An Off-Axis Neutrino Beam

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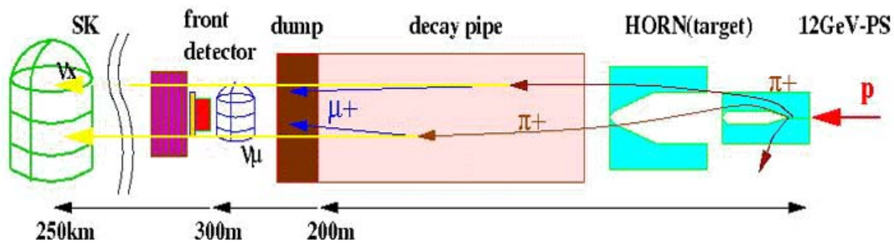
1 Problem

A typical high-energy neutrino beam is made from the decay of π mesons that have been produced in proton interactions on a target, as sketched in the figure below.



Suppose that only positively charged particles are collected by the "horn". The main source of neutrinos is then the decay $\pi^+ \to \mu^+\nu_\mu$.

1. Give a simple estimate of the relative number of other types of neutrinos than ν_μ in the beam.
2. If the decay pions have energy $E_\pi \gg m_\pi$, what is the characteristic angle θ_C of the decay neutrinos with respect to the direction of the π^+?
3. If a neutrino is produced with energy $E_\nu \gg m_\pi$, what is the maximum angle $\theta_{\rm max}(E_\nu)$ between it and the direction of its parent pion (which can have any energy)? What is the maximum energy E_ν at which a neutrino can be produced in the decay of a pion if it appears at a given angle θ with respect to the pion's direction?

   Parts 4 and 6 explore consequences of the existence of these maxima.
4. Deduce an analytic expression for the energy-angle spectrum $d^2N/dE_\nu d\Omega$ for neutrinos produced at angle $\theta \le \theta_C$ to the proton beam. You may suppose that $E_\nu \gg m_\pi$, that the pions are produced with an energy spectrum $dN/dE_\pi \propto (E_p - E_\pi)^5$, where E_p is the energy of the proton beam, and that the "horn" makes all pion momenta parallel to that of the proton beam.
5. At what energy $E_{\nu,\rm peak}$ does the neutrino spectrum peak for $\theta = 0$?
6. Compare the characteristics of a neutrino beam at $\theta = 0$ with an off-axis beam at angle θ such that $E_{\nu,\rm max}(\theta)$ is less than $E_{\nu,\rm peak}(\theta = 0)$.

Facts: m_π = 139.6 MeV/c^2, τ_π = 26 ns, m_μ = 105.7 MeV/c^2, τ_μ = 2.2 μs. In this problem, neutrinos can be taken as massless.

2 Solution

In this solution we use units where $c = 1$.

1. Besides the ν_μ from the decay $\pi^+ \to \mu^+\nu_\mu$, the beam will also contain $\bar{\nu}_\nu$ and ν_e from the subsequent decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$. Both of these decays occur (primarily) in the "decay pipe" shown in the figure. As both the pions and muons of relevance are relativistic in this problem, they both have about the same amount of time to decay before they are absorbed in the "dump". Hence, the ratio of number of muon decays to pion decays is roughly the inverse of the ratio of their lifetimes, *i.e.*, about 0.01. Our simple estimate is therefore,

$$\frac{N_{\nu_e}}{N_{\nu_\mu}} = \frac{N_{\bar{\nu}_\mu}}{N_{\nu_\mu}} \approx 0.01. \tag{1}$$

   Experts may note that an additional source of ν_e is the decay $\pi^+ \to e^+\nu_e$ at the level of 10^{-4}. Also, K^+ mesons will be produced by the primary proton interaction at a rate about 10% that of π^+. About 65% of K^+ decays are to $\mu^+\nu_\mu$, which add to the main ν_μ beam, but about 5% of the decays are to $\pi^+\pi^0\nu_e$, which increases the ν_e component of the neutrino beam by about $0.1 \times 0.05 = 0.005$.

