Laplace and the Speed of Gravity

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

By the 1600's evidence had accumulated that the length of a lunar month in Earth days was slowly decreasing with time, which led to various speculations as to why this might be so, including drag/retardation of the Moon and planets by some kind of æther.[1] In Vol. IV, Book X, Chap. VII of his treatise *Mécanique Céleste* (1805), Laplace [2] considered that if gravity were due a fluid whose particles move towards the force center at a finite speed u of propagation, this would imply a retarding force on the Moon such that the radius of its orbit would "quickly" drop to zero.[2] Use this argument to estimate the ratio u/c, where c is the speed of light in vacuum.[3]

Laplace did not consider gravity to be associated with a retarded scalar potential of the form later introduced by Riemann (1858, but published posthumously only in 1867) [4] and Lorenz [5, 6, 7]. Does a retarded scalar potential lead to a retarding force?

2 Solution

2.1 Laplace's Argument

This section loosely follows prob. 12.4 of [8].

We assume that the force of gravity acts along straight lines, but with finite speed u.

Then, the force exerted at time $t = 0$ on the Moon by the Earth was generated at the earlier time $-t \approx -R/u$, where R is the Earth-Moon separation. In the rest frame of the Sun, the Earth and the Moon are always moving in the same general direction about the

[1]See [1] for commentary on the mix of science and religion in discussions of this by Halley.

[2]Laplace, like Newton, considered that gravity propagated instantaneously, and appears to have offered his model of gravity as due to particles in some kind of fluid mainly to show how implausible it is to suppose that gravity has a finite speed of propagation. Laplace's primary explanation for the changing month was an effect of the Sun on the Earth-Moon system, in the context of Newtonian gravity. Our interest here is that Laplace may have been the first to comment that a finite speed of propagation of gravity would be associated with slow changes in gravitational orbits.

[3]Both gravitational waves and gamma rays were detected from a recent binary-neutron-star merger [3], which provides evidence that the speed of gravitational and electromagnetic waves differs by less that one part in 10^{16}.

Sun,[4] so we see from the figure that the direction of the force on the Moon opposes its motion, assuming that somehow the particles of the gravitational fluid always move towards the instantaneous position of the Earth. Then, the force makes angle $\theta = V_E t/ut = V_E/u$ to the line of centers of the Earth and Moon (when the Moon is aligned with the Earth and Sun), where $V_E \approx 3 \times 10^4$ m/s is the orbital velocity of the Earth about the Sun.

As such, there is a component,

$$F_\theta \approx -\frac{GMm}{R^2}\theta = -\frac{GMm}{R^2}\frac{V_E}{u}\,, \tag{1}$$

of the gravitational force that opposes the Moon's motion, where G is Newton's gravitational constant, and M and m are the masses of the Earth and Moon, respectively.

We now switch to the rest frame of the Earth, where this retarding force changes the energy $E = -GMm/2R$ of the Earth-Moon system at the rate,

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = F_\theta v_m = \frac{GMm}{2R^2}\frac{dR}{dt} = -\frac{GMmv_mV_E}{R^2u}\,, \qquad \frac{dR}{dt} = -\frac{2v_mV_E}{u} = -2\sqrt{\frac{GM}{R}}\frac{V_E}{u}\,, \tag{2}$$

noting that in the rest frame of the Earth the centripetal acceleration of the moon is $v_m^2/R \approx GM/R^2$ for $M \gg m$ and $v_m \ll u$. Integrating eq. (2), we find that,

$$R^{3/2} = R_0^{3/2} - 3\sqrt{GM}\frac{V_E}{u}t = R_0^{3/2} - 3v_0\sqrt{R_0}\frac{V_E}{u}t = R_0^{3/2}\left(1 - \frac{6\pi V_E t}{uT_0}\right), \tag{3}$$

where R_0 is the Earth-Moon distance at time $t = 0$, when the Moon's velocity is $v_0 = 2\pi R_0/T_0$ in terms of the Moon's orbital period T_0.

