Liénard-Wiechert Potentials and Fields via Lorentz Transformations

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Deduce the Liénard-Wiechert potentials and fields [1, 2] in the Lorenz gauge [3][1] for a (point) electric charge q via a Lorentz transformation from the instantaneous (inertial) rest frame of the charge.

2 Solution

2.1 Retarded Fields

Maxwell's equations for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be written (in Gaussian units) in the form,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho, \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

where ρ and $\mathbf{J}$ are the (total) volume densities of electric charge and current, and c is the speed of light in vacuum. These four first-order differential equations can be combined into two second-order wave equations,

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 4\pi\boldsymbol{\nabla}\rho + \frac{4\pi}{c^2}\frac{\partial \mathbf{J}}{\partial t}, \qquad \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{\nabla} \times \mathbf{J}. \tag{2}$$

The method of Lorenz [3] and Riemann [5] of retarded solutions can lead directly to the forms,[2]

$$\mathbf{E}(\mathbf{x}, t) = \int \frac{[\rho]\,\hat{\mathbf{R}}}{R^2}\, d^3\mathbf{x}' + \frac{1}{c}\int \frac{\left([\mathbf{J}] \cdot \hat{\mathbf{R}}\right) + \left([\mathbf{J}] \times \hat{\mathbf{R}}\right) \hat{\mathbf{R}} \times \hat{\mathbf{R}}}{R^2}\, d^3\mathbf{x}' + \frac{1}{c^2}\int \frac{\left([\dot{\mathbf{J}}] \times \hat{\mathbf{R}}\right) \times \hat{\mathbf{R}}}{R}\, d^3\mathbf{x}', \tag{3}$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{c}\int \frac{[\mathbf{J}] \times \hat{\mathbf{R}}}{R^2}\, d^3\mathbf{x}' + \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}] \times \hat{\mathbf{R}}}{R}\, d^3\mathbf{x}', \tag{4}$$

where $\mathbf{R} - \mathbf{x} - \mathbf{x}'$, $\hat{\mathbf{R}} = \mathbf{R}/\left|\mathbf{R}\right| = \mathbf{R}/R$, $[f] = f(\mathbf{x}', t' = t - R/c)$ and $\dot{\mathbf{J}} = \partial\mathbf{J}/\partial t$.

[1]For commentary by the author on the paper of Lorenz, see the Appendix of [4].

[2]The wave equation for $\mathbf{E}$ was given by Lorenz [3] for a medium of electrical conductivity σ, but with $\mathbf{E}$ replaced by $\mathbf{J}$ according to Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$. Lorenz noted that a solution to this equation exists via the method of retarded potentials, but his did not explicitly display this for $\mathbf{J}$. Had he done so, he could have arrived at eq. (3). Instead, the forms (3)-(4) first appeared in eqs. (14.34) and 14.42) of [6]. See also [7] and the Appendix to [4].

2.2 Retarded Potentials

Lorenz [3] also noted that the equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ implies that the magnetic field can be related to a vector potential $\mathbf{A}$ according to,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{5}$$

Using eq. (5) in Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c)\partial \mathbf{B}/\partial t$, we can write

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0, \tag{6}$$

which implies that $\mathbf{E} + (1/c)\partial \mathbf{A}/\partial t$ can be related to a scalar potential V as $-\boldsymbol{\nabla} V$, *i.e.*,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,. \tag{7}$$

Maxwell's equations then lead to wave equations for the potentials V and $\mathbf{A}$,

$$\nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -4\pi\rho, \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t} \right). \tag{8}$$

Lorenz argued that if we enforce the auxiliary condition,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t} \qquad \text{(Lorenz)}, \tag{9}$$

the wave equations (8) take the simpler forms,

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho, \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} \qquad \text{(Lorenz)}, \tag{10}$$

which have the formal, retarded solutions,

$$V(\mathbf{x}, t) = \int \frac{[\rho]}{R}\, d^3\mathbf{x}', \qquad \mathbf{A}(\mathbf{x}, t) = \frac{1}{c}\int \frac{[\mathbf{J}]}{R}\, d^3\mathbf{x}' \qquad \text{(Lorenz)}. \tag{11}$$

In the present context it is useful to note that the retarded potentials (11) depend on the (retarded) charge and current densities, but not on their time derivatives. In particular, this means that the retarded potentials do not depend on the acceleration of the charges (which would correspond to a dependence on $\dot{\mathbf{J}}$). Hence, the form of the retarded potentials of a single accelerating charge have the same form as those for a charge with uniform velocity.