2. Parts 2-6 of this problem are based on the kinematics of charged-pion decay, which are similar to those of neutral-pion decay, $\pi^0 \to \gamma\gamma$ [1].[1]

   Experts may guess that the characteristic angle of the decay neutrinos with respect to the parent pion is $\theta_C = 1/\gamma_\pi = m_\pi/E_\pi$. The details of the derivation are needed in part 3.

   We now consider a generic 2-body decay $a \to b + c$ of a spin-0 particle a of nonzero mass m_a. Such a decay in isotropic in the rest frame of particle a, so it is useful to begin our analysis in this frame in which quantities will be labeled with the superscript $\star$, and then transform the results to the lab frame.

   Energy-momentum conservation for the decay can be written as the 4-vector relation,

$$a_\mu = b_\mu + c_\mu, \tag{2}$$

   where the squares of the 4-vectors are the particle masses, $a^2 = a_\mu a^\mu = a_0^2 - \mathbf{a}^2 = m_a^2$, *etc.*

   If we are not concerned with details of particle b, it is convenient to rewrite eq. (2) as,

$$b_\mu = a_\mu - c_\mu, \tag{3}$$

   and square this to find,

$$m_b^2 = m_a^2 + m_c^2 - 2a_\mu c^\nu. \tag{4}$$

   In the rest frame of particle a, its 4-vector can be written,

$$a_\mu = (m_a, 0, 0, 0). \tag{5}$$

[1] For a pedagogic discussion of 2-body decay for parents of fixed lab energy, see [2].

Taking the z axis to be the direction of particle a in the lab frame, the 4-vector of the particle c in the rest frame of particle a can be written as,

$$c_\mu = (E_c^\star, P_c^\star \sin\theta^\star, 0, P_c^\star \cos\theta^\star). \tag{6}$$

The 4-vector product $a_\mu c^\mu = a_0 c_0 - \mathbf{a}\cdot\mathbf{c}$ is therefore,

$$a_\mu c^\mu = m_a E_c^\star. \tag{7}$$

Hence, from eq. (4) the energy of the particle c in the rest frame of particle a is,

$$E_c^\star = \frac{m_a^2 + m_c^2 - m_b^2}{2m_a}, \tag{8}$$

which implies that $E_\nu^\star$ ($\approx P_\nu^\star$) = 29.8 MeV ($\approx 0.2m_\pi$) for the decay $\pi^+ \to \mu^+\mu_\nu$, using the facts on p. 1. Also, $P_\mu^\star = P_\nu^\star = 29.8$ MeV, and $E_\mu^\star = 109.8$ MeV.

We can now transform the 4-vector (6) of particle c to the lab frame, using the Lorentz boost $\gamma_a = E_a/m_2$, with $\beta_a = P_a/E_a$,

$$\begin{aligned} c_\mu &= (E_c, P_c \sin\theta_c, 0, P_c\cos\theta_c) \\ &= (\gamma_a(E_c^\star + \beta_a P_c^\star \cos\theta_c^\star), P_c^\star \sin\theta_c^\star, 0, \gamma_a(P_c^\star\cos\theta_c^\star + \beta_a E_c^\star)). \end{aligned} \tag{9}$$

A relation for the angle θ_c in the lab frame between the direction of particle c and that of its parent particle a can be obtained from the 1- and 3-components of eq. (9),

$$\tan\theta_c = \frac{P_c^\star \sin\theta_c^\star}{\gamma_a(E_c^\star\beta_a + P_c^\star\cos\theta_c^\star)}. \tag{10}$$

The maximum of angle θ_c in the lab frame is related by $d\tan\theta_c/d\theta_c^\star = 0$, *i.e.*, $\cos\theta_c^\star = -P_c^\star/\beta_a E_c^\star \approx -P_c^\star/E_c^\star$, where the approximation holds when the parent particle a is relativistic in the lab frame. In the decay $\pi^+ \to \mu^+\nu_\mu$, the maximum lab angle of the neutrino is associated with $\cos\theta_\nu^\star \approx -1$, *i.e.*, $\theta_{\nu,\max} \approx 180°$, while for the μ^+ the maximum is associated with $\cos\theta_\mu^\star \approx -P_\mu^\star/E_\mu^\star$, $\sin\theta_\mu^\star \approx m_\mu/E_\mu^\star$, and $\theta_{\mu,\max} \approx P_\mu^\star/\gamma_\pi m_\mu \ll 1$.

