Long Rod with Uniform Magnetization Transverse to its Axis

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 3, 1999; updated January 21, 2017)

1 Problem

A long cylinder of radius a has uniform magnetization $\mathbf{M}$ transverse to its axis. Find the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ everywhere. Show that the field lines outside the cylinder are circles.

A trap for diamagnetic objects based on a pair of finite-length, transversely magnetized cylinders is reported in [1].

Suppose also that the cylinder is given a uniform velocity, $v \ll c$, along its axis, where c is the speed of light in vacuum. Find the resulting charge density and electric field everywhere, ignoring effects of order $(v/c)^2$.

2 Solution Assuming Ampèrian Magnetization

We assume that the magnetization is Ampèrian (associated with electrical currents) and not Gilbertian (associated with pairs of opposite, true magnetic charge). We therefore denote the magnetization as $\mathbf{M}_{\rm A}$ in this section. See the Appendix for the possible case of Gilbertian magnetization.

Let $\hat{\mathbf{z}}$ be the axis of the cylinder and $\hat{\mathbf{x}}$ the direction of the magnetization.

2.1 Cylinder at Rest

2.1.1 Solution via the Magnetic Scalar Potential

Since there are no free currents in this statics problem,

$$\boldsymbol{\nabla} \times \mathbf{H}_{\rm A} = 0, \tag{1}$$

where

$$\mathbf{H}_{\rm A} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}_{\rm A} \tag{2}$$

is the macroscopic magnetic field (in SI units) associated the magnetic induction $\mathbf{B}$ (which obeys $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ in the absence of true magnetic charges) and the Ampèrian magnetization $\mathbf{M}_{\rm A}$.[1] Hence, we can define a magnetic scalar potential such that

$$\mathbf{H}_{\rm A} = -\boldsymbol{\nabla} \phi_H. \tag{3}$$

[1]The field $\mathbf{H}_{\rm A}$ is ordinarily written simply as $\mathbf{H}$, as the usual macroscopic Maxwell equations tacitly assume that the magnetization is Ampèrian.

As the cylinder is very long, we approximate the problem as 2-dimensional: $\phi_H = \phi_H(r, \theta)$ in cylindrical coordinates (r, θ, z).

As discussed in, for example, Appendix A of [2], a fictitious magnetic charge density ,

$$\rho_{m,\mathrm{A}} = -\boldsymbol{\nabla} \cdot \mathbf{M}_\mathrm{A}, \tag{4}$$

can be associated with the Ampèrian magnetization density $\mathbf{M}_\mathrm{A}$, in which case $\boldsymbol{\nabla}\cdot\mathbf{H}_\mathrm{A} = \rho_{m,\mathrm{A}}$ in SI units.

Since $\mathbf{M}_\mathrm{A} = M\,\hat{\mathbf{x}}$ in this example, the volume charge density of (fictitious) magnetic charges is $\rho_{m,\mathrm{A}} = -\boldsymbol{\nabla} \cdot \mathbf{M}_\mathrm{A} = 0$. However, at the surface of the cylinder $(r = a)$, there is a (fictitious) surface magnetic-charge density given by

$$\sigma_{m,\mathrm{A}} = \mathbf{M}_\mathrm{A} \cdot \hat{\mathbf{r}} = M\cos\theta. \tag{5}$$

The potential is continuous at the boundary $r = a$, and Gauss' law, $\boldsymbol{\nabla} \cdot \mathbf{H}_\mathrm{A} = \rho_{m,\mathrm{A}}$, tells us that

$$\sigma_{m,\mathrm{A}} = M\cos\theta = H_{\mathrm{A},r}(r = a^+) - H_{\mathrm{A},r}(r = a^-) = -\frac{\partial\phi_H(r = a^+)}{\partial r} + \frac{\partial\phi_H(r = a^-)}{\partial r}. \tag{6}$$

The potential can be expanded as a harmonic series, but only the term in $\cos\theta$ will contribute in view of eq. (6). Thus,

$$\phi_H = \begin{cases} -Hr\cos\theta, & r \le a, \\ -H\frac{a^2}{r}\cos\theta, & r \ge a, \end{cases} \tag{7}$$

satisfies continuity of the potential at $r = a$. Then, eq. (6) also tells us that $H = -M/2$.

