Torricelli's Law for Large Holes

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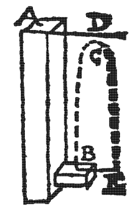


Torricelli's law appeared on pp. 191-192 of [1] (1644), where he observed that a water jet, which emerges from a small, upwards facing hole B in a suitable projection from the bottom of a tank A, rises to the same height C as the water level D in the tank. The upwards velocity at B is the same as the downwards velocity at E, namely $\sqrt{2gh}$, where g is the acceleration due to gravity, and h is the height of the water level D above points B and E.

Torricelli's law was first explained by D. Bernoulli, p. 37 of [2],[1] via an energy argument that will be reviewed below. Nowadays, Torricelli's law is typically presented as an example of the steady-state Bernoulli equation, which does lead to the prediction that the water emerges from a hole with velocity $v = \sqrt{2gh}$ when the hole is at depth h below the water level in the tank. However, if the area a of the hole (in the bottom of the tank) is the same as the cross-sectional area A of the tank, the water simply falls out of the tank with acceleration g due to gravity, and velocity at the bottom of the tank given by $v = \sqrt{2g(h_0 - h)}$, where h_0 is the initial depth of water in the tank.

That is, water flowing from a tank is not a steady process, and so Bernoulli's equation applies only in the limit of very small holes, for which the flow is essentially steady.

1 Bernoulli's Analysis for Large Holes

Bernoulli's original analysis of Torricelli's example did not assume that the area a of the hole was small compared to the cross-sectional area A of the water tank

We follow Bernoulli in assuming that the exit hole is at the center of the bottom of a right-circular-cylindrical tank. We further assume that the flow velocity in the tank is essentially vertical, and ignore the small, horizontal component of the water-flow velocity. As such, Bernoulli's solution (and the alternative solutions presented in this note) cannot be correct in all detail.

We also suppose that the water is incompressible, and inviscid (so that no energy is lost to friction during the flow). Then, the continuity equation relates the (vertical) velocity v of water in the tank to the (vertical) velocity V of the water at the exit hole according to,

$$v = \frac{aV}{A}\,. \tag{1}$$

Also, the velocity v is the rate of change with respect to time t of the depth h of water in the tank,

$$v = -\frac{dh}{dt} \equiv -\dot{h} = \frac{aV}{A}\,. \tag{2}$$

[1] See also [3].

Bernoulli's method (an innovation in 1738) was to set the rate at which work is done by gravity on the water in the tank equal to the rate of change of kinetic energy of the water in the tank plus the rate at which kinetic energy exits the tank through the hole. That is, the method is based on conservation of energy.

The rate dW/dt of gravitational work on the water in the tank at time t is the product of the rate $\rho g v$ of gravitational work per unit volume, and the volume Ah of the water in the tank,

$$\frac{dW}{dt} = \rho g v A h = \rho g V a h, \tag{3}$$

where ρ is the mass density of the water.

The total kinetic energy of the water in the tank is the product of the kinetic energy per unit volume $\rho v^2/2$ and by the volume of the water in the tank,

$$\mathrm{KE}_{\mathrm{tank}} = \frac{\rho v^2}{2} A h = \frac{\rho V^2}{2} \frac{a^2}{A} h, \tag{4}$$

$$\frac{d\,\mathrm{KE}_{\mathrm{tank}}}{dt} = \frac{\rho V^2}{2} \frac{a^2}{A} \frac{dh}{dt} + \rho V \frac{dV}{dt} \frac{a^2}{A} h = -\frac{\rho V^3}{2} \frac{a^3}{A^2} + \rho V \frac{dV}{dt} \frac{a^2}{A} h, \tag{5}$$

using eq. (2) to obtain the last form of eq. (5).

