A Parallelogram Loop Antenna

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(May 28, 2003)

1 Problem

Deduce the far-zone electromagnetic fields and the radiation pattern of a loop antenna in the form of a parallelogram, as shown in Fig. 1, when a spatially uniform current of angular frequency ω flows around the loop.

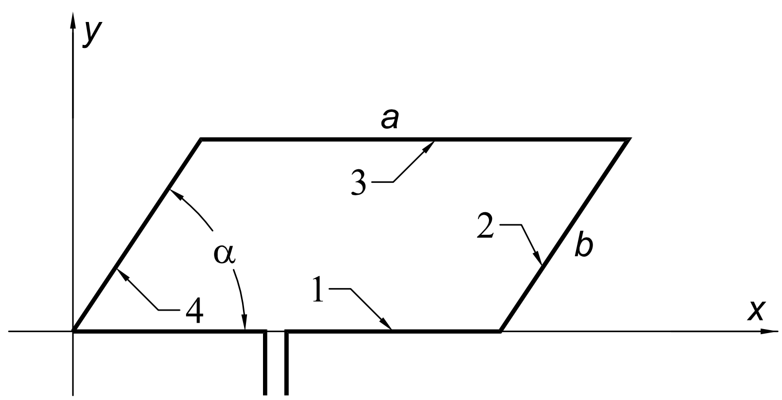

Figure 1: A loop antenna lies in the x-y plane and has the form of a parallelogram of edges a and b with angle α between adjacent sides. Current $I = I_0 e^{-i\omega t}$ flows uniformly around the loop, in which case the location of the feed makes no difference to the radiation pattern.

1.1 Solution

Rhombic antennas, which are a special case of pallaleogram loop antennas, were the subject of considerable discussion in the 1930's. See, for example, [1].

If the current distribution in an antenna is known, there is an "exact" procedure to calculate the far-zone electromagnetic fields and the corresponding radiation pattern.[1]

When the currents oscillate with angular frequency ω, we can write the current density $\mathbf{J}$ as

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}) e^{-i\omega t}. \tag{1}$$

From this, we can calculate the spatial Fourier transform

$$\mathbf{J}_\mathbf{k} = \int \mathbf{J}(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \, d\text{Vol}. \tag{2}$$

[1]While this result is "well-known" it is not often stated crisply in texts on electrodynamics. A reasonably compact statement of the procedure can be found in [2].

The far-zone electromagnetic fields are then given (in Gaussian units) by

$$\mathbf{B} = ik\frac{e^{i(kr-\omega t)}}{cr}\hat{\mathbf{k}} \times \mathbf{J}_{\mathbf{k}}, \qquad \mathbf{E} = \mathbf{B} \times \hat{\mathbf{k}}, \tag{3}$$

where we now interpret $\mathbf{k}$ as the wave vector of magnitude ω/c, where c is the speed of light, whose direction points from the antenna to the receiver. The time-averaged radiation pattern is then

$$\frac{dP}{d\Omega} = \frac{cr^2}{8\pi}\left|B\right|^2. \tag{4}$$

For an observer at angles (θ, ϕ) with respect to the z axis (in a spherical coordinate system with origin at the antenna), the unit wave vector has rectangular components

$$\hat{\mathbf{k}} = \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}. \tag{5}$$

In the present problem, the current flows in the x-y plane, so the current density $\mathbf{J}$ and its Fourier transform $\mathbf{J}_{\mathbf{k}}$ have no z component. Hence,