A characteristic angle of the decay in the lab frame is that associated with $\theta_c^\star = 90°$, Thus,

$$\tan\theta_{c,C} = \frac{P_c^\star}{\gamma_a\beta_a E_c^\star} \approx \frac{P_c^\star}{\gamma_a E_c^\star}, \tag{11}$$

where the approximation holds when the parent particle is relativistic in the lab frame. In the decay $\pi^+ \to \mu^+\nu_\mu$, $\tan\theta_{\nu,C} \approx \theta_C \approx 1/\gamma_a \ll 1$ for the neutrino, while $\theta_{\mu,C} \approx 0.27/\gamma_\pi$ for the μ^+ (which is very close to $\theta_{\mu,\max}$ as $E_\mu^\star \approx m^\mu$). That is, the μ^+ typically is produced at a smaller angle to the π^+ than is the neutrino.

3. We now consider the lab angle (10) between particle c and its parent particle a. Our interest here is in the production of a beam of high-energy particles, so we restrict the

discussion to particles with $E_c \gg m_a$, which implies that $E_a \gg m_a$ also, so $\gamma_a \gg 1$ and $\beta_a \approx 1$. Then, we can write,

$$\tan\theta_c \approx \frac{P_c^\star \sin\theta_c^\star}{\gamma_a(E_c^\star + P_c^\star\cos\theta_c^\star)} \approx \frac{P_c^\star \sin\theta_c^\star}{E_c}, \tag{12}$$

recalling the time component of eq. (9). Since $\sin\theta^\star$ cannot exceed unity, we see that there is a maximum lab angle θ relative to the direction of the particle a at which a particle c with high energy E_c can appear, namely

$$\theta_{c,\max} \approx \frac{P_c^\star}{E_c}, \tag{13}$$

which is small for $m_z \ll E_c$. In the decay $\pi^+ \to \mu^+\nu_\mu$, $P_\nu^\star = P_\mu^\star \approx 30$ MeV.

If instead, the angle θ is given, eq. (12) also tells us that,

$$E_c \approx \frac{P_c^\star \sin\theta_c^\star}{\tan\theta_c} \le \frac{P_c^\star}{\tan\theta_c}. \tag{14}$$

4. We desire the energy-angle spectrum $d^2N/dE_c\,d\Omega_c$ in terms of the laboratory quantities E_c, θ_c and ϕ_c.[2] We expect (for the decay of a spin-0 particle) that the spectrum is uniform in the azimuthal angle ϕ (as well as being flat in $\cos\theta_c^\star$ in the rest frame of the parent particle).

   In the rest of this note, we suppose the parent particle is a charged pion, produced by a proton beam, and that the energy spectrum of the pions has the approximately form

$$\frac{dN}{dE_\pi} \propto (E_p - E_\pi)^5. \tag{15}$$

   Hence, we seek the transformation,

$$\frac{d^2N}{dE_c d\Omega_c} \propto \frac{d^2N}{dE_c d\cos\theta_c} = \frac{d^2N}{dE_\pi d\cos\theta_c^\star} J(E_\pi, \cos\theta_c^\star; E_c, \cos\theta_c) \propto (E_p - E_\pi)^5 J, \tag{16}$$

   where the Jacobian is given by,

$$J(E_\pi, \cos\theta_c^\star; E_c, \cos\theta_c) = \begin{vmatrix} \frac{\partial E_\pi}{\partial E_c} & \frac{\partial\cos\theta_c^\star}{\partial E_c} \\ \frac{\partial E_\pi}{\partial\cos\theta_c} & \frac{\partial\cos\theta_c^\star}{\partial\cos\theta_c} \end{vmatrix}. \tag{17}$$

   The "exact" form of the Jacobian is somewhat lengthy, so we will simplify to the extent we can by noting that when $E_\nu \gg m_\pi$, the parent pion has $E_\pi \gg m_\pi$ also, and so $\beta_\pi \approx 1$. Also, part 3 tells us that θ_c is very small for any value of $\theta_c^\star$ for particle with high energy in the lab frame.