The rate at which kinetic energy exits the tank, with velocity V, is given by,

$$\frac{d\,\mathrm{KE}_{\mathrm{exit}}}{dt} = \frac{dm_{\mathrm{exit}}}{dt} \frac{V^2}{2} = \rho V a \frac{V^2}{2} \,. \tag{6}$$

Conservation of energy now implies that,

$$\frac{dW}{dt} = \rho g V a h = \frac{d\,\mathrm{KE}_{\mathrm{tank}}}{dt} + \frac{d\,\mathrm{KE}_{\mathrm{exit}}}{dt} = -\frac{\rho V^3}{2} \frac{a^3}{A^2} + \rho V \frac{dV}{dt} \frac{a^2}{A} h + \rho V a \frac{V^2}{2} \,. \tag{7}$$

If we divide eq. (7) by $\rho V a$, we obtain,

$$g h = \left(1 - \frac{a^2}{A^2}\right) \frac{V^2}{2} + \frac{a}{A} h \frac{dV}{dt} = -h \ddot{h} + \frac{\dot{h}^2}{2} \left(\frac{A^2}{a^2} - 1\right) \tag{8}$$

Time t can be replaced as the independent variable in this equation by the depth h, by combining the first form of eq. (8) with eq. (2) to yield,[2]

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = -\frac{a}{A} V \frac{dV}{dh} = -\frac{1}{2} \frac{a}{A} \frac{dV^2}{dh} \,, \tag{9}$$

$$2 g h = \left(1 - \frac{a^2}{A^2}\right) V^2 - \frac{a^2}{A^2} h \frac{dV^2}{dh} \,. \tag{10}$$

Equation (10) tells us that the velocity V of the effluent stream from the tank is a function of the area ratio a/A, the water depth h, and the initial conditions $v_0 = 0 = V_0$ at time

[2]The last term in eq. (10) involves the derivative of V^2 with respect to h, which term captures the effect of the fluid acceleration in the tank that is omitted in the steady-flow version of the Bernoulli equation. Equation (10) is consistent with the so-called extended Bernoulli equation presented in Appendix A.

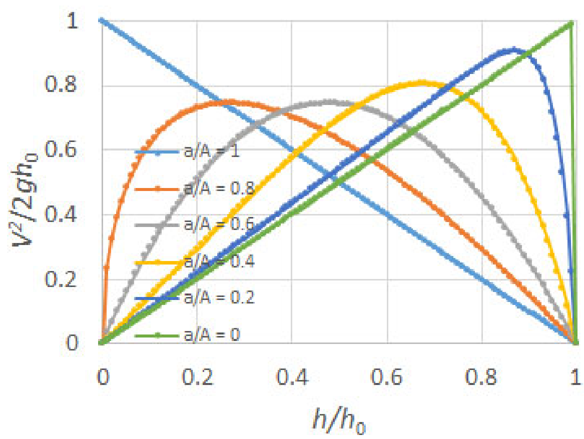
$t = 0$, when depth of the (inviscid) water in the tank is h_0. The solution to this equation, subject to these initial conditions, is given by,[3]

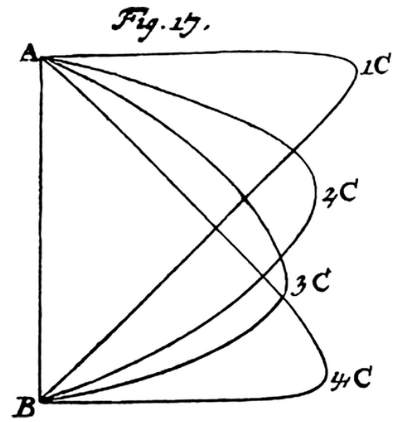
$$V(h) = \sqrt{2gh\frac{1-(h/h_0)^{\frac{1-2r}{r}}}{1-2r}} = \sqrt{2gh_0}\sqrt{\frac{h}{h_0}\frac{1-(h/h_0)^{\frac{1-2r}{r}}}{1-2r}} \qquad (0 \le h \le h_0), \tag{16}$$

where $r = (a/A)^2$. For the limiting cases in which $r = 0$ or 1, this solution reduces to,

$$V(h, r = 0) = \sqrt{2gh}, \qquad V(h, r = 1) = \sqrt{2g(h_0 - h)}. \tag{17}$$

For the case of $r = 1$ (*i.e.*, the case in which the exit-hole area is equal to the tank area), the above equation for the efflux velocity V is, as expected, just that predicted for free fall.