$$\begin{aligned} \hat{\mathbf{k}} \times \mathbf{J}_{\mathbf{k}} &= -J_{\mathbf{k},y}\cos\theta\ \hat{\mathbf{x}} + J_{\mathbf{k},x}\cos\theta\ \hat{\mathbf{y}} + (J_{\mathbf{k},y}\cos\phi - J_{\mathbf{k},x}\sin\phi)\sin\theta\ \hat{\mathbf{z}} \\ &= -\ J_{\mathbf{k},y}\cos\theta\ (\sin\theta\cos\phi\ \hat{\mathbf{r}} + \cos\theta\cos\phi\ \hat{\boldsymbol{\theta}} - \sin\phi\ \hat{\boldsymbol{\phi}}) \\ &\quad +\ J_{\mathbf{k},x}\cos\theta\ (\sin\theta\sin\phi\ \hat{\mathbf{r}} + \cos\theta\sin\phi\ \hat{\boldsymbol{\theta}} + \cos\phi\ \hat{\boldsymbol{\phi}}) \\ &\quad +\ (J_{\mathbf{k},y}\cos\phi - J_{\mathbf{k},x}\sin\phi)\sin\theta(\cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}) \\ &= (J_{\mathbf{k},x}\sin\phi - J_{\mathbf{k},y}\cos\phi)\ \hat{\boldsymbol{\theta}} + (J_{\mathbf{k},x}\cos\phi + J_{\mathbf{k},y}\sin\phi)\cos\theta\ \hat{\boldsymbol{\phi}}. \end{aligned} \tag{6}$$

Thus,

$$E_r = B_r = \hat{\mathbf{k}} \cdot \mathbf{B} = 0, \tag{7}$$

$$E_\theta = B_\phi = ik\frac{e^{i(kr-\omega t)}}{cr}(J_{\mathbf{k},x}\cos\phi + J_{\mathbf{k},y}\sin\phi)\cos\theta \tag{8}$$

$$E_\phi = -B_\theta = -ik\frac{e^{i(kr-\omega t)}}{cr}(J_{\mathbf{k},x}\sin\phi - J_{\mathbf{k},y}\cos\phi), \tag{9}$$

and the radiation pattern is given by

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{k^2}{8\pi c}\Big[\left|J_{\mathbf{k},x}\right|^2(\sin^2\phi + \cos^2\phi\cos^2\theta) + \left|J_{\mathbf{k},y}\right|^2(\cos^2\phi + \sin^2\phi\cos^2\theta) \\ &\qquad -\ 2Re(J_{\mathbf{k},x}J^\star_{\mathbf{k},y})(1-\cos\theta)\sin\phi\cos\phi\Big]. \end{aligned} \tag{10}$$

Clearly, eqs. (7)-(10) apply to any antenna whose currents flow only in the x-y plane.

It remains to evaluate the Fourier transform $\mathbf{J}_{\mathbf{k}}$. Assuming that the antenna is made of a conductor (wire) whose diameter is small compared to the wavelength, we can express the Fourier transform as a line integral around the loop antenna,

$$\mathbf{J}_{\mathbf{k}} = \int \mathbf{J}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\ d\text{Vol} = I_0\oint d\mathbf{l}\ e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{11}$$

where $d\mathbf{l}$ is the differential line element tangent to the conductor. For the case of the parallelogram antenna, we break up the line integral (11) into four integrals along the four sides of the antenna, using labels as shown in Fig. 1. Then,

$$
\begin{aligned}
d\mathbf{l}_1 &= dl\ \hat{\mathbf{x}}, && (12)\\
d\mathbf{l}_2 &= dl(\cos\alpha\ \hat{\mathbf{x}} + \sin\alpha\ \hat{\mathbf{y}}), && (13)\\
d\mathbf{l}_3 &= -dl\ \hat{\mathbf{x}} = -d\mathbf{l}_1, && (14)\\
d\mathbf{l}_4 &= dl(\cos\alpha\ \hat{\mathbf{x}} + \sin\alpha\ \hat{\mathbf{y}}) = -d\mathbf{l}_2. && (15)
\end{aligned}
$$

