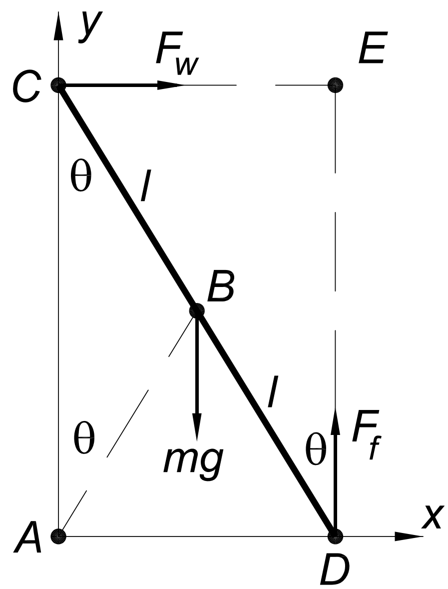
Torque Analyses of a Sliding Ladder

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(May 6, 2007; updated April 15, 2019)

1 Problem

The problem of a ladder that slides without friction while touching a floor and wall is often used to illustrate Lagrange's method for deducing the equation of motion of a mechanical system. Suppose the ladder has mass m, length $2l$, and makes angle θ to the vertical. Deduce the equation of motion via a torque analysis about each of the five points A, B, C, D and E as shown in the figure below. All of these points except A are accelerating in the lab frame.[1]



The usual statement of the ladder problem is to ask at what angle θ does the ladder lose contact with the vertical wall if it starts from rest at $\theta = 0$ (and the bottom of the ladder is given a tiny horizontal velocity).

2 Solution[2]

2.1 Lagrange's Method

We use θ as the single generalized coordinate. The center of mass of the ladder is at point B and moves in a circle of radius l about the fixed point A. The moment of inertia of the ladder about its center of mass is,

$$I_{\rm cm} = \frac{m\,l^2}{3}\,. \tag{1}$$

[1]This problem appears as ex. 2, art. 145 of [1]. This note was inspired in part by comments in [2]-[9].
[2]A solution to this problem that avoids use of the equation of motion is given at [10].

The kinetic energy T of the sliding ladder consists of the kinetic energy of the motion of the center of mass plus the kinetic energy of rotation about the center of mass,

$$T = \frac{m\,v_{\rm cm}^2}{2} + \frac{I_{\rm cm}\,\dot\theta^2}{2} = \frac{m\,(l\dot\theta)^2}{2} + \frac{m\,l^2\,\dot\theta^2}{6} = \frac{2m\,l^2\,\dot\theta^2}{3}\,. \tag{2}$$

The gravitational potential energy V of the ladder relative to the floor is,

$$V = m\,g\,l\,\cos\theta. \tag{3}$$

The equation of motion of the ladder follows from Lagrange's equation,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta}\,, \tag{4}$$

where the Lagrangian is $\mathcal{L} = T - V$. From eqs. (2)-(4) we find that,

$$\ddot\theta = \frac{3g}{4l}\sin\theta. \tag{5}$$

The ladder loses contact with the vertical wall when (horizontal) contact force F_w vanishes. This contact force causes the horizontal acceleration of the center of mass,

$$F_w = m\,a_x. \tag{6}$$

The x and y coordinates of the center of mass (so long as the ladder remains in contact with the wall) are,

$$x_{\rm cm} = l\sin\theta, \qquad y_{\rm cm} = l\cos\theta, \tag{7}$$

so the acceleration **a** of the center of mass has components,

$$a_x = \ddot x_{\rm cm} = l\cos\theta\,\ddot\theta - l\sin\theta\,\dot\theta^2, \qquad a_y = \ddot y_{\rm cm} = -l\sin\theta\,\ddot\theta - l\cos\theta\,\dot\theta^2. \tag{8}$$

The angular velocity $\dot\theta$ of the ladder follows from conservation of energy,

$$E_0 = m\,g\,l = E = T + V, \tag{9}$$

so that[3]

$$\dot\theta^2 = \frac{3g}{2l}(1 - \cos\theta). \tag{10}$$

The ladder loses contact with the vertical wall when $F_w = m\,a_x$ vanishes, which occurs when $\cos\theta = 2/3$, using eqs. (5), (8) and (10).