Results from eq. (16) for the efflux V^2, normalized by $2gh_0$, as a function of the (dimensionless) fluid-depth ratio h/h_0 and the (dimensionless) area ratio a/A are shown in the

[3] We follow, for example, https://en.wikipedia.org/wiki/Linear_differential_equation, in the section on *First-order equation with variable coefficients*, and write eq. (10) as,

$$\frac{dV^2}{dh} = V^2\frac{1-r}{hr} - \frac{2g}{r}, \tag{11}$$

with $r = a^2/A^2$. The solution is,

$$V^2 = e^F\left(C - \frac{2g}{r}\int e^{-F}\,dh\right), \tag{12}$$

where C is a constant and,

$$F = \int\frac{1-r}{hr}\,dh = \frac{1-r}{r}\ln h, \qquad e^F = h^{\frac{1}{r}-1}, \qquad \int e^{-F}\,dh = \int h^{1-\frac{1}{r}}\,dh = \frac{r}{2r-1}h^{2-\frac{1}{r}}. \tag{13}$$

Hence,

$$V^2 = h^{\frac{1}{r}-1}\left(C + \frac{2g}{1-2r}h^{2-\frac{1}{r}}\right) = \left(Ch^{\frac{1}{r}-1} + \frac{2gh}{1-2r}\right) = \left(Ch\,h^{\frac{1}{r}-2} + \frac{2gh}{1-2r}\right). \tag{14}$$

Since $V^2 = 0$ when $h = h_0$, $C = -2gh_0^{2-\frac{1}{r}}/(1-2r)$, and finally,

$$V^2 = \frac{2gh}{1-2r}\left(1-(h/h_0)^{\frac{1-2r}{r}}\right). \tag{15}$$

figure on the left of the previous page (note that time increases from right to left in the figure).[4]

In all cases, the efflux velocity V is equal to zero initially (*i.e.*, when $h = h_0$), and then rises rapidly as the fluid, both inside the tank and in the efflux, accelerates. However, as the depth h of fluid in the tank decreases, the efflux velocity V passes through a maximum and decreases thereafter. Eventually, as the fluid depth h approaches zero, the efflux velocity V, of course, also drops to zero. In the case of $a/A = 1$, the maximum velocity is attained just as the tank reaches empty.[5]

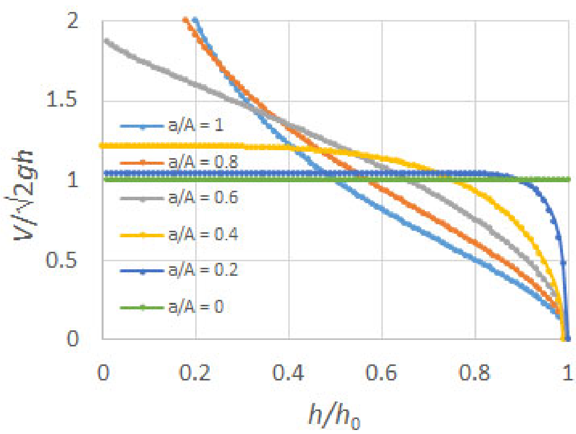
1.1 Draining a Tank through a Small Hole

For a small hole, $a \ll A$, the acceleration $\ddot{h}$ can be neglected in eq. (8), which can then be written as.

$$\frac{\dot{h}}{\sqrt{h}} = -\sqrt{\frac{2g}{A^2/a^2 - 1}}, \qquad \sqrt{h} = \sqrt{h_0} - \sqrt{\frac{g}{2(A^2/a^2 - 1)}}t, \tag{20}$$

[4]The figure on the right is from [2], with horizontal axis V^2 and vertical axis $h_0 - h$. Curve 1 is for a small hole, and curve 4 is for a large one.

[5]It may also be of interest to consider the exit velocity normalized to the instantaneous depth $h(t)$ of fluid in the tank, rather than to the depth h_0 at time zero.



According to the results in the figure, for (small) values of the area ratio a/A less than 0.5, the dimensionless efflux velocity $V/\sqrt{2gh}$ levels off to a constant value as the depth h of fluid in the tank decreases. The smaller the value of a/A, the more rapidly the dimensionless velocity levels off. From our analytic solution, eq. (16), the value to which $V/\sqrt{2gh}$ levels off is given by,

$$\frac{V(h \to 0)}{\sqrt{2gh}} = \sqrt{\frac{1}{1 - 2(a/A)^2}}. \tag{18}$$

For the case of $a/A = 0.2$, for example, we see from the figure above that, once the depth h has decreased to about 80% of the initial depth h_0, the velocity has already leveled off.

