathrm{C})} J_i}{c^2}\, d\mathrm{Vol} - \frac{1}{c^2}\int \partial^j (r_i J_j V^{(\mathrm{C})})\, d\mathrm{Vol}. \tag{35}$$

The latter volume integral becomes a surface integral at infinity, so is zero for any bounded current density. Then,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{A})} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, \mathbf{r}\, d\mathrm{Vol} = \int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\mathrm{Vol} = \mathbf{P}_{\mathrm{EM}}^{(\mathrm{F})}. \tag{36}$$

We could also note that for stationary systems, $\boldsymbol{\nabla}\times\mathbf{E} = 0$, $\mathbf{J} = (c/4\pi)\boldsymbol{\nabla}\times\mathbf{B}$, $J_j = (c/4\pi)\epsilon_{jkl}\partial_k B_l$, so,

$$\begin{aligned}
\mathbf{J}\cdot\mathbf{E}\, r_i &= \frac{c}{4\pi} r_i E_j\, \epsilon_{jkl}\partial_k B_l = \frac{c}{4\pi}\partial_k(r_i\,\epsilon_{jkl}E_j B_l) - \frac{c}{4\pi}\epsilon_{jkl}E_j B_l \partial_k r_i - \frac{c}{4\pi}\epsilon_{jkl} r_i B_l \partial_k E_j \\
&= -\frac{c}{4\pi}\partial_k(r_i\,\epsilon_{kjl}E_j B_l) - \frac{c}{4\pi}\epsilon_{jkl}E_j B_l \delta_{ik} + \frac{c}{4\pi} r_i B_l\,\epsilon_{lkj}\partial_k E_j \\
&= -\frac{c}{4\pi}\partial_k[r_i(\mathbf{E}\times\mathbf{B})_k] + \frac{c}{4\pi}\epsilon_{ijl}E_j B_l + \frac{c}{4\pi} r_i B_l (\boldsymbol{\nabla}\times\mathbf{E})_l \\
&= -\frac{c}{4\pi}\partial_k[r_i(\mathbf{E}\times\mathbf{B})_k] + \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})_i,
\end{aligned} \tag{37}$$

and,

$$P_{\mathrm{EM},i}^{(\mathrm{A})} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, r_i \, d\mathrm{Vol} = -\frac{1}{4\pi c}\int \partial_k[r_i(\mathbf{E}\times\mathbf{B})_k]\, d\mathrm{Vol} + \int \frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi c}\, d\mathrm{Vol}. \tag{38}$$

The volume integral involving ∂_k transforms to a surface integral at infinity, which vanishes for fields that fall off suitably quickly, and we have that,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{A})} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, \mathbf{r}\, d\mathrm{Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})}. \tag{39}$$

However, the moment $\mathbf{r}\times\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}$ of the Aharonov density $\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}$ is identically zero, so the field angular momentum cannot be computed via this form,

$$\mathbf{L}_{\mathrm{EM}}^{(\mathrm{P})} \neq \mathbf{L}_{\mathrm{EM}}^{(\mathrm{A})} = \int \mathbf{r}\times\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,\mathbf{r}\, d\mathrm{Vol} = 0. \tag{40}$$

A Appendix: Rotating Charge Distributions

In quasistatic examples in which all electric charges lie on uniformly charged cylindrical shells that rotate at constant angular velocity $\boldsymbol{\omega}$ about a common axis, the electric and magnetic fields have the forms $\mathbf{E} = E(r_\perp)\,\hat{\mathbf{r}}_\perp$ and $\mathbf{B} = B(r_\perp)\,\hat{\boldsymbol{\omega}}$, where $\mathbf{r}_\perp$ is perpendicular to $\boldsymbol{\omega}$. It is tempting to suppose that the lines of the electric and magnetic fields also rotate with angular velocity $\boldsymbol{\omega}$.