The position vector $\mathbf{r}$ on the four segments of the antenna can be written

$$
\begin{aligned}
\mathbf{r}_1 &= l\ \hat{\mathbf{x}} \qquad (0 \le l \le a), && (16)\\
\mathbf{r}_2 &= (a + l\cos\alpha)\ \hat{\mathbf{x}} + l\sin\alpha\ \hat{\mathbf{y}} \qquad (0 \le l \le b), && (17)\\
\mathbf{r}_3 &= (a + b\cos\alpha - l)\ \hat{\mathbf{x}} + b\sin\alpha\ \hat{\mathbf{y}} \qquad (0 \le l \le a), && (18)\\
\mathbf{r}_4 &= (b - l)\cos\alpha\ \hat{\mathbf{x}} + (b - l)\sin\alpha\ \hat{\mathbf{y}} \qquad (0 \le l \le b). && (19)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbf{k}\cdot\mathbf{r}_1 &= kl\sin\theta\cos\phi, && (20)\\
\mathbf{k}\cdot\mathbf{r}_2 &= k[(a + l\cos\alpha)\sin\theta\cos\phi + l\sin\alpha\sin\theta\sin\phi]\\
&= ka\sin\theta\cos\phi + kl\sin\theta\cos(\alpha - \phi), && (21)\\
\mathbf{k}\cdot\mathbf{r}_3 &= k[(a + b\cos\alpha - l)\sin\theta\cos\phi + b\sin\alpha\sin\theta\sin\phi]\\
&= ka\sin\theta\cos\phi + kb\sin\theta\cos(\alpha - \phi) - kl\sin\theta\cos\phi, && (22)\\
\mathbf{k}\cdot\mathbf{r}_4 &= k[(b - l)\cos\alpha\sin\theta\cos\phi + (b - l)\sin\alpha\sin\theta\sin\phi]\\
&= k(b - l)\sin\theta\cos(\alpha - \phi). && (23)
\end{aligned}
$$

With these forms, the Fourier transform (11) becomes

$$
\begin{aligned}
\mathbf{J}_\mathbf{k} &= I_0\ \hat{\mathbf{x}} \int_0^a dl\ e^{ikl\sin\theta\cos\phi} + I_0(\cos\alpha\ \hat{\mathbf{x}} + \sin\alpha\ \hat{\mathbf{y}})e^{ika\sin\theta\cos\phi} \int_0^b dl\ e^{ikl\sin\theta\cos(\alpha-\phi)}\\
&\quad - I_0\ \hat{\mathbf{x}}\ e^{ika\sin\theta\cos\phi}e^{ikb\sin\theta\cos(\alpha-\phi)} \int_0^a dl\ e^{-ikl\sin\theta\cos\phi}\\
&\quad - I_0(\cos\alpha\ \hat{\mathbf{x}} + \sin\alpha\ \hat{\mathbf{y}})e^{ikb\sin\theta\cos(\alpha-\phi)} \int_0^b dl\ e^{-ikl\sin\theta\cos(\alpha-\phi)}\\
&= 2I_0\ \hat{\mathbf{x}}\ e^{i\frac{ka}{2}\sin\theta\cos\phi} \frac{\sin(\frac{ka}{2}\sin\theta\cos\phi)}{k\sin\theta\cos\phi} \left[1 - e^{ikb\sin\theta\cos(\alpha-\phi)}\right]\\
&\quad + 2I_0(\cos\alpha\ \hat{\mathbf{x}} + \sin\alpha\ \hat{\mathbf{y}})e^{i\frac{kb}{2}\sin\theta\cos(\alpha-\phi)} \frac{\sin(\frac{kb}{2}\sin\theta\cos(\alpha-\phi))}{k\sin\theta\cos(\alpha-\phi)} \left(e^{ika\sin\theta\cos\phi} - 1\right)\\
&= 4iI_0\ \sin\alpha(-\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}})\ e^{i\frac{ka}{2}\sin\theta\cos\phi}e^{i\frac{kb}{2}\sin\theta\cos(\alpha-\phi)}\\
&\qquad \frac{\sin(\frac{ka}{2}\sin\theta\cos\phi)\sin(\frac{kb}{2}\sin\theta\cos(\alpha-\phi))}{k\sin\theta\cos\phi\cos(\alpha-\phi)}\\
&= 4iI_0\ \sin\alpha\ \hat{\boldsymbol{\phi}}\ e^{i\frac{ka}{2}\sin\theta\cos\phi}e^{i\frac{kb}{2}\sin\theta\cos(\alpha-\phi)}\\
&\qquad \frac{\sin(\frac{ka}{2}\sin\theta\cos\phi)\sin(\frac{kb}{2}\sin\theta\cos(\alpha-\phi))}{k\sin\theta\cos\phi\cos(\alpha-\phi)}. && (24)
\end{aligned}
$$