For small values of a/A, eq. (18) can be expressed, to order $r = (a/A)^2$ by,

$$\frac{V(h \to 0; r \ll 1)}{\sqrt{2gh}} \approx 1 + \frac{a^2}{A^2}. \tag{19}$$

noting that $\dot{h}$ is negative. The time t_d for the tank to drain (to $h = 0$) is,

$$t_d = \sqrt{\frac{2h_0}{g}} \sqrt{\frac{A^2}{a^2} - 1} \approx \sqrt{\frac{2h_0}{g}} \frac{A}{a} \,. \tag{21}$$

Many quantitative experimental studies of the draining of a tank exist, including [4]-[23]. Only two experiments [14, 15] report good agreement with eq. (21),[6] while it is more typically found that the tank drains roughly 1.4 times more slowly. This is partly attributable to energy dissipation by viscous effects [6], which we ignore in this note. Also, the tank never actually drains completely but ends with a water level Δh above the outlet, as surface tension exerts a small force to keep water in the tank [18, 23]; this effect can be accommodated by replacing h_0 with $h_0 - \Delta h$ in eq. (21).

Even for liquid with very low viscosity and low surface tension, the empirical evidence [11, 12, 16, 20] is that drainage time is better represented as

$$t_d \approx \sqrt{\frac{2h_0}{g}} \frac{A}{C_d\, a} \,, \tag{22}$$

where the so-called coefficient of discharge C_d is roughly 1/2.

This is related to a comment by Borda (1766) [24] that if one supposes the water at the small hole to be at atmospheric pressure, there is a conflict between the energy analysis of Bernoulli and a momentum analysis, in which the velocity V should be smaller by a factor of $\sqrt{2}$ than that found in the energy analysis.[7] Borda argued that this difficulty is resolved by the *vena contracta*, the reduction in the area of the water jet to $a' \approx 0.64\, a$ (apparently first observed by Torricelli), with the implication that the velocity (and momentum) of the water at the hole is roughly $1/\sqrt{2}$ that a short distance away from the hole.

Stated another way, the velocity $V = \sqrt{2gh}$ found by the energy analysis for a small hole applies not at the hole itself, but at a distance roughly one diameter $\sqrt{a/\pi}$ away from the hole,[8] and is valid for predicting the trajectory of the efflux when the hole is in the vertical wall of the tank. However, if one wishes to emphasize the flow rate, then the energy argument must be corrected for the *vena contracta*, while a momentum analysis can be used without mention of it (although the momentum argument must be corrected for the *vena contracta* to describe the trajectory of an efflux whose initial velocity is horizontal).

[6] In [14], the tank was a right circular cylinder and the drain was a tube of length larger than its diameter. In [15], the tank was an inverted 2-liter soda bottle (whose shape tapers towards the cap) with a 7.2-mm-diameter hole in its cap. It seems that these variants on the nominal case of a right-circular-cylinder tank with a sharp-edged hole happened to compensate for the "corrections" that apply to the nominal case.

[7] For a review of this argument, see [25].

[8] This result was stated by Newton on p. 333 of [26], following an argument in which part of the water in the tank could be ice.

A Appendix: Use of the Extended Bernoulli Equation

The nominal form of Bernoulli's equation is for steady, incompressible, inviscid fluid flow in an inertial frame of reference, relating the fluid pressure P and velocity u at two points along a streamline via conservation of energy,

$$P_1 + \frac{\rho u_1^2}{2} + \rho g h_1 = P_2 + \frac{\rho u_2^2}{2} + \rho g h_2 \qquad \text{(steady Bernoulli)}, \tag{23}$$

where h is the height of a point in a gravitational field with acceleration g. Bernoulli's equation can be extended to the case of nonsteady, compressible, rotational, elasto-viscoplastic flow in a noninertial reference frame by the addition of a "correction" term obtained by an appropriate integration along the streamline,

$$P_1 + \frac{\rho u_1^2}{2} + \rho g h_1 = P_2 + \frac{\rho u_2^2}{2} + \rho g h_2 + \int_1^2 \text{"correction"}, \qquad \text{(extended Bernoulli)}, \tag{24}$$

where the (complicated) "correction" term is displayed in eq. (12) of [27].