Following Einstein [39], we can identify an effective mass density,

$$\rho_{\text{eff}} = \frac{E^2 + B^2}{8\pi c^2}\,, \tag{41}$$

with the density $(E^2 + B^2)/8\pi$ of energy in the electromagnetic fields. Choosing the origin to be on the axis of rotation, the velocity of rotation of a point at location $\mathbf{r}$ is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. This suggests that we identify densities of momentum and angular momentum in the supposedly rotating electromagnetic fields as,

$$\mathbf{p}_{\text{EM,eff}} \stackrel{?}{=} \rho_{\text{eff}}\mathbf{v} = \frac{E^2 + B^2}{8\pi c^2}\boldsymbol{\omega} \times \mathbf{r}, \quad \text{and} \quad \mathbf{l}_{\text{EM,eff}} \stackrel{?}{=} \mathbf{r} \times \rho_{\text{eff}}\mathbf{v} = \mathbf{r} \times \frac{E^2 + B^2}{8\pi c^2}(\boldsymbol{\omega} \times \mathbf{r}). \tag{42}$$

However, in examples where the field lines extend to large distances, the velocity of the rotating field lines can exceed the speed of light, such that it is implausible to associate them with physical momenta and angular momenta.

Indeed, in such examples, the field angular momentum is infinite when computed via the above hypothesis. And in examples where the field lines lie only within bounded volumes, the field angular momentum so computed does not agree with that calculated via the standard form (16) [40].

Of course, Faraday long ago concluded that the magnetic field lines do not rotate in such examples.[19]

The argument of this Appendix is that it's also best to consider that the electric field lines associated with rotating cylinders of charge do not rotate along with the charge.

B Appendix: Lagrangian Approach (May 2019)

The preceding discussion could be characterized as "bottom up", starting from Maxwell's equations and later arriving at the notion of electromagnetic-field momentum. In contrast, a "top down" Lagrangian/Hamiltonian approach, based on the principle of least action, contains within it the notion of conserved "momenta", as emphasized by Noether [44].

Here, we are concerned with the interaction of charged particles with electromagnetic fields, so the Lagrangian should include that of free particles, free electromagnetic fields, and the interaction between them. This has been considered briefly, for example, in chap. 11 of [45], in somewhat more detail in, for example, [46], and at great length in, for example, [47].

For charged particles (P) interacting with electromagnetic fields (F), the Lagrangian has the general form,

$$\mathcal{L} = \mathcal{L}_P + \mathcal{L}_{\text{F}} + \mathcal{L}_{\text{P}-\text{F}}, \tag{43}$$

where $\mathcal{L}_P$ and $\mathcal{L}_{\text{F}}$ are the Lagrangians for particles and fields, respectively, neglecting their interactions, and $\mathcal{L}_{\text{P}-\text{F}}$ is that of their interactions.

[19]See secs. 218 and 220 of [41], and also sec. 3090 of [42]. For a review, see sec. 2 of [43].

Maxwell discussed the "kinetic" and "potential" energies T and U of the electromagnetic fields in Arts. 630-635 of [48], with the (understated) implication that, in Gaussian units,

$$\mathcal{L}_{\rm F} = T - U = \int \frac{B^2 - E^2}{8\pi}\, d{\rm Vol}, \tag{44}$$

but he did not consider the interaction Lagrangian $\mathcal{L}_{\rm P-F}$. As noted on pp. 376-380 of [45], the canonical momentum associated with the Lagrangian (44) is,

$$\mathbf{P}_{\rm F,free} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}, \tag{45}$$

for the idealized case of "free" fields not coupled to any charges.

In 1892, both Helmholtz [49] and Lorentz [50] considered Lagrangians that included interactions between electric charges and electromagnetic fields, but an interaction Lagrangian $\mathcal{L}_{\rm P-F}$ was not crisply identified.[20]

The interaction Lagrangian,

$$\mathcal{L}_{\rm P-F} = e\frac{\mathbf{v}}{c}\cdot\mathbf{A} - eV, \tag{46}$$

for an electric charge e with mass m and velocity $\mathbf{v}$ in electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ that can be deduced from the potentials V and $\mathbf{A}$, was first displayed by Schwarzschild(1903) [51]. While he did not do so, if one combines this with the (nonrelativistic) free-particle Lagrangian $\mathcal{L}_{\rm P} = mv^2/2$, then the canonical momentum is (p. 357 of [45], p. 921 of [46]),

$$\mathbf{p} = \frac{\partial\mathcal{L}}{\partial\mathbf{v}} = m\mathbf{v} + \frac{e\mathbf{A}}{c}\,. \tag{47}$$

The electromagnetic part of this momentum is, for an electric charge density ρ,

$$\mathbf{P}_{\rm P-F} = \int \frac{\rho\mathbf{A}}{c}\, d{\rm Vol}, \tag{48}$$

which is the form due to Faraday and Maxwell, if the vector potential is in the Coulomb gauge (as tacitly assumed by Schwarzschild).