[2] As shown in part 2 of [2], dN/dE_c is constant, with limiting energies $E_a(E_c^\star \pm P_c^\star)/2m_a$, for a given energy of the parent particle a . For the decay $\pi^+ \to \mu^+\nu_\mu$ we have $E_\nu^\star$ ($\approx P_\nu^\star = P_\mu^\star$) = 29.8 MeV $\approx 0.21 m_{\pi^+}$, $E_\mu^\star = 109.8$ MeV $\approx 0.79 m_{\pi^+}$, so the laboratory distributions of the neutrino and muons energies are flat with $0 < E_\nu < 0.42 E_\pi$ and $0.58 E_\pi < E_\mu < E_\pi$ (for monoenergetic parent pions).

We already have relation (12) between E_c, $\tan\theta_c$ and $\sin\theta_c^\star$, so we can write,

$$\cos\theta_c^\star = \sqrt{1-\sin^2\theta_c^\star} \approx \sqrt{1-\frac{E_c^2}{P_c^{\star 2}}\tan^2\theta_c} = \sqrt{1-\frac{E_c^2}{P_c^{\star 2}}\left(\frac{1}{\cos^2\theta_c}-1\right)}. \tag{18}$$

Thus,

$$\frac{\partial\cos\theta_c^\star}{\partial E_c} \approx -\frac{\frac{E_c}{P_c^{\star 2}}\tan^2\theta_c}{\sqrt{1-\frac{E_c^2}{P_c^{\star 2}}\tan^2\theta_c}} \approx -\frac{E_c\theta_c^2}{P_c^{\star 2}\cos\theta_c^\star}, \tag{19}$$

for small θ_c, and,

$$\frac{\partial\cos\theta_c^\star}{\partial\cos\theta_c} \approx \frac{\frac{E_c^2}{P_c^{\star 2}\cos^3\theta_c}}{\sqrt{1-\frac{E_c^2}{P_c^{\star 2}}\tan^2\theta_c}} \approx \frac{E_c^2}{P_c^{\star 2}\cos\theta_c^\star}. \tag{20}$$

We can also use time components of eq. (9) to write,

$$\gamma_\pi = \frac{E_\pi}{m_\pi} = \frac{E_c}{P_c^\star(1+\beta_\pi\cos\theta^\star)} \approx \frac{E_c}{P_c^\star(1+\cos\theta_c^\star)}. \tag{21}$$

Hence,

$$\frac{\partial E_\pi}{\partial E_c} \approx \frac{m_\pi}{P_c^\star(1+\cos\theta_c^\star)} - \frac{m_\pi E_c}{P_c^\star(1+\cos\theta_c^\star)^2}\frac{\partial\cos\theta_c^\star}{\partial E_c} \approx \frac{E_\pi}{E_c} + \frac{E_\pi^2\theta_c^2}{m_\pi P_c^\star\cos\theta_c^\star}, \tag{22}$$

and,

$$\frac{\partial E_\pi}{\partial\cos\theta_c} \approx -\frac{m_\pi E_c}{P_c^\star(1+\cos\theta_c^\star)^2}\frac{\partial\cos\theta_c^\star}{\partial\cos\theta_c} \approx -\frac{E_\pi^2 E_c}{m_\pi P_c^\star\cos\theta_c^\star}. \tag{23}$$

The Jacobian (17) is therefore,

$$J \approx \begin{vmatrix} \frac{E_\pi}{E_c} + \frac{E_\pi^2\theta_c^2}{m_\pi P_c^\star\cos\theta_c^\star} & -\frac{E_c\theta_c^2}{P_c^{\star 2}\cos\theta_c^\star} \\ -\frac{E_\pi^2 E_c}{m_\pi P_c^\star\cos\theta_c^\star} & \frac{E_c^2}{P_c^{\star 2}\cos\theta_c^\star} \end{vmatrix} = \frac{E_\pi E_c}{P_c^{\star 2}\cos\theta_c^\star}, \tag{24}$$

and hence the energy-angle spectrum can be written, from eq. (16) as, for small θ_c,

$$\frac{d^2N}{dE_c d\cos\theta_c} \propto (E_p - E_\pi)^5\frac{E_\pi E_c}{\cos\theta_c^\star}. \tag{25}$$

While eq. (25) has the same form, $\propto E_c/\cos\theta_c^\star$, for both the neutrino and the muon, the forms are, in general, different as functions of E_c and θ_c. However, for relativistic muons and small lab angles, we see from eqs. (18)-(20) that $\cos\theta_c^\star(E_c,\theta_C)$ is very similar for muons and neutrinos, since $P_\mu^\star = P_\nu^\star$, so the energy-angle spectra are almost identical for both particles.