In the present example of unsteady, but incompressible flow, still in an inertial frame where rotation of the fluid is neglected, only a single "correction" applies,[9]

$$P_1 + \frac{\rho u_1^2}{2} + \rho g h_1 = P_2 + \frac{\rho u_2^2}{2} + \rho g h_2 + \int_1^2 \rho \frac{\partial \mathbf{u}}{\partial t} \cdot d\mathbf{l}, \tag{25}$$

where we make the approximation that $\mathbf{u}(\mathbf{r}, t) = \dot{h}\,\hat{\mathbf{z}}$, with the z-axis vertical and upwards, is the unsteady velocity of the fluid in the system (which ignores the small horizontal velocity of the water inside the tank). Taking point 1 at the center of the upper surface of the water in the tank ($z = h$), and point 2 at the center of the hole at the bottom of the tank ($z = 0$), we ignore the tiny difference in atmospheric pressure between these points, and note that $u_1 = \dot{h} = -aV/A$, $u_2 = -V$, and,

$$\int_1^2 \rho \frac{\partial \mathbf{u}}{\partial t} \cdot d\mathbf{l} = \rho \int_h^0 \frac{d^2 h}{dt^2}\, dz = -\rho h \frac{d^2 h}{dt^2} = \rho \frac{a}{A} h \frac{dV}{dt}\,. \tag{26}$$

Then, eq. (25) becomes, after dividing by ρ,

$$gh = \left(1 - \frac{a^2}{A^2}\right) \frac{V^2}{2} + \frac{a}{A} h \frac{dV}{dt}\,. \tag{27}$$

as previously found in eq. (8). As remarked after eq. (10) above, the "correction" term in the extended Bernoulli equation is the last term of eq. (27), which is negligible for a small hole in the tank is small, $a/A \ll 1$, in which case $V^2 \approx 2gh$, as first found by Torricelli [1], and argued in most textbooks on the basis of the (steady) Bernoulli equation (23).

[9]This relatively simple form of the extended/unsteady Bernoulli equation is deduced from Euler's equation at https://ocw.mit.edu/courses/mechanical-engineering/2-25-advanced-fluid-mechanics-fall-2013/inviscid-flow-and-bernoulli/MIT2_25F13_Unstea_Bernou.pdf

B Appendix: A Lagrangian Approach

A Lagrangian approach to variable-mass problems has been given in [28, 29].

For the present example, it seems appropriate to consider the system to be only the water still in the tank, which can be characterized by a single coordinate $q = h$. The velocity V of the efflux of water from the tank is related by the continuity equation for incompressible fluids, as in eq. (1) above,

$$V = \frac{av}{A} = -\frac{a\dot{h}}{A}\,. \tag{28}$$

The kinetic energy of the system is,

$$T = \frac{\rho A h \dot{h}^2}{2}\,. \tag{29}$$

While one can give an expression for the gravitational potential energy of this system, the force on the system is not simply related to this potential energy, so the latter is not used in the method of [28]. Rather, one uses a generalized force, Q_h as was introduced by Lagrange.

We recall that for a system with a set of coordinates q_k (which could be functions of time t) and kinetic energy $T(q_k, \dot{q}_k, t)$, Lagrange's equations can be written as,[10]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k = \sum_i \mathbf{F}_i^{\rm ext} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_i \mathbf{F}_i^{\rm ext} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \tag{30}$$

where $\mathbf{r}_i$ is the (x, y, z) coordinate of the $i^{\rm th}$ particle in the system, and $\mathbf{F}_i^{\rm ext}$ is the external force on particle i.[11] The forms for the generalized force given in eq. (30) follow from arguments by d'Alembert [31]. If the external forces are deducible from potentials, $\mathbf{F}_i = -\partial V_i/\partial \mathbf{r}_i$, then the first form of eq. (24) simplifies to

$$Q_k = -\frac{\partial V}{\partial q_k}\,, \tag{31}$$

where $V = \sum_i V_i$.