If we now consider the full Lagrangian (31) for electric charges interacting with electromagnetic fields, the canonical electromagnetic momentum will be the sum of $\mathbf{P}_{\rm P-F}$ and $\mathbf{P}_{\rm F}$. We might naïvely suppose that this is the sum of eqs. (45) and (48), but this is not so. The total field momentum is just that given by eq. (45), although now the fields are not "free".

One way to see this is to recall a key step in the argument on p. 380 of [45], that for free fields, $\int \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{E})\, d{\rm Vol}/4\pi c = 0$, so that this quantity can be added to the (free) field momentum without affecting it, leading (after some "algebra") to the form (45). When charges are present, this term is $\int \rho\mathbf{A}\, d{\rm Vol}/c = \mathbf{P}_{\rm P-F}$, and nonzero in general. But then, when we add it to the free-field momentum to get the total electromagnetic momentum, the "algebra" (not displayed on p. 380 of [45]) again leads to eq. (45), which is now $\mathbf{P}_{\rm P-F} + \mathbf{P}_{\rm F}$.

[20] Of great historical importance is that Lorentz deduced the equation of motion of a charged particle (the Lorentz force law) from his Lagrangian.

This seems quite satisfactory, but we should recall the result of sec. 2.2 above, that for quasistatic examples (where radiation is ignored), the total field momentum is also given by eq. (48). Of course, if we ignore radiation, we are ignoring the possible contribution of free fields to the field momentum, so this momentum is entirely the "interaction" momentum $\mathbf{P}_{\text{P}-\text{F}}$ of eq. (48).

C Appendix: If Magnetic Charges Existed (June 2020)

In this Appendix, we consider various possible forms of the electromagnetic field momentum $\mathbf{P}_{\text{EM}}$ if magnetic charges and currents existed in addition to electric charges and currents. As above, we restrict our attention to static examples.

The total electromagnetic fields can now be written as,

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m, \qquad \mathbf{B} = \mathbf{B}_e + \mathbf{B}_m, \tag{49}$$

where $\mathbf{E}_e$ and $\mathbf{B}_e$ are due only to electric charges and currents, while $\mathbf{E}_m$ and $\mathbf{B}_m$ are due only to emagnetic charges and currents.

Maxwell's equations could then be written as (see, for example Appendix D.4 of [30]),

$$\boldsymbol{\nabla} \cdot \mathbf{E}_e = 4\pi\rho_e, \qquad \boldsymbol{\nabla} \cdot \mathbf{B}_e = 0, \qquad \boldsymbol{\nabla} \times \mathbf{E}_e = -\frac{1}{c}\frac{\partial \mathbf{B}_e}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{B}_e = \frac{1}{c}\frac{\partial \mathbf{E}_e}{\partial t} + \frac{4\pi}{c}\mathbf{J}_e, \tag{50}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E}_m = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{B}_m = 4\pi\rho_m, \quad \boldsymbol{\nabla} \times \mathbf{E}_m = -\frac{1}{c}\frac{\partial \mathbf{B}_m}{\partial t} - \frac{4\pi}{c}\mathbf{J}_m, \quad \boldsymbol{\nabla} \times \mathbf{B}_m = \frac{1}{c}\frac{\partial \mathbf{E}_m}{\partial t}, \tag{51}$$

Furthermore, the electromagnetic fields can be related to potentials according to,[21]

$$\mathbf{E}_e = -\boldsymbol{\nabla} V_e - \frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t}, \quad \mathbf{E}_m = -\boldsymbol{\nabla} \times \mathbf{A}_m, \quad \mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e, \quad \mathbf{B}_m = -\boldsymbol{\nabla} V_m - \frac{1}{c}\frac{\partial \mathbf{A}_m}{\partial t}. \tag{52}$$