That is, the Fourier transform $\mathbf{J_k}$ (which is proportional to the vector potential) has only an azimuthal component in the present case.

The far-zone electromagnetic fields follow from eqs. (7)-(9),

$$\begin{aligned} E_\theta = B_\phi &= 0, && (25) \\ E_\phi = -B_\theta &= -4I_0 \sin\alpha \frac{e^{i(kr-\omega t)}}{cr} e^{i\frac{ka}{2}\sin\theta\cos\phi} e^{i\frac{kb}{2}\sin\theta\cos(\alpha-\phi)} \\ & \qquad \frac{\sin(\frac{ka}{2}\sin\theta\cos\phi)\sin(\frac{kb}{2}\sin\theta\cos(\alpha-\phi))}{\sin\theta\cos\phi\cos(\alpha-\phi)} . && (26) \end{aligned}$$

The radiation is linearly polarized, parallel to the x-y plane.

The time-averaged radiation pattern for the general parallelogram loop antenna follows from eq. (4),

$$\frac{dP}{d\Omega} = \frac{2I_0^2 \sin^2\alpha \sin^2(\frac{ka}{2}\sin\theta\cos\phi)\sin^2(\frac{kb}{2}\sin\theta\cos(\alpha-\phi))}{\pi c \sin^2\theta \cos^2\phi \cos^2(\alpha-\phi)} . \qquad (27)$$

For angles θ very near 0° or 180° the radiation pattern simplifies to

$$\frac{dP}{d\Omega} \approx \frac{k^4 I_0^2 (ab\sin\alpha)^2 \sin^2\theta}{8\pi c} = \frac{\omega^4 I_0^2 \mathrm{Area}^2 \sin^2\theta}{8\pi c^5} , \qquad (28)$$

which is the same as for a circular loop of the same area as the parallelogram. Of course, the radiation is very weak for θ near 0° and 180°.

While the denominator of eq. (27) also vanishes when $\phi = \pm 90°$ and $\alpha \pm 90°$, the numerator vanishes there also, such that the radiation pattern actually is nowhere divergent. However, the radiation pattern has zeroes at other angles (θ, ϕ) such that $ka\sin\theta\cos\phi = 2m\pi$ and $kb\sin\theta\cos\phi = 2n\pi$, for integers m and n.

2 Examples

2.1 "Small" Loop

For a "small" loop, where $ka \ll 1$ and $kb \ll 1$ (*i.e.*, $a \ll \lambda$ and $b \ll \lambda$), the electric field simplifies to

$$\mathbf{E} = -\hat{\boldsymbol{\phi}}\, k^2 I_0 ab \sin\alpha \frac{e^{i(kr-\omega t)}}{cr} \sin\theta = -\hat{\boldsymbol{\phi}}\, m_0 \omega^2 \frac{e^{i(kr-\omega t)}}{c^2 r} \sin\theta \qquad (\text{“small” loop}), \qquad (29)$$

where

$$m_0 = \frac{I_0 ab \sin\alpha}{c} = \frac{I_0 \mathrm{Area}}{c} \qquad (30)$$

is the magnetic moment of the current distribution. Thus, a "small" loop is completely equivalent to a circular loop of the same area, which emits pure magnetic dipole radiation. While we have shown this only for a parallelogram loop, it is "clearly" true for any small planar loop, including a "fractal" loop antenna [3].