In a variable-mass problem such as the present example, the flow of water out of the hole in tank is associated with a reaction force on the water still in the tank. In the Newtonian approach, this reaction force must be included in the equation(s) of motion, but in Lagrangian approach the reaction force is not considered to be an external force, and so is not to be included in the generalized forces.

In the present example, a water molecule i has position (x_i, y_i, z_i) that does not depend directly on h, so the generalized force would be $Q_h = 0$ according to the first form of eq. (30). The velocity of a water molecule is, to a good approximation $(0, 0, \dot{h})$, and the external force

[10] See, for example, p. 41 of [30]. The fact that $\partial \mathbf{r}_i/\partial q_k = \partial \dot{\mathbf{r}}_i = \partial \dot{q}_k$ is deduced on p. 40 of [30].

[11] We recall that Lagrange's method distinguishes between external and constraint forces. In the present example, the upward normal force on the bottom of the tank, which holds it at rest, is a constraint force, and so is not included in the computation of the generalized force.

on this molecule is $\mathbf{F}^{\rm ext} = -m_{\rm mol}\, g\,\hat{\mathbf{z}}$, so the generalized force Q_h according to the second form of eq. (30) is given by,

$$Q_h = -\sum_i m_{{\rm mol},i}\, g\,\hat{\mathbf{z}} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{h}} = -\sum_i m_{{\rm mol},i}\, g\,\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = -mg = -\rho A g h. \tag{32}$$

The potential energy of the water in the tank is $V = \rho A h^2 g/2$ with respect to the bottom of the tank, so according to eq. (31), the generalized force is $Q_h = -\rho A h g = -mg$ as in eq(32). Since taking $Q_h = 0$ would lead to no dependence of the motion on g, we accept that the generalized force is given by eq. (32).[12]

In the method of [28, 29], the left side of eq. (30) is modified for a variable-mass system, whose (control) volume has velocity $\mathbf{w}$, according to eq. (5.6) of [28] and eq. (1) of [29],[13]

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{q}_k} - \frac{\partial T_w}{\partial q_k} + \int \frac{\partial \tilde{T}}{\partial \dot{q}_k}(\mathbf{v}-\mathbf{w})\cdot d\mathbf{Area} - \int \tilde{T}\frac{\partial(\mathbf{v}-\mathbf{w})}{\partial \dot{q}_k}\cdot d\mathbf{Area} = Q_k, \tag{33}$$

where T_w is the kinetic energy within the control volume, $\tilde{T}$ is the kinetic energy per unit volume, and $\mathbf{v}$ is the velocity of the material at a point in the system.

In the present example, the control volume is the bucket and water therein, $\mathbf{w} = 0$, $\mathbf{v} = \dot{h}\,\hat{\mathbf{z}}$ and $\tilde{T} = \rho\dot{h}^2/2$ in the interior of the system, but on its surface $\mathbf{v}$ and $\tilde{T}$ are zero except at the hole, where $\mathbf{v} = -V\,\hat{\mathbf{z}} = A\dot{h}\,\hat{\mathbf{z}}/a$ and $\tilde{T} = \rho V^2/2 = \rho\dot{h}^2 A^2/2a^2$. Then, from eq. (29),

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{h}} = \rho A h\ddot{h} + \rho A\dot{h}^2, \qquad \frac{\partial T_w}{\partial h} = \frac{\rho A\dot{h}^2}{2}\,, \tag{34}$$

and at the hole, where the area vector is direction outwards with $d\mathbf{Area} = -a\,\hat{\mathbf{z}}$,

$$\tilde{T} = \rho\dot{h}^2\frac{A^2}{2a^2}\,, \qquad \frac{\partial \tilde{T}}{\partial \dot{h}} = \rho\dot{h}\frac{A^2}{a^2}\,, \qquad \mathbf{v} = \frac{A}{a}\dot{h}\,\hat{\mathbf{z}}, \qquad \frac{\partial \mathbf{v}}{\partial \dot{h}} = \frac{A}{a}\,\hat{\mathbf{z}}, \qquad \mathbf{w} = 0 = \frac{\partial \mathbf{w}}{\partial \dot{h}}\,. \tag{35}$$