Inside the cylinder we have

$$\phi_H(r < a) = \frac{Mx}{2}, \tag{8}$$

$$\mathbf{H}_\mathrm{A}(r < a) = -\boldsymbol{\nabla}\phi_H = -\frac{M}{2}\hat{\mathbf{x}} = -\frac{\mathbf{M}_\mathrm{A}}{2}, \tag{9}$$

$$\mathbf{B}_\mathrm{A}(r < a) = \mu_0(\mathbf{H}_\mathrm{A} + \mathbf{M}_\mathrm{A}) = \mu_0\frac{\mathbf{M}_\mathrm{A}}{2}. \tag{10}$$

Outside the cylinder there is no magnetization, and

$$\phi_H(r > a) = Ma^2\frac{\cos\theta}{2r} = \frac{Ma^2}{2}\frac{x}{x^2 + y^2}, \tag{11}$$

$$\mathbf{H}_\mathrm{A}(r > a) = \frac{\mathbf{B}_\mathrm{A}(r > a)}{\mu_0} = \frac{Ma^2}{2r^2}(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{Ma^2}{2r^4}\left[(x^2 - y^2)\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}\right]. \tag{12}$$

The potential (11) and the fields (12) outside the cylinder are equivalent to those of a line with magnetic-moment density $\pi a^2 M\,\hat{\mathbf{x}}$ per unit length in z.[2]

[2]The magnetic scalar potential of a point magnetic dipole $\mathbf{m}$ is $\mathbf{m}\cdot\mathbf{R}/R^3$, where $\mathbf{R}$ is the distance from the dipole to the observation point. In case of density $m\,\hat{\mathbf{x}}$ per unit length of magnetic dipoles along the z-axis, the magnetic scalar potential at (r, θ, z) in cylindrical coordinates is then given by

$$\phi_H(r, \theta, z) = \phi_H(r, \theta, 0) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\mathbf{m}\cdot\mathbf{R}}{R^3}\,dz' = \frac{mr\cos\theta}{4\pi}\int_{-\infty}^{\infty}\frac{dz'}{(z'^2 + r^2)^{3/2}} = \frac{m\cos\theta}{2\pi r}. \tag{13}$$

Field Lines outside the Cylinder

The exterior magnetic field lines are orthogonal to the equipotential surfaces given by eq. (11).

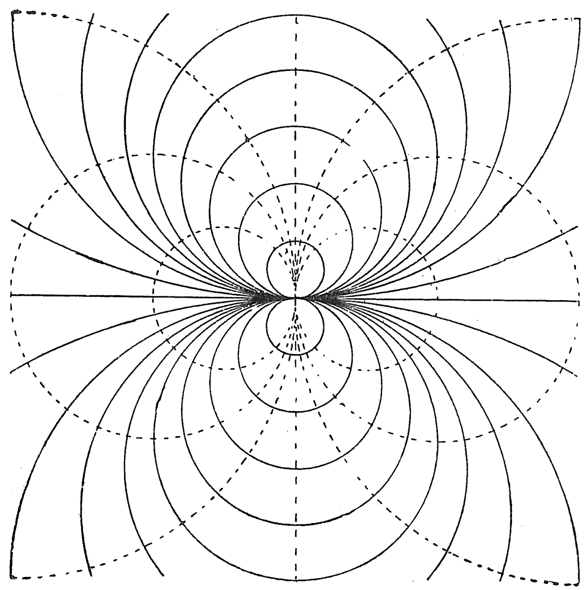
This problem is 2-dimensional, so we can take advantage of the fact that any analytic function $f(z) = \phi(z) + i\lambda(z)$ of a complex variable $z = x + iy$ obeys $\nabla^2\phi = 0 = \nabla^2\lambda$, and that lines of constant ϕ are orthogonal to lines of constant λ.[3] In particular, we consider

$$f = \frac{Ma^2}{2z} = \frac{Ma^2}{2}\frac{x - iy}{x^2 + y^2}, \qquad \phi = \frac{Ma^2}{2}\frac{x}{x^2 + y^2}, \qquad \lambda = -\frac{Ma^2}{2}\frac{y}{x^2 + y^2}. \tag{14}$$