[3]The equation of motion (5) can also be deduced by taking the time derivative of eq. (10), or conversely the energy equation (10) could be obtained by integrating the equation of motion (5).

2.2 Torque Analysis About Point A

A Newtonian approach to the equation of motion of the sliding ladder is based on a torque analysis about some point. If that point is at rest, the torques are due only to the forces that are apparent in the lab frame. However, if the point is accelerating, there are additional ("fictitious") torques due to apparent forces associated with the use of an accelerated coordinate system.[4]

First, we consider an analysis about point A, which is fixed in the lab frame.

The torque equation is,

$$\frac{dL_A}{dt} = \tau_A = 2F_f\, l \sin\theta - m\, g\, l \sin\theta - 2F_w\, l \cos\theta, \tag{11}$$

where L_A is the angular momentum of the ladder about point A, and a torque is positive if its vector points along the $+z$ axis in a right-handed coordinate system.

A subtlety of the torque analysis about point A (and about point E) is that the rotation of the ladder is not rigid body rotation about this point, so the angular momentum is not the product of the momentum of inertia about point A times the angular velocity $\dot\theta$.

In general, the angular momentum of a rigid body with respect to a point equals the angular momentum of the center of mass motion with respect to that point, plus the angular momentum of the body relative to the center of mass.

The angular momentum of the center of mass motion of the ladder relative to point A is $-ml^2\,\dot\theta$, while the angular momentum of the ladder relative to the center of mass is $I_{\rm cm}\,\dot\theta = ml^2\,\dot\theta/3$, recalling eq. (1) and noting that the senses of these two rotations are opposite. Thus, the total angular momentum of the ladder about point A is,

$$L_A = -m\, l^2\,\dot\theta + \frac{m\, l^2\,\dot\theta}{3} = -\frac{2m\, l^2\,\dot\theta}{3}\,. \tag{12}$$

We note that the moment of inertia of the ladder about point A (as well as that about points C, D and E) follows from the parallel axis theorem,

$$I_A = I_C = I_D = I_E = I_{\rm cm} + m\, l^2 = \frac{4}{3} m\, l^2. \tag{13}$$

But this moment of inertia is relevant only if the rotation of the center of mass about the point of reference and the rotation with respect to the center of mass are equal in magnitude and sign. In the present example this is true for points C and D but not points A and E. In particular, L_A of eq. (12) does not equal $-I_A\dot\theta$.

As previously discussed in eq. (6), the horizontal force F_w equals the mass of the ladder times the horizontal acceleration of its center of mass, so that,

$$F_w = m\, a_x = m\, l \cos\theta\,\ddot\theta - m\, l \sin\theta\,\dot\theta^2. \tag{14}$$

Similarly, the vertical force F_f is related to the vertical acceleration of the center of mass according to $F_f - m\, g = m\, a_y$, so that,

$$F_f = m\, g + m\, a_y = m\, g - m\, l \sin\theta\,\ddot\theta - m\, l \cos\theta\,\dot\theta^2. \tag{15}$$

[4]Other forms of torque analyses are possible, as reviewed in the Appendix below.

Combining the torque equation (11) with eqs. (12), (14) and (15) we find,

$$\begin{aligned}\frac{dL_A}{dt} = -\frac{2}{3} m\, l^2\, \ddot{\theta} = \tau_A &= 2m\, g\, l \sin\theta - 2m\, l^2 \sin^2\theta\, \ddot{\theta} - 2m\, l^2 \cos\theta \sin\theta\, \dot{\theta}^2 \\ &\quad - m\, g\, l \sin\theta - 2m\, l^2 \cos^2\theta\, \ddot{\theta} + 2m\, l^2 \cos\theta \sin\theta\, \dot{\theta}^2 \\ &= m\, g\, l \sin\theta - 2m\, l^2\, \ddot{\theta}, \end{aligned} \tag{16}$$

which leads to the equation of motion (5).