The radiation pattern is

$$\frac{dP}{d\Omega} = \frac{\omega^4 I_0^2 \mathrm{Area}^2 \sin^2\theta}{8\pi c^5} . \tag{31}$$

The time-averaged radiated power is the integral of this with respect to solid angle $d\Omega$,

$$P = \frac{\omega^4 I_0^2 \mathrm{Area}^2}{3c^5} = \frac{1}{2} I_0^2 \frac{32\pi^4}{3c} \frac{\mathrm{Area}^2}{\lambda^4} = \frac{1}{2} I_0^2 R_{\mathrm{antenna}}, \tag{32}$$

where

$$R_{\mathrm{antenna}} = \frac{32\pi^4}{3c} \frac{\mathrm{Area}^2}{\lambda^4} = 31,170\Omega \frac{\mathrm{Area}^2}{\lambda^4} \tag{33}$$

is the radiation resistance of the antenna, recalling that $1/c = 30\Omega$.

2.2 Rectangular Loop ($\alpha = 90°$)

$$\frac{dP}{d\Omega} = \frac{8 I_0^2 \sin^2(\frac{ka}{2}\sin\theta\cos\phi)\sin^2(\frac{kb}{2}\sin\theta\sin\phi)}{\pi c \sin^2\theta \sin^2 2\phi} . \tag{34}$$

2.3 Square Loop

Here, $a = b$. The radiation pattern in, say, the x-z plane ($\phi = 0$) is

$$\frac{dP(\phi = 0)}{d\Omega} = \frac{k^2 a^2 I_0^2 \sin^2(\frac{ka}{2}\sin\theta)}{2\pi c} . \tag{35}$$

Of course, the pattern is the same in the y-z plane.

The radiation pattern for a square with $a = 4.4\lambda$ is shown in the left side of Fig. 2. For comparison, the right side of that figure shows the pattern for a circular loop whose diameter is 5λ (see also sec. 2.4). As anticipated in our general discussion, these two radiation patterns are only closely similar near the z (vertical) axis.

2.4 Circular Loop

For comparison, we briefly consider that case of a circular loop of radius a that carries current $I_0 e^{-i\omega t}$. The loop lies in the x-y plane and is centered on the origin. Then the current element is $d\mathbf{l} = a d\phi_0\ \hat{\boldsymbol{\phi}}_0$ where ϕ_0 is the azimuth to the current element; the phase is $\mathbf{k}\cdot\mathbf{r} = ka\sin\theta\cos(\phi_0 - \phi) \equiv ka\sin\theta\cos\psi$, so the Fourier transform is

$$\begin{aligned} \mathbf{J_k} &= aI_0 \int_0^{2\pi} d\phi_0\ \hat{\boldsymbol{\phi}}_0 e^{ika\sin\theta\cos(\phi_0-\phi)} = aI_0 \int_0^{2\pi} d\phi_0\ (-\sin\phi_0\ \hat{\mathbf{x}} + \cos\phi_0\ \hat{\mathbf{y}}) e^{ika\sin\theta\cos(\phi_0-\phi)} \\ &= aI_0 \int_0^{2\pi} d\psi\ (-\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}})\cos\psi e^{ika\sin\theta\cos\psi} = -2\pi i a I_0 J_1(ka\sin\theta)\ \hat{\boldsymbol{\phi}}, \end{aligned} \tag{36}$$

where J_1 is the Bessel function of order 1. The magnetic field is

$$\mathbf{B} = \frac{2\pi k a I_0 e^{i(kr-\omega t)}}{cr} J_1(ka\sin\theta)\ \hat{\boldsymbol{\theta}}. \tag{37}$$