Hence, the effect of the finite speed u of propagation of gravity in Laplace's model is that the Moon would fall onto the Earth after time,

$$\Delta t = \frac{cT_0}{6\pi V_E}\frac{u}{c} \approx \frac{3\times 10^8 \text{ m/s}\cdot 2.5\times 10^6 \text{ s}}{20\cdot 3\times 10^4 \text{ m/s}}\frac{u}{c} \approx 10^9\frac{u}{c}\text{ s} \approx 300\frac{u}{c}\text{ years.} \tag{4}$$

If we suppose the lifetime of the Moon is, say, 30 billion years, we infer that $u/c \approx 10^8$.

On p. 645 of [2], Laplace stated "we must suppose that the gravitating fluid has a velocity which is at least a hundred millions of times greater than that of light."

Laplace was unaware that the Earth-Moon distance actually increases slowly with time,[5] *and that R was essentially zero about 4 billion years ago, when presumably the Moon was ejected from the Earth by a collision of the proto-Earth an another large body. The increasing Earth-Moon distance (and the lengthening of an Earth day) is a consequence of tidal friction, as first predicted by G.H. Darwin (1879) [10].*

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[4]See [9] for discussion of some counterintuitive aspects of the Moon's motion relative to the Sun.

[5]Laplace knew that the number of Earth days in a lunar month was slowly decreasing and supposed this was because the Moon's velocity was increasing, whereas the Moon's velocity is decreasing, while the length of an Earth day is increasing at a faster rate.

2.2 Argument via a Retarded Scalar Potential

Suppose the gravitational force on mass m is $\mathbf{F} = m\mathbf{g}$, where the vector field $\mathbf{g} = -\boldsymbol{\nabla}\phi$ can be deduced from a scalar potential ϕ.[6] If we also suppose that gravity propagates at speed u, it would seem best to consider ϕ as the retarded potential,

$$\phi(\mathbf{x}, t) = G \int \frac{\rho(\mathbf{x}', t' = t - r/u)}{r}\, d\mathrm{Vol}' = G \int \frac{[\rho]}{r}\, d\mathrm{Vol}', \quad \text{with} \quad [\rho] = \rho(\mathbf{x}', t' = t - r/u), \tag{5}$$

where ρ is the source mass density, and $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. As Lorenz remarked on p. 291 of [7] regarding the retarded electromagnetic scalar potential (which has the same form as eq. (5)), this form *expresses further that the entire action between the free electricity requires time to propagate itself – an assumption not strange in science, and which may in itself be assumed to have a certain degree of probability. For in accordance with the formula found, the action in the point* $\mathbf{x}$ *at the moment* t *does not depend on the simultaneous condition in the point* $\mathbf{x}'$, *but on the condition in which it was at the moment* $t - r/u$; *that is, so much time in advance as is required to traverse the distance* r *with the constant velocity* u.[7]

For a point source M with velocity $\boldsymbol{\beta} = \mathbf{v}/u$, the retarded potential becomes,[8]

$$\phi = \frac{GM}{[r - \boldsymbol{\beta} \cdot \mathbf{r}]}\,. \tag{6}$$

Then, the gravitational field is,[9]

$$\mathbf{g} = -\boldsymbol{\nabla}\phi = \frac{GM}{[r - \boldsymbol{\beta} \cdot \mathbf{r}]^3}[\mathbf{r}(1/\gamma^2 + \dot{\boldsymbol{\beta}} \cdot \mathbf{r}) - \boldsymbol{\beta}(r - \boldsymbol{\beta} \cdot \mathbf{r})], \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}\,. \tag{7}$$

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[6]Maxwell noted in sec. 82 of [11] that in his electromagnetic theory, which is deducible from what we now call a 4-vector potential, like charges repel, so such a (4-vector-potential) theory cannot explain gravity. He also noted that in a theory such as eq. (5) with a scalar potential, the field energy would be negative. Hence, Maxwell considered such a theory to be nonphysical. Much later, Einstein gave a 4-tensor theory of gravity [12] in which all masses attract and the issue of negative field energy is (largely) evaded.