Lines of constant ϕ and λ (which constants can be negative) obey the relations

$$\left(x - \frac{a^2}{M\phi}\right)^2 + y^2 = \left(\frac{a^2}{M\phi}\right)^2, \qquad x^2 + \left(y - \frac{a^2}{M\lambda}\right)^2 = \left(\frac{a^2}{M\lambda}\right)^2, \tag{15}$$

which are circles of radii $a^2/M\,|\phi|$ and $a^2/M\,|\lambda|$ centered at $(x, y) = (a^2/M\phi, 0)$ and $(0, a^2/M\lambda)$. The figure below is from p. 262 of [3]; for horizonal magnetization, the field lines are solid and the equipotentials are dashed.



2.1.2 Solution via the Vector Potential

Instead of invoking the magnetic scalar potential ϕ_H, we can analyze this problem by consideration of the bound currents associated with the magnetization $\mathbf{M}_\mathrm{A}$.

The bulk magnetization current density is $\mathbf{J}_{\mathrm{A,bound}} = \boldsymbol{\nabla} \times \mathbf{M}_\mathrm{A}$, which vanishes for uniform magnetization. However, there is a nonzero bound surface current density,

$$\mathbf{K}_\mathrm{A}(r = a, \theta, z) = \hat{\mathbf{r}} \times \mathbf{M}_\mathrm{A}(r = a^-) = -M \sin\theta\,\hat{\mathbf{z}}. \tag{16}$$

[3]See, for example, secs. 306-318 of [3] or sec. 7.2 of [4].

The vector potential $\mathbf{A}_\mathrm{A}$ generated by this surface current has only a z-component, which is independent of z for a long cylinder,

$$A_{\mathrm{A},z}(r,\theta,z) \;=\; -\frac{\mu_0 M a}{4\pi}\int\!\!\int \frac{\sin\theta'}{\sqrt{a^2+r^2-2ar\cos(\theta'-\theta)+z'^2}}\,d\theta'\,dz' \tag{17}$$

The magnetic field $\mathbf{B}_\mathrm{A} = \boldsymbol{\nabla}\times\mathbf{A}_\mathrm{A}$ has only $r-$ and θ-components, with

$$\begin{aligned} B_{\mathrm{A},r} &= \frac{1}{r}\frac{\partial A_z}{\partial\theta} = \frac{\mu_0 M a^2}{4\pi}\int\!\!\int \frac{\sin\theta'\sin(\theta'-\theta)}{[a^2+r^2-2ar\cos(\theta'-\theta)+z'^2]^{3/2}}\,d\theta'\,dz' \\ &= \frac{\mu_0 M a^2}{2\pi}\int \frac{\sin(\theta'+\theta)\sin\theta'}{a^2+r^2-2ar\cos\theta'}\,d\theta' \\ &= \begin{cases} \frac{\mu_0 M\cos\theta}{2} & (r<a), \\ \frac{\mu_0 M a^2\cos\theta}{2r^2} & (r>a), \end{cases} \end{aligned} \tag{18}$$

using Dwight 859.131. Similarly,

$$\begin{aligned} B_{\mathrm{A},\theta} &= -\frac{\partial A_z}{\partial r} = -\frac{\mu_0 M a^2}{4\pi}\int\!\!\int \frac{\sin\theta'\cos(\theta'-\theta)}{[a^2+r^2-2ar\cos(\theta'-\theta)+z'^2]^{3/2}}\,d\theta'\,dz' \\ &= -\frac{\mu_0 M a^2}{2\pi}\int \frac{\sin(\theta'+\theta)\cos\theta'}{a^2+r^2-2ar\cos\theta'}\,d\theta' \\ &= \begin{cases} \frac{\mu_0 M\sin\theta}{2} & (r<a), \\ \frac{\mu_0 M a^2\sin\theta}{2r^2} & (r>a). \end{cases} \end{aligned} \tag{19}$$

Thus, the magnetic field $\mathbf{B}_\mathrm{A}$ is the same as that found in eqs. (10) and (12).

