Maximum Height of Mud Thrown From a Wheel

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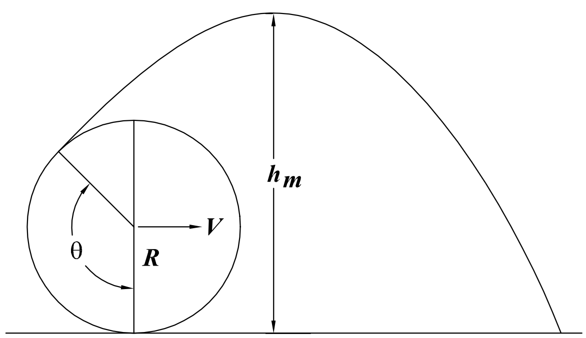
1 Problem

What is the maximum height that mud can attain if it is thrown off a wheel of radius R moving at velocity V?

2 Solution

The mud that reaches the greatest height will be thrown from the rim of the wheel. The initial velocity of the mud will be tangential to the wheel. However, if suffices to analyze only the vertical motion of the mud.

For the sake of this problem the angle θ will be measured from the point of contact of the wheel with the road, as shown in the figure.



The angular velocity $\omega = d\theta/dt$ of the wheel about its axis is V/R.

The vertical height y above the road of a point on the rim of the wheel at angle θ is

$$y = R(1 - \cos\theta).$$

The initial vertical velocity is

$$\frac{dy}{dt} = R\sin\theta\frac{d\theta}{dt} = V\sin\theta.$$

After the mud leaves the wheel it rises an additional height d that can be deduced from the formula $V_f^2 = V_0^2 - 2gd$ (conservation of energy), where g is the acceleration to gravity. We find

$$d = \frac{V^2\sin^2\theta}{2g},$$

using $V_f = 0$ and $V_0 = dy/dt$ as above. The horizontal velocity of the mud is constant once it has left the wheel, and so can be ignored in the analysis of d.

Adding in the initial height of the mud, its maximum height is

$$h = \frac{V^2 \sin^2 \theta}{2g} + R(1 - \cos \theta).$$

Now we maximize h by setting its derivative with respect to θ equal to 0:

$$\frac{dh}{d\theta} = \frac{V^2 \sin \theta \cos \theta}{g} + R \sin \theta = 0.$$

The extremes of h occur when θ is equal to 0 or π or when

$$\cos \theta = -\frac{gR}{V^2}.$$

The mud leaves the wheel at zero height when θ is equal to 0, and stays at zero height because $\theta = 0$ corresponds to the point of contact of the wheel with the road, which point is instantaneously at rest.

At $\theta = \pi$ the mud is thrown horizontally from initial height $h = 2R$, reaching no additional height. This is the maximum height attainable when $V^2 < gR$.

When $V^2 \geq gR$, $\cos \theta = -gR/V^2$ is defined and the mud flies to a height greater than $2R$. Plugging in to the formula for the maximum height h, we get:

$$h_m = R \left[1 + \frac{1}{2} \left(\frac{gR}{V^2} + \frac{V^2}{gR} \right) \right].$$

Note that when $V^2 = gR$ then $h_m = 2R$, in agreement with the solution when $\theta = \pi$.