A weak-field approximation to Einstein's theory can be expressed in terms of a 4-vector potential (see, for example, [13, 14]), which leads to an approximate understanding of "gravitomagnetic" effects between moving/rotating masses. The gravitomagnetic force is noncentral and of order $1/c^2$, but too small to affect the dynamics of the solar system. A gravitomagnetic effect was detected (at a cost of $\approx$ \$1B) in the Gravity Probe B satellite experiment [15, 16].

[7]Although the retarded potential implies that the "action" between source and observer propagates at speed u, the retarded potential is not the superposition of waves of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ that propagate only at speed $k/\omega = u$. For example, the potential associated with a source at rest has no time dependence, so if this potential is considered to be composed of plane waves, these waves have zero angular frequency ω, corresponding to infinite wave speed (infinite phase velocity).

Thus, the concept of retarded potentials is quite subtle (and was never well accepted by Maxwell; see, for example, [17]). In the quantum view, we recognize the retarded potential as being associated both with real gravitons (or photons in the case of electrodynamics) that propagate with speed $u = c$, as well as with virtual gravitons that can propagate at essentially any speed.

In the classical view, it is possible to give a decomposition of a general, time-dependent field in terms of plane waves that propagate both at speed c (corresponding to real quanta) and at other speeds (corresponding to virtual quanta). For the case of electromagnetic fields, see, for example, [18].

[8]See, for example, sec. 10.3.1 of [19].

[9]See, for example, eq. (10.69) of [19].

For constant source velocity, $\dot{\boldsymbol\beta} = 0$, the field is,

$$\mathbf{g} = GM\frac{[\mathbf{r}/\gamma^2 - \boldsymbol\beta(r - \boldsymbol\beta\cdot\mathbf{r})]}{[r - \boldsymbol\beta\cdot\mathbf{r}]^3}, \tag{8}$$

which does not point along the vector $\mathbf{r} = [\mathbf{r} - r\boldsymbol\beta]$ from the present source position to the observer.[10] At large distance from an accelerated mass, the gravitational field is,

$$\mathbf{g} \approx GM\frac{[\mathbf{r}(\dot{\boldsymbol\beta}\cdot\mathbf{r})]}{[r - \boldsymbol\beta\cdot\mathbf{r}]^3}\,, \tag{9}$$

which points away from the retarded location of the mass, and varies inversely with distance. That is, the gravitational "radiation" field is longitudinal in the model of gravity as associated with a scalar potential (as also occurs for (scalar) sound waves).[11]

For orbital motion with small velocity and acceleration, it is useful to approximate the retarded quantities in terms of present quantities, to order $1/c^2$. This was done for the electric field $\mathbf{E} = -\boldsymbol\nabla\phi - \partial\mathbf{A}/\partial ct$ for an accelerated charge on p. 303 of [20]. Omitting the contribution from the vector potential, we obtain the desired approximation for the gravitational field,

$$\phi \approx GM\left(\frac{1}{r} + \frac{\ddot{r}}{2c^2}\right), \qquad \mathbf{g} \approx GM\left\{\hat{\mathbf{r}}\left(\frac{1}{r^2} + \frac{v^2 - 3(\mathbf{v}\cdot\hat{\mathbf{r}})^2}{2u^2r^2} - \frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{2u^2r}\right) - \frac{\mathbf{v}(\mathbf{v}\cdot\hat{\mathbf{r}})}{u^2r^2} + \frac{\mathbf{a}}{2u^2r}\right\}, \tag{10}$$

where $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ are the present position, velocity and acceleration of the source mass M.[12,13]

In contrast to Laplace's model, the components of the gravitational force (10) that are not along the line of centers between the Earth and Moon are of order v_E^2/u^2, rather than v_E/u. Further, the terms not along $\hat{\mathbf{r}}$ are suppressed in that $\mathbf{v}_E\cdot\hat{\mathbf{r}}$ averages to zero over a month; and $\mathbf{a}_E$ points to the Sun, which direction is nearly perpendicular at all times to the velocity $\mathbf{v}_m$ of the Moon, so the rate of change of the Moon's energy due to this force component is very small. In addition $a_E/u^2R = v_E^2/u^2R_{EM}R_{ES}$ so this term is smaller than eq. (1) by the factors $(V_E/u)(R_{EM}/R_{ES})$. Altogether, the rate of change of the Moon's energy if $u = c$ is about 10^9 times smaller in case of a retarded scalar potential than in Laplace's model.

