

# Maximal Gravity at the Surface of an Asteroid

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## 1 Problem

What is the shape of an asteroid of given uniform density and given total mass such that the force of gravity is maximal for one point on its surface? Compare the maximal gravity with that of a sphere of the same mass and density.

*This problem is from Yakov Kantor's Physics Quiz site, Feb. 2002,*

<http://www.tau.ac.il/~kantor/QUIZ/>

See also prob. 15 at <http://webdev.physics.harvard.edu/academics/undergrad/problems.html>

## 2 Solution

If the asteroid is spherical with radius  $a$  and mass  $M$ , the force of gravity on a test mass  $m$  is

$$F = \frac{GMm}{a^2}, \quad (1)$$

everywhere on the surface, where  $G$  is Newton's constant of gravitation.

Can some other shape of the asteroid result in a larger force? Let the test mass be at the origin, which is on the surface of the asteroid, and define the  $z$  axis to be along the direction of the desired maximal force on the test mass. It is "obvious" that this  $z$  axis is an axis of symmetry of the asteroid: if the asteroid were not symmetric about the  $z$  axis, the  $z$  component of the vector force  $Gmm'\hat{\mathbf{r}}/r^2$  would be increased by moving material from larger to smaller transverse distances from the axis, until the asteroid is axially symmetric.

We seek the functional form  $x = x(z)$ , where  $0 < z < z_0$ , that generates a surface of revolution about the  $z$  axis, the surface of the asteroid, that maximizes the (axial) force on test mass  $m$  at the origin. The axial force due to a ring of extent  $dx dz$  that passes through point  $(x, z)$  is

$$Gm\rho 2\pi x dx dz \frac{1}{x^2 + z^2} \frac{z}{\sqrt{x^2 + z^2}}, \quad (2)$$

so the total force is

$$F = 2\pi Gm\rho \int_0^{z_0} dz \int_0^{x(z)} x dx \frac{z}{(x^2 + z^2)^{3/2}} = \frac{3GMm}{2a^3} \int_0^{z_0} dz \left( 1 - \frac{z}{(x^2(z) + z^2)^{1/2}} \right), \quad (3)$$

where we suppose that the asteroid has the same density as a sphere of radius  $a$  and mass  $M$ , *i.e.*,  $\rho = 3M/4\pi a^3$ . Hence, the volume of the asteroid is constrained to have value

$$V = \pi \int_0^{z_0} dz x^2(z) = \frac{4\pi a^3}{3}. \quad (4)$$

This suggests use of the calculus of variations, although there is the complication that the value of the endpoint  $z_0$  is unknown.

In principle, the shape  $x(z)$  might involve nonzero values at  $z = 0$  and  $z_0$ , *i.e.*, planar surfaces that bound the asteroid. However, we readily convince ourselves that such a geometry could not maximize the  $z$  component of the gravitational force; material in the plane  $z = 0$  contributes nothing to the  $z$  component of the force, while material at  $x > 0$  in the plane  $z = z_0$  would be more useful if it were moved to a smaller  $z$ . Thus, we understand that  $x(0) = 0 = x(z_0)$  for the desired shape function.

## 2.1 An Intuitive Solution

However, we first consider the related question: what is the shape of a thin shell of matter such that all points on the shell contribute the same axial force on our test mass at the origin? For a shell described by  $x(z)$ , the answer is already contained in the form of integral (3). Namely, we desire that

$$\frac{z}{(x^2(z) + z^2)^{3/2}} = \text{const.} = \frac{1}{z_0^2}, \quad (5)$$

where we have evaluated the constant at the intercept  $z_0$  of the shell with the  $z$  axis. On rearranging eq. (5), we have

$$x^2(z) = z^{2/3} z_0^{4/3} - z^2. \quad (6)$$

It is noteworthy that this form also intersects the  $z$  axis at the origin, so is a satisfactory candidate for the shape of the asteroid (whose surface surely must touch the test mass at the origin). Inserting this form in eq. (4) for the volume, we find that  $z_0 = \sqrt[3]{5}a = 1.7a$ . Using eq. (6) in eq. (3) for the axial force, we find

$$F = \frac{3GMmz_0}{5a^3} = \frac{3\sqrt[3]{5}GMm}{5a^2} = 1.026 \frac{GMm}{a^2}. \quad (7)$$

This argument does not prove that eq. (6) describes the shape of the maximal gravity asteroid, but it does show that the shape is not a sphere. The simplicity of the argument suggests that the result is in fact correct, but a proof of this is still desired.

## 2.2 Solution via the Calculus of Variations

We wish to find the form  $x(z)$  that maximizes integral  $F$  of eq. (3) subject to the constraint that integral  $V$  of eq. (4) has the stated value, while permitting the endpoint  $z_0$  to vary. To proceed via the calculus of variations, we consider the integral

$$I = \int_0^{z_0} f(z, x(z)) dz, \quad (8)$$

where

$$f = 1 - \frac{z}{(x^2 + z^2)^{1/2}} - \lambda x^2, \quad (9)$$

which combines the integrands of eqs. (3) and (4) using the Lagrange multiplier  $\lambda$ .