Because the factor $\cos\theta_c^\star$ in the denominator of the Jacobian can go to zero, it is possible that the flux of particle c is higher for nonzero values of the lab angle θ_c.

5. On the axis, $\theta_c = 0$, $\theta_c^\star = 0$, and $E_\pi \approx m_\pi E_c/2P_c^\star \approx 2.5E_c$ according to eq. (21), and recalling that $P_\nu^\star = P_\mu^\star \approx 0.2m_\pi$. In this case, the spectrum (25) is,

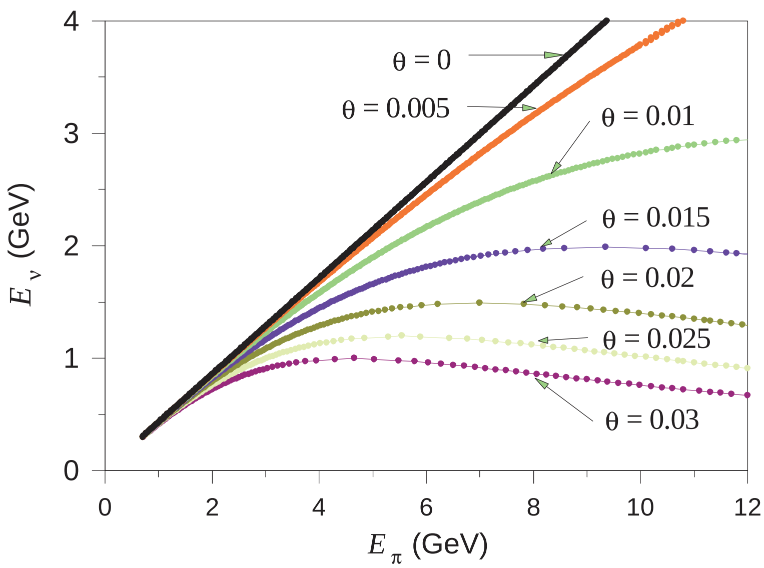
$$\frac{d^2N(\theta_c = 0)}{dE_c\, d\cos\theta_c} \propto (E_p - 2.5E_c)^5\, E_c^2. \tag{26}$$

The peak of the on-axis spectrum occurs at,

$$E_{c,\mathrm{peak}} \approx \frac{2E_p}{7}\,, \tag{27}$$

for both neutrinos and for muons.

6. For an off-axis neutrino beam (at a nonzero value of angle θ) we must evaluate the spectrum (25) using relations (18) and (21). This is readily done numerically. For example, a plot of the pion energy E_π needed to produce a neutrino of energy E_ν at various angles θ is shown below.