2.3 Liénard-Wiechert Potentials via Lorentz Transformations

The argument of this section was first given by Minkowski [8].[3]

Textbook derivations of the Liénard-Wiechert potentials tend to follow the original arguments [1, 2] that evaluate the retarded potentials (11) for a "point" electric charge.[4]

[3] A version by the author on pp. 222-223 of [9], written in 1979, was perhaps too brief.

[4] See, for example, sec. 19-1 of [6] and sec. 14.1 of [10].

A lesson of the retarded potentials (11) is that their value at $(\mathbf{x}, t)$ in the lab frame depends on the position $[\mathbf{x}']$ of the charge at the retarded time $[t'] = t - [R]/c$ where $[\mathbf{R}] = \mathbf{x} - [\mathbf{x}']$ is the retarded distance between the charge and the observer. We expect the retarded potentials $V(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ for a single electric charge to depend on the retarded distance $[R]$ between the observer and the charge and on the normalized velocity, $[\mathbf{v}]/c \equiv [\boldsymbol{\beta}]$, of the charge at the retarded time.

In the case of a uniformly moving charge, for which $[\boldsymbol{\beta}] = \boldsymbol{\beta} = \mathbf{v}/c =$ constant, we wish to relate the retarded potentials in lab frame to those in a frame (called the * frame) in which the charge is at rest.

In the * frame, the potentials according to an observer at $(\mathbf{x}^\star, t^\star)$ of a charge q that is uniformly moving in the lab frame are simply,

$$V^\star(\mathbf{x}^\star, t^\star) = \frac{q}{[R^\star]}, \qquad \mathbf{A}^\star(\mathbf{x}^\star, t^\star) = 0, \tag{12}$$

where $[\mathbf{R}^\star] = \mathbf{x}^\star - [\mathbf{x}'^\star]$, the charge is at rest at position $[\mathbf{x}'^\star]$ in the * frame, and the retarded time is $[t'^\star] = t^\star - [R^\star]/c$.

It seems reasonable that the potentials (12) also hold in the instantaneous, inertial rest frame of an arbitrarily accelerated charge. This view seems to have been held by Minkowski [8], and was endorsed by Landau in sec. 63 of [11]. However, in the case of a charge that is uniformly accelerated, the potentials do not obey eq. (12), as reviewed in sec. 2.3.1 below.

The potentials (12) in the * frame are related to those in the lab frame by a Lorentz transformation involving $[\boldsymbol{\beta}]$ and $[\gamma] = 1/\sqrt{1 - [\beta]^2}$,

$$V = [\gamma](V^\star + [\boldsymbol{\beta}] \cdot \mathbf{A}^\star) = q\left[\frac{\gamma}{R^\star}\right], \quad \mathbf{A}_\parallel = [\gamma](\mathbf{A}^\star_\parallel + [\boldsymbol{\beta}]V^\star) = q\left[\frac{\gamma\boldsymbol{\beta}}{R^\star}\right], \quad \mathbf{A}_\perp = \mathbf{A}^\star_\perp = 0, \tag{13}$$

where $\mathbf{A}_\parallel = \left(\mathbf{A} \cdot [\hat{\boldsymbol{\beta}}]\right)[\hat{\boldsymbol{\beta}}]$ and $\mathbf{A}_\perp = \mathbf{A} - \mathbf{A}_\parallel = \mathbf{A} - \left(\mathbf{A} \cdot [\hat{\boldsymbol{\beta}}]\right)[\hat{\boldsymbol{\beta}}]$.