The various fields associated with electric charges and currents are related to those associated with magnetic charges and currents by duality relations (see, for example, Appendix D.5 of [30]),

$$\varrho_e \to \varrho_m, \qquad \varrho_m \to -\varrho_e, \qquad \mathbf{J}_e \to \mathbf{J}_m, \qquad \mathbf{J}_m \to -\mathbf{J}_e, \qquad \mathbf{E} \to \mathbf{B}, \qquad \mathbf{B} \to -\mathbf{E}. \tag{53}$$

$$V_e \to V_m, \qquad V_m \to -V_e, \qquad \mathbf{A}_e \to \mathbf{A}_m, \qquad \mathbf{A}_m \to -\mathbf{A}_e. \tag{54}$$

The Poynting form of the field momentum remains,

$$\mathbf{P}_{\text{EM}}^{(\text{P})} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\text{Vol}. \tag{55}$$

Using eq. (49), we can expand this as,

$$\begin{aligned} \mathbf{P}_{\text{EM}}^{(\text{P})} &= \int \frac{\mathbf{E}_e \times \mathbf{B}_e}{4\pi c}\, d\text{Vol} + \int \frac{\mathbf{E}_m \times \mathbf{B}_m}{4\pi c}\, d\text{Vol} + \int \frac{\mathbf{E}_e \times \mathbf{B}_m}{4\pi c}\, d\text{Vol} + \int \frac{\mathbf{E}_m \times \mathbf{B}_e}{4\pi c}\, d\text{Vol} \\ &\equiv \mathbf{P}_{e,e}^{(\text{P})} + \mathbf{P}_{m,m}^{(\text{P})} + \mathbf{P}_{e,m}^{(\text{P})} + \mathbf{P}_{m,e}^{(\text{P})}. \end{aligned} \tag{56}$$

[21]The potentials V_m and $\mathbf{A}_m$ were prehaps first discussed in [52]. Potentials V_m and $\mathbf{A}_m$ were discussed for static fields in eq. (24) of [53], supposing that $\mathbf{E}_e = \boldsymbol{\nabla} \times \mathbf{A}_m$.

C.1 $\mathbf{P}_{e,e}^{(\mathrm{P})} = \int \mathbf{E}_e \times \mathbf{B}_e \, d\mathrm{Vol}/4\pi c$

The result (1) above, which assumed only electric charges and currents, can be rewritten as,

$$\mathbf{P}_{e,e}^{(\mathrm{P})} = \mathbf{P}_{e,e}^{(\mathrm{M})} = \mathbf{P}_{e,e}^{(\mathrm{F})} = \mathbf{P}_{e,e}^{(\mathrm{A})}, \tag{57}$$

where,

$$\mathbf{P}_{e,e}^{(\mathrm{M})} = \int \frac{\varrho_e \mathbf{A}_e^{(\mathrm{C})}}{c} \, d\mathrm{Vol}, \qquad \mathbf{P}_{e,e}^{(\mathrm{F})} = \int \frac{V_e^{(\mathrm{C})} \mathbf{J}_e}{c^2} \, d\mathrm{Vol}, \qquad \mathbf{P}_{e,e}^{(\mathrm{A})} = \int \frac{\mathbf{J}_e \cdot \mathbf{E}_e}{c^2} \mathbf{r} \, d\mathrm{Vol}. \tag{58}$$

C.2 $\mathbf{P}_{m,m}^{(\mathrm{P})} = \int \mathbf{E}_m \times \mathbf{B}_m \, d\mathrm{Vol}/4\pi c$

Applying the duality relations (53)-(54) to eqs. (57)-(58), we have that,

$$\mathbf{P}_{m,m}^{(\mathrm{P})} = \mathbf{P}_{m,m}^{(\mathrm{M})} = \mathbf{P}_{m,m}^{(\mathrm{F})} = \mathbf{P}_{m,m}^{(\mathrm{A})}, \tag{59}$$