2.3 Torque Analysis about Point B

Points B-E are accelerating, so the torque analyses about these points must include the effect of the "coordinate" force,

$$\mathbf{F}_P = -m\, \mathbf{a}_P, \tag{17}$$

that appears to act on the center of mass according to an observer at a point P that is accelerating with respect to the lab frame [11].

However, if the point of reference is the center of mass of the system, the "coordinate" force (17) causes no torque, so a torque analysis about the center of mass has the same form whether or not the center of mass is accelerating.

Point B is the center of mass of the ladder. The angular momentum about point B is therefore,

$$L_B = I_{\rm cm}\, \dot{\theta}, \tag{18}$$

and the torque equation is,

$$\begin{aligned}\frac{dL_B}{dt} = \frac{m\, l^2\, \ddot{\theta}}{3} &= \tau_B = F_f\, l \sin\theta - F_w\, l \cos\theta \\ &= m\, g\, l \sin\theta - m\, l^2 \sin^2\theta\, \ddot{\theta} - m\, l^2 \cos\theta \sin\theta\, \dot{\theta}^2 \\ &\quad - m\, l^2 \cos^2\theta\, \ddot{\theta} + m\, l^2 \cos\theta \sin\theta\, \dot{\theta}^2 \\ &= m\, g\, l \sin\theta - m\, l^2\, \ddot{\theta}, \end{aligned} \tag{19}$$

which again leads to the equation of motion (5).

2.4 Torque Analysis About Point C

Point C has coordinates $(0, 2y_{\rm cm})$ so the "coordinate" force associated with taking this point as our reference point for a torque analysis is,

$$\mathbf{F}_C = (0, -2m\, \ddot{y}_{\rm cm}) = \left(0, 2m\, l \sin\theta\, \ddot{\theta} + 2m\, l \cos\theta\, \dot{\theta}^2\right), \tag{20}$$

recalling eq. (8).

Since point C is located on the ladder, the angular momentum about point C is simply,

$$L_C = I_C\, \dot{\theta} = \frac{4m\, l^2\, \dot{\theta}}{3}, \tag{21}$$

recalling eq. (13). The torque analysis about point C is then,

$$\begin{aligned}
\frac{dL_C}{dt} = \frac{4m\,l^2\,\ddot{\theta}}{3} &= \tau_C + F_{C,y}\,l\sin\theta = -m\,g\,l\sin\theta + 2F_f\,l\sin\theta + F_{C,y}\,l\sin\theta \\
&= -m\,g\,l\sin\theta \\
&\quad +2m\,g\,l\sin\theta - 2m\,l^2\sin^2\theta\,\ddot{\theta} - 2m\,l^2\cos\theta\sin\theta\,\dot{\theta}^2 \\
&\quad +2m\,l^2\sin^2\theta\,\ddot{\theta} + 2m\,l^2\cos\theta\sin\theta\,\dot{\theta}^2 \\
&= m\,g\,l\sin\theta,
\end{aligned} \tag{22}$$

which again leads to the equation of motion (5).

2.5 Torque Analysis About Point D

Point D has coordinates $(2x_{\rm cm}, 0)$ so the "coordinate" force associated with taking this point as our reference point for a torque analysis is,

$$\mathbf{F}_D = (-2m\,\ddot{x}_{\rm cm}, 0) = \left(-2m\,l\cos\theta\,\ddot{\theta} + 2m\,l\sin\theta\,\dot{\theta}^2, 0\right), \tag{23}$$

recalling eq. (8).