To complete the analysis, we need the value of $R^\star$ in the lab frame. For this, we combine the retarded distance $[\mathbf{R}] = \mathbf{x} - [\mathbf{x}']$ with the time difference $t - [t'] = [R]/c$ into a 4-vector,

$$\Delta x_\mu = (ct - c[t'], \mathbf{x} - [\mathbf{x}']) - ([R], [\mathbf{R}]), \tag{14}$$

whose invariant length is zero. That is, Δx_μ is a lightlike 4-vector, whose components in the * frame are $([R^\star], [\mathbf{R}^\star])$. Then, the Lorentz transformation of the time component of Δx_μ tells us that

$$[R^\star] = \Delta x_0^\star = [\gamma](\Delta x_0 - [\boldsymbol{\beta}] \cdot \Delta\mathbf{x}) = [\gamma(R - \boldsymbol{\beta} \cdot \mathbf{R})] = [\gamma(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})R]. \tag{15}$$

Using this in eq. (13), we obtain the retarded potentials in the lab frame as,

$$V(\mathbf{x}, t) = \frac{q}{[R - \boldsymbol{\beta} \cdot \mathbf{R}]} = \frac{q}{[(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})R]}, \qquad \mathbf{A}(\mathbf{x}, t) = \frac{q[\boldsymbol{\beta}]}{[R - \boldsymbol{\beta} \cdot \mathbf{R}]} = \frac{q[\boldsymbol{\beta}]}{[(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})R]}. \tag{16}$$

While the result (16) has been obtained for the special case of a uniformly moving "point" charge, it also applies to a "point" charge with arbitrary motion, according to the argument at the end of sec. 2.2. Thus, we have arrived at the Liénard-Wiechert potentials via Lorentz transformations.[5]

[5]This derivation has also been given in [11, 12, 13, 14]. Another work, [15], should also have arrived at eq. (16) via a Lorentz transformation, but due to algebraic errors, concluded that the famous Lorentz contraction of lengths should instead be an expansion.

2.3.1 Uniformly Accelerated Charge

Analytic calculations for the potentials and fields of a point charge with uniform acceleration of infinite duration can be given (see, for example, [16]). If the charge moves along the positive z-axis with constant, positive acceleration $a^\star$ in the instantaneous rest frame of the charge, we define the charge to be momentarily at rest at $t = 0$. Then, the position of the charge at time t is given by,

$$z(t) = \sqrt{b^2 + c^2t^2}, \qquad x = 0 = y. \tag{17}$$

where $b = c^2/a^\star$. The potentials according to an observer at cylindrical coordinates $\mathbf{x} = (\rho, \phi, z)$ are,[6]

$$V(\mathbf{x}, t) = q\frac{z(z^2 + b^2 + \rho^2 - c^2t^2) - cts}{s(z^2 - c^2t^2)}, \qquad \mathbf{A}(\mathbf{x}, t) = q\frac{ct(z^2 + b^2 + \rho^2 - c^2t^2) - zs}{s(z^2 - c^2t^2)}\,\hat{\mathbf{z}}, \tag{18}$$

where,

$$s = \sqrt{(z^2 + \rho^2 + b^2 - c^2t^2)^2 - 4b^2(z^2 - c^2t^2)} = \sqrt{(z^2 + \rho^2 - z_b^2)^2 + 4b^2\rho^2}. \tag{19}$$

The potentials (and fields) are zero for $z < -ct$ (and diverge on the plane $z = -ct$).[7] The (moving) plane $z = -ct$ can be called an event horizon, in that an observer at $z < -ct$ can detect no fields from the charge q at time t.