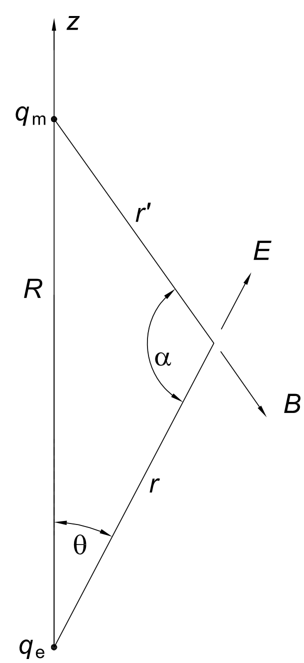
where,

$$\mathbf{P}_{m,m}^{(\mathrm{M})} = \int \frac{\varrho_m \mathbf{A}_m^{(\mathrm{C})}}{c} \, d\mathrm{Vol}, \qquad \mathbf{P}_{m,m}^{(\mathrm{F})} = \int \frac{V_m^{(\mathrm{C})} \mathbf{J}_m}{c^2} \, d\mathrm{Vol}, \qquad \mathbf{P}_{m,m}^{(\mathrm{A})} = \int \frac{\mathbf{J}_m \cdot \mathbf{E}_m}{c^2} \mathbf{r} \, d\mathrm{Vol}. \tag{60}$$

C.3 $\mathbf{P}_{e,m}^{(\mathrm{P})} = \int \mathbf{E}_e \times \mathbf{B}_m \, d\mathrm{Vol}/4\pi c$

This case is self dual in that $e \to m$ and $m \to e$ under duality transformations.

For $\mathbf{P}_{e,m}^{(\mathrm{P})}$ we note that in static examples, the electric field $\mathbf{E}_e$ is only due to the (static) electric charge density ϱ_e. Likewise, the (static) magnetic $\mathbf{B}_m$ is only due to the (static) magnetic charge density ϱ_m. Then, the field momentum $P_{e,m}^{(\mathrm{P})}$ is the sum of the field momenta of all pairs of electric and magnetic charges (at rest) in the system. This momentum was considered in 1904 by J.J. Thomson [24], who found it to be zero.



Suppose the electric charge q_e is at the origin, and the magnetic charge q_m is at distance R away along the positive z-axis, as shown in the figure above. Then, the electromagnetic momentum density $\mathbf{E}_e \times \mathbf{B}_m/4\pi c$ circulates around the z-axis and is independent of azimuth, such that the total electromagnetic momentum $\mathbf{P}_{\mathrm{EM}}$ is zero,

$$\mathbf{P}_{\mathrm{EM}}(q_e, q_m) = \int \mathbf{p}_{\mathrm{EM}}(q_e, q_m) \, d\mathrm{Vol} = 0. \tag{61}$$

Hence, the total electromagnetic-field momentum $\mathbf{P}_{e,m}^{(\mathrm{P})}$ for any configuration of static magnetic poles and electric charges is zero, being the sum of the momenta of all pairs of such particles,[22]

$$\mathbf{P}_{e,m}^{(\mathrm{P})} = 0. \tag{62}$$

C.4 $\mathbf{P}_{m,e}^{(\mathrm{P})} = \int \mathbf{E}_m \times \mathbf{B}_e \, d\mathrm{Vol}/4\pi c$

This case is also self dual in that $m \to e$ and $e \to m$ under duality transformations.

Recalling eqs. (50)-(52), for static fields $\boldsymbol{\nabla} \cdot \mathbf{E}_m = 0$, $\boldsymbol{\nabla} \times \mathbf{E}_m = -4\pi \mathbf{J}_m/c$, and $\mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e$, so we can write,