Since point D is located on the ladder, the angular momentum about point C is simply,

$$L_D = I_D\,\dot{\theta} = \frac{4m\,l^2\,\dot{\theta}}{3}, \tag{24}$$

recalling eq. (13). The torque analysis about point D is then,

$$\begin{aligned}
\frac{dL_D}{dt} = \frac{4m\,l^2\,\ddot{\theta}}{3} &= \tau_D - F_{D,x}\,l\cos\theta = m\,g\,l\sin\theta - 2F_w\,l\cos\theta - F_{D,x}\,l\cos\theta \\
&= m\,g\,l\sin\theta \\
&\quad -2m\,l^2\cos^2\theta\,\ddot{\theta} + 2m\,l^2\cos\theta\sin\theta\,\dot{\theta}^2 \\
&\quad +2m\,l^2\cos^2\theta\,\ddot{\theta} - 2m\,l^2\cos\theta\sin\theta\,\dot{\theta}^2 \\
&= m\,g\,l\sin\theta,
\end{aligned} \tag{25}$$

which again leads to the equation of motion (5).

2.6 Torque Analysis About Point E

The ordinary torques from the floor and wall about point E vanish, but we must still consider the torques due to gravity and to the "coordinate" forces.

Point E has coordinates $(2x_{\rm cm}, 2y_{\rm cm})$ so the "coordinate" force associated with taking this point as our reference point for a torque analysis is,

$$\mathbf{F}_E = (-2m\,\ddot{x}_{\rm cm}, -2m\,\ddot{y}_{\rm cm}) = \left(-2m\,l\cos\theta\,\ddot{\theta} + 2m\,l\sin\theta\,\dot{\theta}^2, 2m\,l\sin\theta\,\ddot{\theta} + 2m\,l\cos\theta\,\dot{\theta}^2\right), \tag{26}$$

recalling eq. (8).

Point E is not on the ladder, so we calculate its angular momentum as the sum of the angular momentum of the center of mass relative to point E plus the angular momentum relative to the center of mass,

$$L_E = -m\,l^2\,\dot\theta + \frac{m\,l^2\,\dot\theta}{3} = -\frac{2m\,l^2\,\dot\theta}{3}\,, \tag{27}$$

as for the analysis about point A. The torque analysis about point E is then,

$$\begin{aligned} \frac{dL_E}{dt} = -\frac{2m\,l^2\,\ddot\theta}{3} &= \tau_E - F_{E,y}\,l\sin\theta = m\,g\,l\sin\theta + F_{E,x}\,l\cos\theta - F_{E,y}\,l\sin\theta \\ &= m\,g\,l\sin\theta \\ &\quad -2m\,l^2\cos^2\theta\,\ddot\theta + 2m\,l^2\cos\theta\sin\theta\,\dot\theta^2 \\ &\quad -2m\,l^2\sin^2\theta\,\ddot\theta - 2m\,l^2\cos\theta\sin\theta\,\dot\theta^2 \\ &= m\,g\,l\sin\theta - 2m\,l^2\,\ddot\theta, \end{aligned} \tag{28}$$

which again leads to the equation of motion (5).

2.7 Comments

We have deduced the equation of motion of the sliding ladder by six methods. Of these, Lagrange's method is perhaps the simplest. If a torque analysis is desired, it is simplest to use the center of mass as the reference point so that no "coordinate" forces appear in the calculation. The use of reference points not on the ladder, such as points A and E is complicated by the fact that the ladder is not in simple rigid-body rotation about these points, so the angular momentum must be calculated as the sum of that of the center of mass plus that relative to the center of mass.

A Appendix: The 5 Methods of Jensen *(March 2019)*

The theme of this Appendix is elaborated upon in [12].

An interesting paper by Jensen [13] comments that many torque analyses in textbooks are misguided, but that there are (at least) 5 valid methods that can be used.[5] In this Appendix we illustrate these methods with analyses of the ladder problem based on point C of contact of the ladder with the vertical wall.