The lab frame is the instantaneous rest frame of the charge at time $t = 0$, when the charge is at $(0, 0, b)$. At this time, the potentials at $\mathbf{x} = (\rho, \phi, z)$ are,

$$V(\mathbf{x}, 0) = q\frac{z^2 + b^2 + \rho^2}{z\sqrt{(z^2 + \rho^2 + b^2)^2 - 4b^2z^2}}, \qquad \mathbf{A}(\mathbf{x}, 0) = -\frac{q}{z}\,\hat{\mathbf{z}}. \tag{20}$$

Unlike the case of a nonaccelerated charge, the vector potential $\mathbf{A}$ is nonzero in the instantaneous rest frame of the charge. Also, the scalar potential V does not have the form q/r where $r = \sqrt{(z - b)^2 + \rho^2}$ is the distance from the observer to the charge.

That is, the potentials of a uniformly accelerated charge do not have the form of eq. (12) in the instantaneous rest frame of the charge.[8]

2.4 Liénard-Wiechert Fields via Lorentz Transformations

The Liénard-Wiechert potentials are used to deduce the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of an accelerated charge using eqs. (5) and (7), via notoriously intricate computations. Hence, it is useful to consider whether the electromagnetic fields could be deduced in a simpler manner, perhaps via Lorentz transformations.

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[6]Neither V nor $\mathbf{A}$ depend directly on the acceleration $a^\star$.

[7]This singular plane can be thought of as the location at time t of the plane of Čerenkov radiation associated with the charge $t = -\infty$ when its velocity was $\mathbf{v} = -c\,\hat{\mathbf{z}}$.

[8]However, I believe that form eq. (12) holds except for charges whose velocity was formally equal to c at $t = -\infty$ and was decelerated in some manner to $v < c$ thereafter. That is, I believe eq. (12) holds for all physical motions of charges, so it is perhaps less surprising that the potentials (16) apply to the general case, and not just for motion with uniform velocity.

This has been reported in [17],[9] based on a model of the fields $\mathbf{E}^\star$ and $\mathbf{B}^\star$ in the instantaneous rest frame of the charge. In this model, the fields are nonzero throughout all space, whereas in the case of a uniformly accelerated charge, as in sec. 2.3.1, the fields are nonzero only for $z > -ct$. Hence, the argument of [17] cannot be considered as generally applicable, although the objection to this argument is mainly that it does not apply to the mathematically interesting, but physically unrealistic, case of a uniformly accelerated charge.

The argument can be considered as an application of eq. (4) in the instantaneous rest frame (the * frame) of a point charge q, where it seems appealing to suppose that $[\mathbf{J}^\star] = 0$ while $[\dot{\mathbf{J}}^\star]/c = q[\dot{\boldsymbol{\beta}}^\star]$, such that the fields at the observer in this frame would be,

$$\mathbf{E}^\star = q\left[\frac{\mathbf{R}^\star}{R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\left(\dot{\boldsymbol{\beta}}^\star \times \mathbf{R}^\star\right) \times \mathbf{R}^\star}{R^{\star 3}}\right] = \frac{q}{c}\left[\frac{\mathbf{R}^\star}{[R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\left(\mathbf{R}^\star \cdot \dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star}{R^{\star 3}}\right] - \frac{q}{c}\left[\frac{\dot{\boldsymbol{\beta}}^\star}{R^\star}\right] \tag{21}$$

$$\mathbf{B}^\star = \frac{q}{c}\left[\frac{\dot{\boldsymbol{\beta}}^\star \times \mathbf{R}^\star}{R^{\star 2}}\right] = [\hat{\mathbf{R}}^\star] \times \mathbf{E}^\star. \tag{22}$$

The Lorentz transformation to the lab frame has boost with $[\boldsymbol{\beta}]$ and $[\gamma] = 1/\sqrt{1-[\beta]^2}$, as on p. 3. The electric field transforms according to,

$$\mathbf{E}_\parallel = \mathbf{E}^\star_\parallel, \qquad \mathbf{E}_\perp = [\gamma]\mathbf{E}^\star_\perp - [\gamma\boldsymbol{\beta}] \times \mathbf{B}^\star, \tag{23}$$