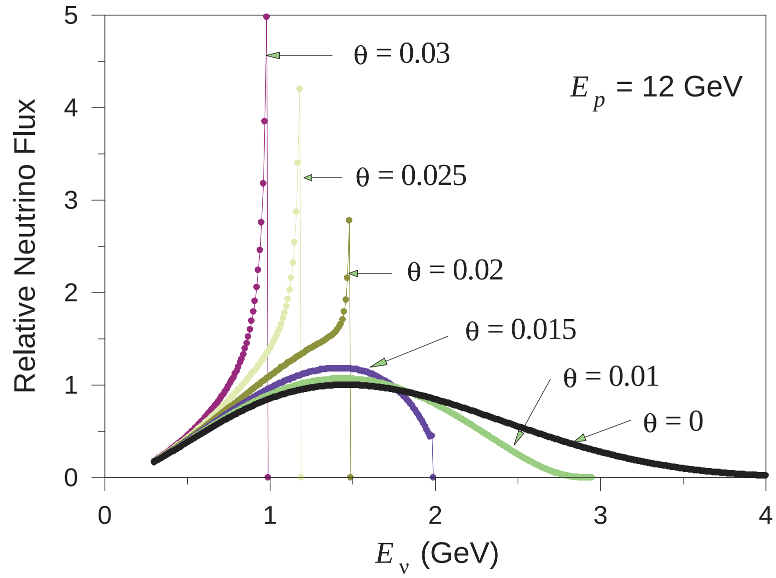


As expected from part 3, we see that for a given angle θ, there is a maximum possible neutrino energy, and as the neutrino energy approaches this value, a large range of pion energies contributes to a small range of neutrino energies. This will result in an enhancement of the neutrino spectrum. If we desire the enhancement at a particular neutrino energy, we should look for the neutrinos close to the angle $\theta_{\max}$ given in eq. (13), which is independent of the proton/pion energy.

A numerical evaluation of the neutrino spectrum (25) for several values of angle θ with respect to the proton/pion beam is shown on the next page.

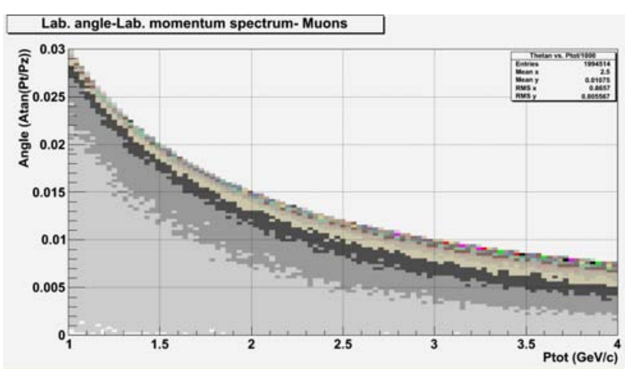
We see that the spectrum of neutrinos at a nonzero angle is peaked at a lower energy, and is narrower, than that at zero degrees, due to the existence of a maximum possible neutrino energy (14) in decays at a given angle to the direction of the parent pion. This effect is especially prominent when $E_{\nu,\max}(\theta) \approx (30\ \mathrm{MeV})/\theta$ is less than $E_{\nu,\mathrm{peak}}(\theta = 0)$, as then there is a substantial rate of higher energy pions all of which decay into a narrow band of neutrino energies at this angle.

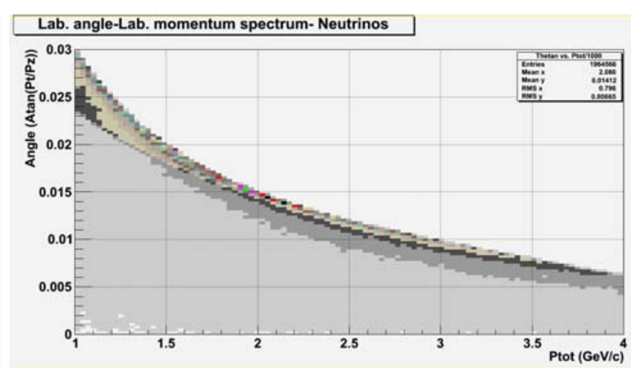
The spectral narrowing of an off-axis neutrino beam remains in more complete calculation [3, 4] that includes the nonzero transverse momenta of the pions before and after passing through the "horn", although the spectrum will not have such hard edges, and the favored angle-energy combination is $\theta \approx (50\ \text{MeV})/E_\nu$.



In sum, the existence of a maximum energy for neutrinos that decay at a given angle to their parent pions implies that many different pion energies contribute to the this neutrino energy, which enhances the neutrino spectrum at this angle-energy combination, $\theta \approx (30\text{-}50\ \text{MeV})/E_\nu$.

As noted after eq. (25), the angle-energy spectra from pion decay are very similar for neutrinos and relativistic muons at small angles, so the above two figures apply approximately for muons (on changing E_ν to E_μ). The two figures below are from a Monte Carlo simulation (by N. Charitonidis), for parent pions with a uniform energy distribution between 1 and 12 GeV, for which $E_{\nu,\max} = 5$ GeV and $E_{\mu,\min} = 0.6$ GeV.





3 Acknowledgment

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