Hence, the equation of motion (33) for the coordinate h is,

$$\rho A h\ddot{h} + \rho A\dot{h}^2 - \frac{\rho A\dot{h}^2}{2} - \rho\frac{A^3}{a^2}\dot{h}^2 + \frac{\rho A^3\dot{h}^2}{2a^2} = -\rho A g h, \tag{36}$$

$$h\ddot{h} - \left(\frac{A^2}{a^2} - 1\right)\frac{\dot{h}^2}{2} = -gh, \tag{37}$$

as previously found in eq. (8). If we make the substitutions $\dot{h} = -(a/A)V$ and $\ddot{h} = -(a/A)dV/dt$, we arrive at,

$$\frac{a}{A}h\frac{dV}{dt} + \left(1 - \frac{a^2}{A^2}\right)\frac{V^2}{2} = gh, \tag{38}$$

[12] It appears that when eq. (31) it applicable, it should be used. For an example where use of either form of eq. (30) leads to zero generalized force, in contrast to use of eq. (31), see sec. 2.4 of [33].

[13] An earlier discussion of Lagrange's equations for systems of variable mass was given in [32] (1947), where the context was rocket motion. It was noted that although the system of rocket plus fuel has variable mass, the center of mass of this system remains constant to a reasonable approximation, relative to the system, which permits a simpler form of the equations of motion than eq. (33).

as previously found in eqs. (8) and (27).

While the approach of [28, 29] leads to a reasonable analysis of the present problem, it seems that there is no single Lagangian approach that applies to all variable-mass problems.[14] *However, other "Lagrangian" approches have been discussed.*

B.1 Another Variant of the Lagrangian Approach

A recent paper [34] presents a Lagrangian approach close to the above, but with a slightly different notation, and traces this approach to an early paper on variable-mass systems by Cayley [35] (1857).

The equations of motion are written in [34] in the form,

$$\frac{d}{dt}\frac{\dot{\partial} S}{\dot{\partial}\dot{q}_k} - \frac{\dot{\partial} S}{\dot{\partial} q_k} + \frac{\dot{\partial}\sigma}{\dot{\partial}\dot{q}_k} = Q_k, \tag{40}$$

where S is the same as the kinetic energy of the system called T previously, $\sigma = \dot{m}_{\rm e} V^2/2$ describes the rate of kinetic energy ejected from the system (at velocity V), the partial derivative operations $\dot{\partial}/\dot{\partial} q_k$ and $\dot{\partial}/\dot{\partial}\dot{q}_k$ on $f(m, q_k, \dot{q}_k, t)$ act only on the dependence of f on q_k and $\dot{q}_k$ (and not on the dependence on mass m), and Q_k is the generalized (external) force associated with coordinate k.

For the present example of the leaky tank, with only a single coordinate h,

$$S = \frac{m\dot{h}^2}{2} = \frac{\rho A h\dot{h}^2}{2} = T, \quad m = \rho A h, \quad \sigma = \frac{\dot{m}_{\rm e} V^2}{2} = \frac{\dot{m}_{\rm e}\dot{h}^2 A^2}{2a^2}, \quad \dot{m}_{\rm e} = \rho a V = -\rho A\dot{h}, \tag{41}$$

with the convention that $\dot{m}_{\rm e} > 0$ when mass is ejected from the system. As before, the generalized force is,

$$Q_h = -mg = -\rho A g h. \tag{42}$$

The equation of motion according to eq. (40) is then,

$$\frac{d}{dt} m\dot{h} - 0 + \dot{m}_{\rm e}\dot{h}\frac{A^2}{a^2} = \rho A h\ddot{h} + \rho A\dot{h}^2 - \rho A\dot{h}^2\frac{A^2}{a^2} = -\rho A g h, \tag{43}$$

$$h\ddot{h} + \dot{h}^2\left(1 - \frac{A^2}{a^2}\right) = -gh, \tag{44}$$

$$\frac{a}{A} h\frac{dV}{dt} + \left(1 - \frac{a^2}{A^2}\right) V^2 = gh, \tag{45}$$

[14] An example where the method of [28, 29] fails has been pointed out by the author of [34]. A snowball slides on snow-covered ice, subject to no external force,and accumulates mass at rate $\dot{m} = kv$, *i.e.*, $m(t) = m_0 + kx$. Conservation of horizonatal momentum implies that $m(t)v(t) = m(t + dt)v(t + dt)$ $= (m + kv\,dt)(v + \dot{v}\,dt) \approx m(t)v(t) + (m\dot{v} + kv^2)\,dt$, such that the equation of motion is $m\dot{v} = -kv^2$.