where $\mathbf{E}_\parallel = \left(\mathbf{E}\cdot[\hat{\boldsymbol{\beta}}]\right)[\hat{\boldsymbol{\beta}}]$ and $\mathbf{E}_\perp = \mathbf{E} - \mathbf{E}_\parallel = \mathbf{E} - \left(\mathbf{E}\cdot[\hat{\boldsymbol{\beta}}]\right)[\hat{\boldsymbol{\beta}}]$. In the transformation of the magnetic field, the transformation of $[\hat{\mathbf{R}}^\star]$ is just $[\hat{\mathbf{R}}]$, so we have that,

$$\mathbf{B} = [\hat{\mathbf{R}}] \times \mathbf{E}. \tag{24}$$

To complete the transformation (23) we need the parallel and perpendicular components of $\left[\dot{\boldsymbol{\beta}}^\star\right]$ and of $[\mathbf{R}^\star]$. The transformation of the acceleration $\left[\dot{\boldsymbol{\beta}}^\star\right]$ is,

$$\left[\dot{\boldsymbol{\beta}}^\star_\parallel\right] = \left[\frac{d^2\mathbf{x}'^\star_\parallel}{dt'^{\star 2}}\right] = \left[\gamma^3\dot{\boldsymbol{\beta}}_\parallel\right], \qquad \left[\dot{\boldsymbol{\beta}}^\star_\perp\right] = \left[\frac{d^2\mathbf{x}'^\star_\perp}{dt'^{\star 2}}\right] = \left[\gamma^2\dot{\boldsymbol{\beta}}_\perp\right], \tag{25}$$

since the Lorentz transformation of spacetime quantities is $d[\mathbf{x}'^\star_\parallel] = [\gamma]d[\mathbf{x}'_\parallel]$, $d[\mathbf{x}'^\star_\perp] = d[\mathbf{x}'_\perp]$ and $d[t'^\star] = d[t']/[\gamma]$, *i.e.*, a Lorentz contraction for the parallel interval $d[\mathbf{x}'_\parallel]$ and a time dilation for $d[t']$. Then, recalling the Lorentz transformation of the lightlike 4-vector Δx_μ, eqs. (14)-(15), we have,

$$[R^\star] = [\gamma(R - \boldsymbol{\beta}\cdot\mathbf{R})], \qquad [\mathbf{R}^\star_\parallel] = [\gamma(\mathbf{R}_\parallel - R\boldsymbol{\beta})], \qquad [\mathbf{R}^\star_\perp] = [\mathbf{R}_\perp]. \tag{26}$$

Note that $[\mathbf{R}^\star_\parallel]$ *does not equal* $[\gamma\mathbf{R}_\parallel]$ *as would hold if* $[\mathbf{R}_\parallel]$ *were part of a spacelike 4-vector* $(0, [\mathbf{R}_\parallel])$ *for which the Lorentz contraction would apply.*

[9] The argument of [17] is an elaboration of the "kink" model of Thomson [18] for the electric field of a charge that has constant velocity except for a brief interval of acceleration. This argument was also used in [19]. That the argument is not "simple" is illustrated by [20], where computational errors led to a claim that the Liénard-Wiechert fields are incorrect.

We next display three relations needed below (following sec. IV of [19]):