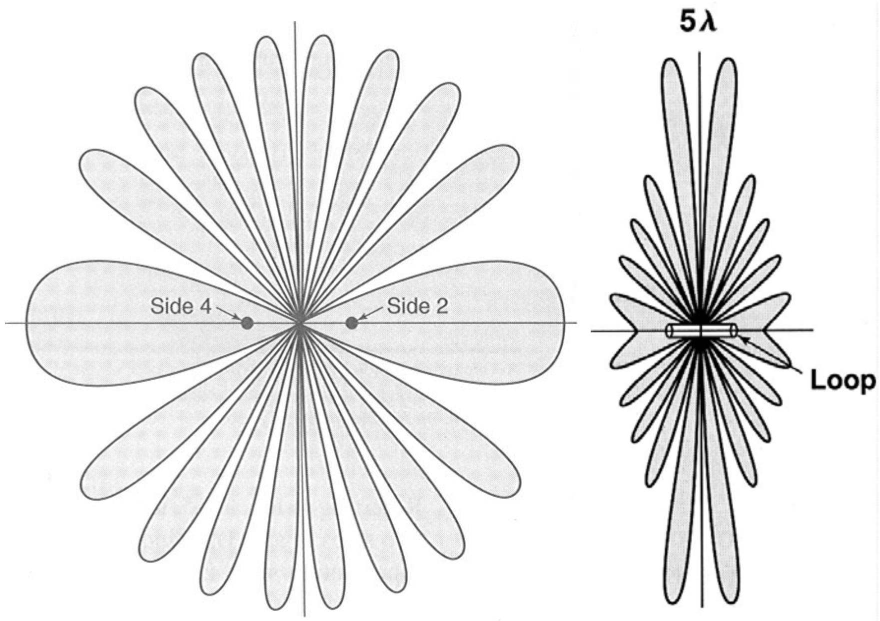

Figure 2: Radiation patterns in the x-z planes for two loops antennas that lie in the x-y plane. Left: square loop with edge = 4.4 λ. Right: circular loop with diameter = 5 λ. From [4].

The time-averaged radiation pattern is

$$\frac{dP}{d\Omega} = \frac{\pi k^2 a^2 I_0^2}{2c} J_1^2(ka \sin\theta). \tag{38}$$

Since the Bessel function J_1 is something like a damped sine wave, the radiation pattern is stronger near the z axis than near the x-y plane, as shown in the right side of Fig. 2.

For a small loop, $ka \ll 1$, we have $J_1(ka\sin\theta) \approx (ka/2)\sin\theta$, so that

$$\frac{dP}{d\Omega} = \frac{\pi k^4 a^4 I_0^2}{8c}\sin^2\theta = \frac{2\pi^5 a^4 I_0^2}{\lambda^4 c}\sin^2\theta \qquad \text{(small circular loop)}, \tag{39}$$

and the total radiated power is

$$P = \frac{1}{2} I_0^2 \frac{32\pi^6}{3c}\frac{a^4}{\lambda^4} = \frac{1}{2} I_0^2\ 306,645\Omega \frac{a^4}{\lambda^4} = \frac{1}{2} I_0^2 R_{\rm antenna} \qquad \text{(small circular loop)}, \tag{40}$$

in agreement with sec. 2.1. To have, say, $R_{\rm antenna} = 50\Omega$, use $a \approx \lambda/9$.

For a large loop, $ka \gg 1$, the total radiated power is given by[2]

$$\begin{aligned} P &= \frac{\pi k^2 a^2 I_0^2}{2c} 2\pi \int_0^\pi J_1^2(ka\sin\theta)\sin\theta\ d\theta = \frac{\pi^2 k^2 a^2 I_0^2}{c}\frac{1}{ka}\int_0^{2ka} J_2(t)\ dt \\ &\approx \frac{\pi^2 ka I_0^2}{c} = \frac{1}{2} I_0^2 \frac{2\pi^3}{c}\frac{a}{\lambda} = \frac{1}{2} I_0^2\ 1,860\Omega\frac{a}{\lambda} \qquad \text{(large circular loop)}. \end{aligned} \tag{41}$$

[2]Kraus [4] claims that the integral relation $\int_0^\pi J_1^2(ka\sin\theta)\sin\theta\ d\theta = (1/ka)\int_0^{2ka} J_2(t)\ dt$ is given somewhere in [5], but I haven't been able to find it there. Schwinger [6] deduces this from Neumann's integral (p. 32 of [5]). Thanks to J. Castro for pointing this out.