Taking the control volume to be the moving mass, with coordinate $q = x$ and $\dot{q} = v$, the velocity of the control volumes is $\mathbf{w} = \mathbf{v}$, the generalized force Q is zero, and the kinetic energy of the control volume is $T_w = m(t)v^2/2$. Then, the equation of motion according to eq. (33) would be just

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{v}} = m\dot{v} + kv^2 = \frac{\partial T_w}{\partial x} = \frac{kv^2}{2}, \qquad m\dot{v} = -\frac{kv^2}{2}, \tag{39}$$

in disagreement with the momentum analysis.

The second terms in eqs. (44)-(45) are a factor of 2 larger than that found in eqs. (8), (27) and (37), and implies that $V^2 = gh$ rather than $2gh$ as in the energy analysis. That is, application of eq. (40) to the present example seems to lead to the result of an momentum analysis, rather than an energy analysis. If one then considers the *vena contracta* at the outlet, which increase the velocity of the efflux by $\approx \sqrt{2}$ over a short distance, agreement with the energy analysis is obtained.

The limitations of the method of [34] are illustrated, for example, by the uncoiling of a tape [33], including the variant in sec. V.D of [34] which we consider to be misanalyzed there.

B.2 Yet Another Variant of a Lagrangian Approach

Yet another Lagrangian approach is given in [36] for systems in which the variable mass depends on position, as is the case of the present example if we take the system to be the mass $m = \rho A h$ of water in the leaky bucket.

According to eq. (19) of [36] the equation(s) of motion of the system can be written as,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = \hat{Q}_k = \sum_i \left\{ (\mathbf{f}_i + \dot{m}_i \mathbf{w}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_k}\mathbf{v}_i^2\right) - \frac{1}{2}\frac{\partial m_i}{\partial q_k}\mathbf{v}_i^2 \right\}, \tag{46}$$

where T is the kinetic energy of the system, whose particles at positions $\mathbf{r}_i$ have masses $m_i(q_k, \dot{q}_k, t)$, velocities $\mathbf{v}_i$, $\mathbf{f}_i$ is the "active" force on particle i, and $\dot{m}_i \mathbf{w}_i$ is a "nonconservative force, proportional to the rate of variation of mass with respect to time and to the velocity of the expelled (or gained) mass".[15]

In the present example with a single coordinate h, $T = \rho A h \dot{h}^2/2$ as before, the first term of $\hat{Q}_h$ is the generalized force $Q_h = -\rho A g h$, the velocity of all particles in the system is $\mathbf{v}_i = \dot{h}\,\hat{\mathbf{z}}$, and $\sum_i m_i = \rho A h$. Possibly,

$$\sum_i \dot{m}_i \mathbf{w}_i \cdot \frac{\partial \mathbf{r}_i}{\partial h} = \rho A \dot{h}(-V\hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = -\rho A \dot{h} V = -\frac{\rho A^2 \dot{h}^2}{a}. \tag{47}$$

If so, the equation of motion according to eq. (46) would be,

$$\rho A h \ddot{h} + \rho A \dot{h}^2 - \frac{\rho A \dot{h}^2}{2} = -\rho A g h - \frac{\rho A^2 \dot{h}^2}{a} + 0 - \frac{\rho A \dot{h}^2}{2} \tag{48}$$

$$h\ddot{h} + \dot{h}^2\left(1 + \frac{A}{a}\right) = -gh. \tag{49}$$

Or, if we ignore the term in $\dot{m}_i \mathbf{w}_i$, the equation of motion would be,

$$h\ddot{h} + \dot{h}^2 = -gh. \tag{50}$$

??? However, neither eq. (49) nor (50) agree with the equation of motion found previously in eqs. (8) and (37)??

[15] The meaning of this phrase is unclear to the authors.

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