$$\begin{aligned}
\left[\left(\boldsymbol{\beta}^\star\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\parallel\right] &= \left[\beta^\star\dot{\beta}^\star_\parallel R^\star_\parallel\,\hat{\boldsymbol{\beta}}^\star\right] = \left[(\beta^\star R^\star_\parallel)\dot{\beta}^\star_\parallel\,\hat{\boldsymbol{\beta}}^\star\right] = \left[(\boldsymbol{\beta}\cdot\mathbf{R}^\star_\parallel)\dot{\boldsymbol{\beta}}^\star_\parallel\right], && (27)\\
\left[(\mathbf{R}^\star + R^\star\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}^\star\right] &= \left[(\mathbf{R}^\star_\parallel + R^\star\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}^\star_\parallel + \mathbf{R}^\star_\perp\cdot\dot{\boldsymbol{\beta}}^\star_\perp\right]\\
&= \left[\gamma^4\left\{\mathbf{R}_\parallel - R\boldsymbol{\beta} + (R-\boldsymbol{\beta}\cdot\mathbf{R})\boldsymbol{\beta}\right\}\cdot\dot{\boldsymbol{\beta}}_\parallel + \gamma^2\mathbf{R}_\perp\cdot\dot{\boldsymbol{\beta}}_\perp\right]\\
&= \left[\gamma^4(1-\beta^2)\mathbf{R}_\parallel\cdot\dot{\boldsymbol{\beta}}_\parallel + \gamma^2\mathbf{R}_\perp\cdot\dot{\boldsymbol{\beta}}_\perp\right] = \left[\gamma^2\mathbf{R}\cdot\dot{\boldsymbol{\beta}}\right], && (28)\\
\left[\boldsymbol{\beta}\times\left(\mathbf{R}^\star\times\dot{\boldsymbol{\beta}}^\star\right)\right] &= \left[\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star - (\boldsymbol{\beta}\cdot\mathbf{R}^\star)\dot{\boldsymbol{\beta}}^\star\right] = \left[\beta\dot{\beta}^\star_\parallel(\mathbf{R}^\star_\parallel + \mathbf{R}^\star_\perp) - \beta R^\star_\parallel\left(\dot{\boldsymbol{\beta}}^\star_\parallel + \dot{\boldsymbol{\beta}}^\star_\perp\right)\right]\\
&= \left[\beta\dot{\beta}^\star_\parallel\mathbf{R}^\star_\perp - \beta R^\star_\parallel\dot{\boldsymbol{\beta}}^\star_\perp\right] = \left[\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\perp - (\boldsymbol{\beta}\cdot\mathbf{R}^\star)\dot{\boldsymbol{\beta}}^\star_\perp\right]. && (29)
\end{aligned}$$

Of these, eq. (29) is readily anticipated in that $\left[\boldsymbol{\beta}\times\left(\mathbf{R}^\star\times\dot{\boldsymbol{\beta}}^\star\right)\right]$ *is perpendicular to* $[\boldsymbol{\beta}]$.

The ingredients for the Lorentz transformation of the electric field can be written as,