Although I couldn't evaluate eq. (17) directly, we can now integrate $\mathbf{B}_\mathrm{A} = \boldsymbol{\nabla}\times\mathbf{A}_\mathrm{A}$ to find the vector potential as

$$A_{\mathrm{A},z} = \begin{cases} \frac{\mu_0 M y}{2} = \frac{\mu_0 M r\sin\theta}{2} & (r<a), \\ \frac{\mu_0 M a^2\sin\theta}{2r} & (r>a), \end{cases} \tag{20}$$

which is continuous across the surface $r = a$.

2.2 Cylinder Moving with Velocity v

In case of a moving cylinder, the analysis of sec. 2.1 holds in the rest frame of the cylinder. When the cylinder has velocity $\mathbf{v} = v\hat{\mathbf{z}}$ in the lab frame, the fields in that frame appear to be

$$\mathbf{E}_\mathrm{A} = -\gamma\mathbf{v}\times\mathbf{B}'_\mathrm{A} \approx -\mathbf{v}\times\mathbf{B}_\mathrm{A}, \qquad \mathbf{B}_\mathrm{A} = \gamma\left(\mathbf{B}'_\mathrm{A} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}'_\mathrm{A}\right) \approx \mathbf{B}'_\mathrm{A}, \tag{21}$$

where we ignore terms of order v^2/c^2, so the magnetic field $\mathbf{B}_\mathrm{A}$ in the lab frame is the same as the field $\mathbf{B}'_\mathrm{A}$ in the rest frame (given by eqs. (10) and (12)).[4]

[4]Regarding the sign in (21), we note that a hypothetical electric charge which is at rest in the lab frame would move with velocity $-\mathbf{v}$ in the rest frame of the magnetized cylinder, and so in the latter frame would experience a Lorentz force $-\mathbf{v}\times\mathbf{B}$.

Thus,

$$\mathbf{E}_{\rm A}(r<a) \approx -\mu_0 M \frac{v}{2}\hat{\mathbf{y}} = -\mu_0 M \frac{v}{2}(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}), \tag{22}$$

$$\mathbf{E}_{\rm A}(r>a) \approx \mu_0 \frac{Mva^2}{2r^2}(\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}). \tag{23}$$

There is an electric charge density on the surface of the cylinder given by

$$\sigma_e = \epsilon_0 \left[E_r(r=a^+) - E_r(r=a^-)\right] = \frac{Mv\sin\theta}{c^2}. \tag{24}$$

This can be thought of as arising from an electric polarization $\mathbf{P}_{\rm A}$ *related to the moving Ampèrian magnetization by*

$$\mathbf{P}_{\rm A} \approx \frac{\mathbf{v}}{c^2} \times \mathbf{M}'_{\rm A} \approx \frac{\mathbf{v}}{c^2} \times \mathbf{M}_{\rm A} = \frac{Mv}{c^2}\,\hat{\mathbf{y}}, \tag{25}$$

according to

$$\sigma_e(r=a) = \mathbf{P}_{\rm A}\cdot\hat{\mathbf{r}} \approx P_{\rm A}\sin\theta = \frac{Mv\sin\theta}{c^2}. \tag{26}$$

This is an illustration of the fact that the polarization densities $\mathbf{M}_{\rm A}$ *and* $\mathbf{P}_{\rm A}$ *are components of the 4-tensor*

$$\mathsf{M}_{\rm A} = \begin{pmatrix} 0 & cP_{{\rm A},x} & cP_{{\rm A},y} & cP_{{\rm A},x} \\ -cP_{{\rm A},x} & 0 & -M_{{\rm A},z} & M_{{\rm A},y} \\ -cP_{{\rm A},y} & M_{{\rm A},z} & 0 & -M_{{\rm A},x} \\ -cP_{{\rm A},z} & -M_{{\rm A},y} & M_{{\rm A},x} & 0 \end{pmatrix}. \tag{27}$$

Appendix: Fields in the Case of Gilbertian Magnetization

If the magnetization were due to Gilbertian magnetic dipoles rather than Ampérian ones, we suppose the existence of true magnetic charges with volume density $\rho_{m,\rm true}$, such that the third Maxwell equation becomes[5]

$$\boldsymbol{\nabla}\cdot\mathbf{B}_{\rm G} = \mu_0\,\rho_{m,\rm true}. \tag{28}$$

In the present, static example with no free electric or magnetic currents, we also have that

$$\boldsymbol{\nabla}\times\mathbf{B}_{\rm G} = 0, \tag{29}$$

so the magnetic field can be related to a scalar potential,

$$\mathbf{B}_{\rm G} = -\boldsymbol{\nabla}\phi_B. \tag{30}$$

[5]See, for example, [2].