It is noteworthy that the integral of $J_1^2(ka \sin\theta)$ varies as $1/ka$, such that the radiation resistance of a large circular loop varies only as a/λ, in contrast to a^4/λ^4 for a small loop.

2.5 "Large" Square Loop

If $a \approx b \gg \lambda$, then the radiation pattern (27) has ka/π "lobes" as a function of θ. If we average over these rapid variations in angle θ, the angular distribution is approximately isotropic. Comparing with eq. (35), we see that

$$\frac{dP(\phi = 0)}{d\Omega} \approx \frac{k^2 a^2 I_0^2}{4\pi c}. \tag{42}$$

As the azimuthal angle ϕ is varied, the radiation pattern oscillates in a complicated manner, as discussed briefly at the end of sec. 1. We estimate that the average effect of these oscillations is to cut the radiated power in half again, so we write

$$\left\langle \frac{dP}{d\Omega} \right\rangle \approx \frac{k^2 a^2 I_0^2}{8\pi c}. \tag{43}$$

The time-averaged radiated power is approximately 4π times this, *i.e.*,

$$P \approx \frac{k^2 a^2 I_0^2}{2c} = \frac{1}{2} I_0^2 \frac{4\pi^2}{3c} \frac{a2}{\lambda^2} = \frac{1}{2} I_0^2 R_{\rm antenna}, \tag{44}$$

where

$$R_{\rm antenna} = \frac{4\pi^2}{3c} \frac{a^2}{\lambda^2} = 395\Omega \frac{a^2}{\lambda^2} \tag{45}$$

is the radiation resistance. Since we have assumed that $a \gg \lambda$, the radiation resistance is large.

Is it really correct that the radiation resistance of a large square loop of edge a varies as a^2/λ^2 while that for a large circular loop of radius a varies as a/λ? If correct, this is because the pattern for a square loop "bulges" out at the "equator" much more than does the pattern for a circular loop. However, it could be that the averaging over aziumuthal angle ϕ does not yield a factor 1/2 as claimed, but rather a much smaller factor $1/ka$, which could bring the result for a large square loop into better agreement with that for a large circular loop.

2.6 Folded Dipole

We take this to be a rectangular loop antenna with side $b \ll \lambda$. Then,

$$\frac{dP}{d\Omega} = \frac{k^2 b^2 I_0^2 \sin^2(\frac{ka}{2} \sin\theta \cos\phi)}{8\pi c \cos^2\phi}. \tag{46}$$

If the current is uniform around the antenna, the radiation is very weak for small b, since the radiation from the two long arms of the antenna very nearly cancels. Of more interest would be the case when the current is nonuniform, such that the currents in the two long arms flow in the same direction. For this, the antenna must be considered as a kind of resonant cavity that is excited in a desirable mode, which possibility is beyond the scope of this note.

References

[1] D. Foster, *Radiation from Rhombic Antennas*, Proc. I.R.E. **25**, 1327 (1937), http://physics.princeton.edu/~mcdonald/examples/EM/foster_pire_25_1327_37.pdf

[2] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), sec. 14-4, http://physics.princeton.edu/~mcdonald/examples/EM/panofsky-phillips.pdf

[3] K.T. McDonald, *Small Fractal Antennas* (Dec. 22, 2003), http://physics.princeton.edu/~mcdonald/examples/fractal_antenna.pdf

[4] J.D. Kraus and R. Marhefka, *Antennas for All Applications*, 3rd ed. (McGraw-Hill, Boston, 2002), chap. 7.

[5] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge U. Press, 1996).

[6] See p. 14 of J. Schwinger, *On Radiation of Electrons in a Betatron* (1945), http://physics.princeton.edu/~mcdonald/accel/schwinger.pdf