For a fixed endpoint  $z_0$ , integral  $I$  is at an extremum provided  $f$  obeys the Euler-Lagrange equations

$$\frac{\partial f}{\partial x} = \frac{d}{dz} \frac{\partial f}{\partial x'}. \quad (10)$$

Since  $f$  does not depend on  $x' = dx/dz$  in this problem, we simply have

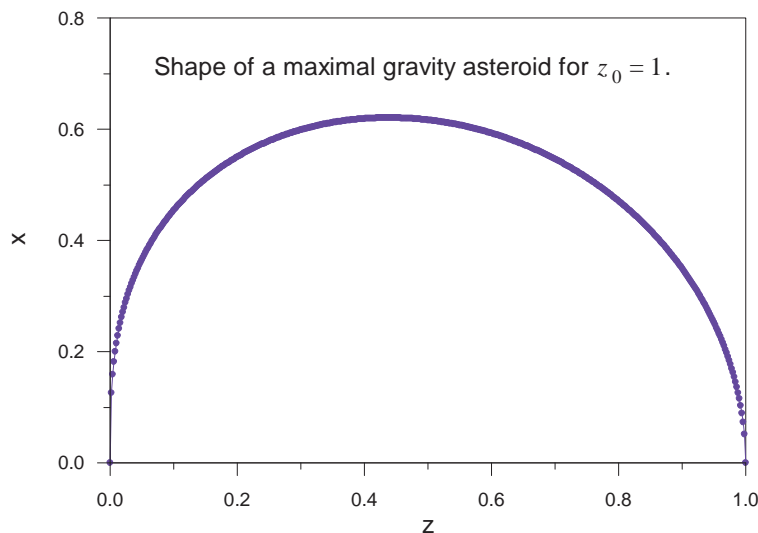
$$0 = \frac{\partial f}{\partial x} = \frac{xz}{(x^2 + z^2)^{3/2}} - 2\lambda x, \quad (11)$$

or

$$\frac{z}{(x^2 + z^2)^{3/2}} = 2\lambda. \quad (12)$$

As in eq. (5), the constant  $2\lambda$  is clearly  $1/z_0^2$ , since  $x(z_0) = 0$ . Again, we obtain eq. (6) as the shape of the asteroid. And again, the constant  $z_0$  (*i.e.*, the multiplier  $\lambda$ ) must be set to  $\sqrt[3]{5}a$  to satisfy the volume constraint (4), which completes the solution.

The shape (6) of the maximal gravity asteroid is shown below. It is noteworthy that there is no cusp at either  $z = 0$  or  $z_0$ .



### 2.3 A Mathematical Footnote

We have not directly included in our calculus of variations the fact that the endpoint  $z_0$  is free to move along the  $z$  axis, *i.e.*, along the curve  $g(z) = 0$ . In, for example, sec. 4.3 of [1] we read that the freedom to vary the endpoint in the maximization of integral  $I$  leads to the additional relation

$$f + (g' - x') \frac{\partial f}{\partial x'} = 0, \quad (13)$$

at the variable endpoint  $z_0$ , where  $g(z)$  describes the curve on which the endpoint  $z_0$  can vary. Applying this to the present problem, we must have  $f(z_0) = 0$ , which requirement is satisfied by the condition  $x(z_0) = 0$  that has already been included in the analysis of sec. 2.2. However, it is pleasant to have a formal confirmation of our earlier “intuitive” justification of this condition.

## 2.4 Further Comment on the Shape the Asteroid

The form (6) for the shape of a maximal gravity asteroid differs only slightly from that of a sphere, as shown in the figure on p. 3. Starting from a sphere, the force of gravity at, say, the “north” pole is increased by moving material from the “southern” hemisphere into the “northern” hemisphere.

We verify that this is the right thing to do by a simple calculation. Namely, we compare the  $z$  component of the force of gravity at the north pole due to a point mass on the surface of a sphere of radius  $a$  at polar angle  $\theta$  with that due to a point mass at angle  $\pi - \theta$ . The claim is that the point at  $\theta$  contributes more to the force than the point at  $\pi - \theta$ , so that it would be favorable to move some of the mass to  $\theta < \pi/2$ .

The distance from the pole to the point at angle  $\theta$  is  $r = 2a \sin \theta/2$ . The  $z$  component of the force of gravity due to a point mass at angle  $\theta$  is proportional to

$$F_z(\theta) \propto \frac{\Delta z}{r^3} = \frac{a(1 - \cos \theta)}{8a^3 \sin^3 \theta/2} = \frac{1}{4a^2 \sin \theta/2}, \quad (14)$$

so that the force due to a point at angle  $\pi - \theta$  is related by

$$F_z(\pi - \theta) \propto \frac{1}{4a^2 \cos \theta/2}. \quad (15)$$

The ratio,

$$\frac{F_z(\pi - \theta)}{F_z(\theta)} = \tan \theta/2, \quad (16)$$

is less than one for all  $\theta < \pi/2$ . Thus, the force of gravity at the north pole would be increased by moving mass from angle  $\pi - \theta$  to angle  $\theta$  on the surface of the asteroid, as claimed.

While this argument gives a qualitative feel as to how a spherical asteroid should be reshaped to become a maximal-gravity asteroid, it does not provide a quantitative prescription of how much mass should be moved. For that, the arguments of secs. 2.1 and 2.2 are better.

## References

- [1] C. Fox, *An Introduction to the Calculus of Variations* (Oxford U.P., Oxford, 1950; Dover Publications, New York, 1987).