The source of the magnetic scalar potential is the true magnetic charges associated with the (true, Gilbertian) magnetization $\mathbf{M}_{\rm G}$. Again we suppose that $\mathbf{M}_{\rm G} = M\,\hat{\mathbf{x}}$, so the true volume charge density is $\rho_{m,{\rm G}} = -\boldsymbol{\nabla}\cdot\mathbf{M}_{\rm G} = 0$. At the surface of the cylinder ($r = a$), there is a surface density of true magnetic charge given by

$$\sigma_{m,{\rm G}} = \mathbf{M}_{\rm G}\cdot\hat{\mathbf{r}} = M\cos\theta. \tag{31}$$

These conditions are identical to those in sec. 2.1.1 with the substitution of $\mathbf{B}_{\rm G}/\mu_0$ for $\mathbf{H}_{\rm A}$ and $\sigma_{\rm m,G}$ for $\sigma_{m,{\rm A}}$. Hence, we conclude that the magnetic field for the case of Gilbertian magnetization follows from eqs. (10) and (12) as

$$\mathbf{B}_{\rm G}(r<a) = -\mu_0\frac{M}{2}\hat{\mathbf{x}} = -\mu_0\frac{\mathbf{M}_{\rm G}}{2}\,, \tag{32}$$

$$\mathbf{B}_{\rm G}(r>a) = \mu_0\frac{Ma^2}{2r^2}(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{33}$$

The magnetic field $\mathbf{B}$ outside the cylinder is the same whether the magnetization is Ampérian or Gilbertian, but the $\mathbf{B}$-field inside the cylinder is opposite in the two cases.

If we accept that the Lorentz force density on an electric current density $\mathbf{J}_e$ *is* $\mathbf{f} = \mathbf{J}_e\times\mathbf{B}$,[6] *then, for example, the Hall voltage difference between* $y = a$ *and* $y = -a$ *(for electrical current flowing parallel to the axis of the cylinder) would have opposite signs for Ampérian and Gilbertian magnetization. This permits a simple test as to the character of the magnetization in a conducting, permanent, cylindrical magnet with transverse magnetization.*

References

[1] O. Gunawan, Y. Virgus and K.-F. Tai, *A parallel dipole line system*, Appl. Phys. Lett. **106**, 062407 (2015), http://physics.princeton.edu/~mcdonald/examples/EM/gunawan_apl_106_062407_15.pdf

[2] K.T. McDonald, *Poynting's Theorem with Magnetic Monopoles* (Mar. 13, 2013), http://physics.princeton.edu/~mcdonald/examples/poynting.pdf

[3] J.H. Jeans, *The Mathematical Theory of Electrcity and Magnetism* (Cambridge U. Press, 1908), http://physics.princeton.edu/~mcdonald/examples/EM/jeans_electricity.pdf

[4] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, vol. 2 (Addison-Wesley, 1964), http://www.feynmanlectures.caltech.edu/II_07.html

[6] See, for example, [2]. An alternative Lorentz force density is $\mathbf{J}_e\times\mu_0\mathbf{H}_{\rm G}$ where the Gilbertian magnetic field is defined as $\mathbf{H}_{\rm G} = \mathbf{B}_{\rm G}/\mu_0 + \mathbf{M}_{\rm G}$. In the present example, $\mathbf{H}_{\rm G} = \mathbf{B}_{\rm G}/\mu_0 = \mathbf{M}_{\rm A}/2 = M\,\hat{\mathbf{x}}/2$ inside the cylinder for the case of Ampèrian magnetization, while $\mathbf{H}_{\rm G} = \mathbf{M}_{\rm G}/2 = M\,\hat{\mathbf{x}}/2$ for Gilbertian magnetization, so the Lorentz force on an electric current would be the same for both types of magnetization.

However, the alternative form of the Lorentz force is not consistent with energy conservation, as discussed in [2].