$$\begin{aligned}
\mathbf{E}^\star_\parallel &= q\left[\frac{\mathbf{R}^\star_\parallel}{R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\left(\mathbf{R}^\star\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\parallel}{R^{\star 3}}\right] - \frac{q}{c}\left[\frac{\dot{\boldsymbol{\beta}}^\star_\parallel}{R^\star}\right] && (30)\\
&= q\left[\frac{\mathbf{R}^\star_\parallel}{R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\left\{(\mathbf{R}^\star + R^\star\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}^\star\right\}\mathbf{R}^\star_\parallel}{R^{\star 3}}\right] - \frac{q}{c}\left[\frac{(R^\star + \boldsymbol{\beta}\cdot\mathbf{R}^\star_\parallel)\dot{\boldsymbol{\beta}}^\star_\parallel}{R^{\star 2}}\right]\\
&= q\left[\frac{\mathbf{R}_\parallel - R\boldsymbol{\beta}}{\gamma^2(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right] + \frac{q}{c}\left[\frac{\left(\mathbf{R}\cdot\dot{\boldsymbol{\beta}}\right)(\mathbf{R}_\parallel - R\boldsymbol{\beta})}{(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right] - \frac{q}{c}\left[\frac{R\dot{\boldsymbol{\beta}}_\parallel}{(R-\boldsymbol{\beta}\cdot\mathbf{R})^2}\right]\\
&= q\left[\frac{\mathbf{R}_\parallel - R\boldsymbol{\beta}}{\gamma^2(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right] + \frac{q}{c}\left[\frac{\left(\mathbf{R}\cdot\dot{\boldsymbol{\beta}}\right)(\mathbf{R}_\parallel - R\boldsymbol{\beta}) - R(R-\boldsymbol{\beta}\cdot\mathbf{R})\dot{\boldsymbol{\beta}}_\parallel}{(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right],\\
[\gamma]\,\mathbf{E}^\star_\perp - [\gamma\boldsymbol{\beta}]\times\mathbf{B}^\star &= q\left[\frac{\gamma\mathbf{R}^\star_\perp}{R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\gamma\left(\mathbf{R}^\star\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\perp}{R^{\star 3}}\right] - \frac{q}{c}\left[\frac{\gamma\dot{\boldsymbol{\beta}}^\star_\perp}{R^\star}\right] + \frac{q}{c}\left[\frac{\gamma\left\{\boldsymbol{\beta}\times\left(\mathbf{R}^\star\times\dot{\boldsymbol{\beta}}^\star\right)\right\}}{R^{\star 2}}\right]\\
&= q\left[\frac{\gamma\mathbf{R}^\star_\perp}{R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\gamma\left(\mathbf{R}^\star\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\perp}{R^{\star 3}}\right] - \frac{q}{c}\left[\frac{\gamma\dot{\boldsymbol{\beta}}^\star_\perp}{R^\star}\right]\\
&\quad + \frac{q}{c}\left[\frac{\gamma\left\{\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\perp - (\boldsymbol{\beta}\cdot\mathbf{R}^\star)\dot{\boldsymbol{\beta}}^\star_\perp\right\}}{R^{\star 2}}\right] && (31)\\
&= q\left[\frac{\gamma\mathbf{R}^\star_\perp}{R^{\star 3}}\right] + \frac{q}{c}\left[\frac{\gamma\left\{\left(\{\mathbf{R}^\star + R^\star\boldsymbol{\beta}\}\cdot\dot{\boldsymbol{\beta}}^\star\right)\mathbf{R}^\star_\perp - R^\star(R^\star + \boldsymbol{\beta}\cdot\mathbf{R}^\star)\dot{\boldsymbol{\beta}}^\star_\perp\right\}}{R^{\star 3}}\right]\\
&= q\left[\frac{\mathbf{R}_\perp}{\gamma^2(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right] + \frac{q}{c}\left[\frac{\left(\mathbf{R}\cdot\dot{\boldsymbol{\beta}}\right)\mathbf{R}_\perp - R(R-\boldsymbol{\beta}\cdot\mathbf{R})\dot{\boldsymbol{\beta}}_\perp}{(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right].
\end{aligned}$$

Finally, we arrive at the Liénard-Wiechert electric field,

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_{\parallel}^{\star} + [\gamma]\,\mathbf{E}_{\perp}^{\star} - [\gamma\boldsymbol{\beta}] \times \mathbf{B}^{\star} \\
&= q\left[\frac{\mathbf{R} - R\boldsymbol{\beta}}{\gamma^2(R - \boldsymbol{\beta}\cdot\mathbf{R})^3}\right] + \frac{q}{c}\left[\frac{\left(\mathbf{R}\cdot\dot{\boldsymbol{\beta}}\right)(\mathbf{R} - R\boldsymbol{\beta}) - R(R - \boldsymbol{\beta}\cdot\mathbf{R})\dot{\boldsymbol{\beta}}}{(R - \boldsymbol{\beta}\cdot\mathbf{R})^3}\right] \\
&= q\left[\frac{\mathbf{R} - R\boldsymbol{\beta}}{\gamma^2(R - \boldsymbol{\beta}\cdot\mathbf{R})^3}\right] + \frac{q}{c}\left[\frac{\mathbf{R}\times\left\{(\mathbf{R} - R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(R - \boldsymbol{\beta}\cdot\mathbf{R})^3}\right], \qquad (32)
\end{aligned}$$

noting that $R(R - \boldsymbol{\beta}\cdot\mathbf{R}) = \mathbf{R}\cdot(\mathbf{R} - R\boldsymbol{\beta})$. We had previously seen in eq. (24) that the Liénard-Wiechert expression for the magnetic field is,

$$\mathbf{B} = [\hat{\mathbf{R}}] \times \mathbf{E}. \qquad (33)$$

It is interesting that we find the general forms for the Liénard-Wiechert fields by an argument that is not completely general.

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