$$\begin{aligned}
\mathbf{P}_{m,e}^{(\mathrm{P})} &= \int \frac{\mathbf{E}_m \times \mathbf{B}_e}{4\pi c} \, d\mathrm{Vol} \\
&= \int \frac{\mathbf{E}_m \times (\boldsymbol{\nabla} \times \mathbf{A}_e^{(\mathrm{C})}) + \mathbf{A}_e^{(\mathrm{C})} \times (\boldsymbol{\nabla} \times \mathbf{E}_m)}{4\pi c} \, d\mathrm{Vol} + \int \frac{\mathbf{A}_e^{(\mathrm{C})} \times \mathbf{J}_m}{c^2} \, d\mathrm{Vol} \\
&= \int \frac{\mathbf{A}_e^{(\mathrm{C})} \times \mathbf{J}_m}{c^2} \, d\mathrm{Vol} + \int \frac{\boldsymbol{\nabla}(\mathbf{E}_m \cdot \mathbf{A}_e^{(\mathrm{C})}) - (\mathbf{E}_m \cdot \boldsymbol{\nabla})\mathbf{A}_e^{(\mathrm{C})} - (\mathbf{A}_e^{(\mathrm{C})} \cdot \boldsymbol{\nabla})\mathbf{E}_m}{4\pi c} \, d\mathrm{Vol} \\
&= \int \frac{\mathbf{A}_e^{(\mathrm{C})} \times \mathbf{J}_m}{c^2} \, d\mathrm{Vol} + \int \frac{\mathbf{E}_m \cdot \mathbf{A}_e^{(\mathrm{C})}}{4\pi c} \, d\mathbf{Area} = \int \frac{\mathbf{A}_e^{(\mathrm{C})} \times \mathbf{J}_m}{c^2} \, d\mathrm{Vol}
\end{aligned} \tag{63}$$

assuming that all fields fall off suffciently quickly at large distances such that the surface integrals are negligible, using $\boldsymbol{\nabla} \cdot \mathbf{E}_m = 0$ to see that,

$$\int (\mathbf{E}_m \cdot \boldsymbol{\nabla})\mathbf{A}_e^{(\mathrm{C})} \, d\mathrm{Vol} = \int \mathbf{A}_e^{(\mathrm{C})}(\mathbf{E}_m \cdot d\mathbf{Area}) - \int \mathbf{A}_e^{(\mathrm{C})}(\boldsymbol{\nabla} \cdot \mathbf{E}_m) \, d\mathrm{Vol} = 0, \tag{64}$$

and similarly $\int (\mathbf{A}_e^{(\mathrm{C})} \cdot \boldsymbol{\nabla})\mathbf{E}_m \, d\mathrm{Vol} = 0$, noting that in the Coulomb gauge, vector potentials obey $\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{C})} = 0$.[23]

For what it's worth, with $\boldsymbol{\nabla} \cdot \mathbf{B}_e = 0$, and $\boldsymbol{\nabla} \times \mathbf{B}_e = 4\pi \mathbf{J}_e/c$ for static fields, and $\mathbf{E}_m = -\boldsymbol{\nabla} \times \mathbf{A}_m$, we could also write,

$$\begin{aligned}
\mathbf{P}_{m,e}^{(\mathrm{P})} &= -\int \frac{\mathbf{B}_e \times \mathbf{E}_m}{4\pi c} \, d\mathrm{Vol} \\
&= \int \frac{\mathbf{B}_e \times (\boldsymbol{\nabla} \times \mathbf{A}_m^{(\mathrm{C})}) + \mathbf{A}_m^{(\mathrm{C})} \times (\boldsymbol{\nabla} \times \mathbf{B}_e)}{4\pi c} \, d\mathrm{Vol} - \int \frac{\mathbf{A}_m^{(\mathrm{C})} \times \mathbf{J}_e}{c^2} \, d\mathrm{Vol} \\
&= \int \frac{\mathbf{J}_e \times \mathbf{A}_m^{(\mathrm{C})}}{c^2} \, d\mathrm{Vol} + \int \frac{\boldsymbol{\nabla}(\mathbf{B}_e \cdot \mathbf{A}_m^{(\mathrm{C})}) - (\mathbf{B}_e \cdot \boldsymbol{\nabla})\mathbf{A}_m^{(\mathrm{C})} - (\mathbf{A}_m^{(\mathrm{C})} \cdot \boldsymbol{\nabla})\mathbf{B}_e}{4\pi c} \, d\mathrm{Vol} \\
&= \int \frac{\mathbf{J}_e \times \mathbf{A}_m^{(\mathrm{C})}}{c^2} \, d\mathrm{Vol} + \int \frac{\mathbf{B}_e \cdot \mathbf{A}_m^{(\mathrm{C})}}{4\pi c} \, d\mathbf{Area} = \int \frac{\mathbf{J}_e \times \mathbf{A}_m^{(\mathrm{C})}}{c^2} \, d\mathrm{Vol}
\end{aligned} \tag{65}$$

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