Hence, while the gravitational field associated with a scalar potential that propagates at a finite speed u does destabilize the Moon's motion (as argued by Laplace), the slowness of the evolution of this motion cannot be used as evidence that the speed of gravity is different that the speed of light.[14]

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[10] This contrasts with electromagnetism, where for constant source velocity, the electric field $\mathbf{E} = e[\mathbf{r} - r\boldsymbol\beta]/\gamma^2[r - \boldsymbol\beta\cdot\mathbf{r}]^3$ due to charge e does point away from the present location of the charge, as in this case the present separation $\mathbf{r}$ between observer and source is related by $\mathbf{r} = [\mathbf{r} - r\boldsymbol\beta]$.

[11] In contrast, the "radiation" fields are transverse in vector and tensor theories of gravity.

[12] The scalar potential is given in eq. (65.5) of [21].

[13] For electromagnetism at order $1/c^2$, the term along $\mathbf{v}$ is absent, and the term along $\mathbf{a}$ has the opposite sign.

[14] There exists an ongoing literature of "alternative" theories of gravity in which Laplace's analysis is cited as supporting evidence. See, for example, [22, 23]. Paper [22] is discussed in the blog [24], and perhaps more insightfully in [25]. Paper [23] makes an "elementary" error in its eq. (2.7), supposing there that the quantity r is the distance in the lab frame, whereas it is the distance in the rest frame; this error propagates (at what speed?) throughout the rest of the paper.

2.3 Gerber and the Speed of Gravity

This section follows http://en.wikipedia.org/wiki/Paul_Gerber.

Among the many attempts around 1890 to explain the precession of the perihelion of Mercury,[15] Lévy [29] proposed that gravity is deducible from a scalar potential,

$$\phi = \frac{GM}{r}\left(1 - \frac{\dot{r}^2}{u^2}\right), \tag{11}$$

where r is the present distance from the source to the observer, $\dot{r}$ is the speed of the source, and u is the speed of gravity. Apparently Lévy was inspired by Weber's electrodynamics, and hoped that $u = c$ would explain the data; however it did not quite.

In 1898, Gerber [30, 31] gave a model of gravity based on the scalar potential,

$$\phi = \frac{GM}{r(1 - \dot{r}/u)^2} \approx \frac{GM}{r}\left(1 + \frac{2\dot{r}}{u} + \frac{3\dot{r}^2}{u^2}\right), \tag{12}$$

From this potential he computed the rate Ω of precession of the perihelion of Mercury, finding,

$$\Omega = \frac{24\pi^3 a^2}{T^2 u^2 (1 - \epsilon^2)}, \tag{13}$$

where a is the semi-major axis, T is the period, and ϵ is the eccentricity of Mercury's orbit around the Sun. Based on the data, Gerber inferred that $u = c$ to good accuracy.

In retrospect, this is less surprising in that eq. (13) is identical to the result of Einstein [32], computed via his theory of general relativity with $u = c$.[16]

Acknowledgment

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[15] The famous data are due to Le Verrier (1959) [26, 27]. For a review as of 1903, see secs. 23-24 of [28].

[16] This "coincidence" led to accusations that Einstein plagarized Gerber's result, which reverberate to this day. See [33] for extensive comments on various approximations to gravity in the solar system.

Gerber's effort is often dismissed as naïve, but is perhaps better appreciated as an attempt at an effective theory of gravity beyond that of Newton (as discussed in sec. 6.3 of [34]; see also [35]).

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