We first recall that Newton's second law for the momentum $\mathbf{p} = m\mathbf{v}$ of a particle of time-independent mass m at position $\mathbf{x}$ with velocity $\mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}}$ is,

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}m\,\mathbf{v} = m\,\mathbf{a} = \mathbf{F}, \tag{29}$$

where $\mathbf{a} = d\mathbf{v}/dt$ and $\mathbf{F}$ is the force on the particle. This permits one to introduce the angular momentum $\mathbf{L}$ (with respect to the origin),

$$\mathbf{L} = \mathbf{x}\times\mathbf{p} = \mathbf{x}\times m\,\mathbf{v}, \tag{30}$$

[5]Jensen makes no mention of Lagrange, or of the simple method of differentiating the conserved energy, although his methods concern only examples where these techniques could be used.

which for time-independent mass obeys the relation,

$$\frac{d\mathbf{L}}{dt} = \mathbf{v} \times m\,\mathbf{v} + \mathbf{x} \times \frac{d\mathbf{p}}{dt} = \mathbf{x} \times \mathbf{F} \equiv \boldsymbol{\tau}, \tag{31}$$

where the torque τ is defined with respect to the origin.

For a set of particles, labeled by subscript i, of time-independent masses we can then write,

$$m = \sum_i m_i, \quad m\mathbf{x}_{\rm cm} = \sum_i m_i \mathbf{x}_i, \quad m\mathbf{v}_{\rm cm} = \sum_i m_i \mathbf{v}_i, \quad m\,\mathbf{a}_{\rm cm} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i = \mathbf{F}, \tag{32}$$

which introduces quantities related to the center of mass of the system. Similarly, the total angular momentum $\mathbf{L}$ of the system with respect to the origin can be written,

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{L}_i = \sum_i m_i\, \mathbf{x}_i \times \mathbf{v}_i \\
&= \sum_i m_i\,(\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times (\mathbf{v}_i - \mathbf{v}_{\rm cm}) + \sum_i m_i\, \mathbf{x}_i \times \mathbf{v}_{\rm cm} + \mathbf{x}_{\rm cm} \times \sum_i m_i\, \mathbf{v}_i - \sum_i m_i\, \mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm} \\
&= \mathbf{L}_{\rm cm} + m\,\mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm} + \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_{\rm cm} - m\,\mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm} = \mathbf{L}_{\rm cm} + \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_{\rm cm},
\end{aligned} \tag{33}$$

where the angular momentum $\mathbf{L}_{\rm cm}$ with respect to the center of mass is defined by,

$$\mathbf{L}_{\rm cm} = \sum_{-} m_i\,(\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times (\mathbf{v}_i - \mathbf{v}_{\rm cm}). \tag{34}$$

Equation (33) is the familiar decomposition of the total angular momentum with respect to the origin as the sum of the ("spin") angular momentum $\mathbf{L}_{\rm cm}$ with respect to the center of mass plus the ("orbital") angular momentum $\mathbf{x}_{\rm cm} \times m\mathbf{v}_{\rm cm}$ of the system (with respect to the origin) as if the mass were concentrated at the center of mass.

The time dependence of the total angular momentum $\mathbf{L}$ is related by,

$$\frac{d\mathbf{L}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \mathbf{x}_i \times \mathbf{F}_i = \sum_i \boldsymbol{\tau}_i = \boldsymbol{\tau}, \tag{35}$$

where the total torque $\boldsymbol{\tau}$ is defined with respect to the origin.

In addition to considering angular momentum with respect to the origin and to the center of mass, it is useful to consider it relative to a general point P that may or may not be in motion in the inertial lab frame.

The torque $\boldsymbol{\tau}_P$ about point P is related to the torque $\boldsymbol{\tau}$ about the origin by,

$$\boldsymbol{\tau}_P = \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times \mathbf{F}_i = \boldsymbol{\tau} - \mathbf{x}_P \times \mathbf{F} = \boldsymbol{\tau} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm}. \tag{36}$$

Jensen (following an interesting discussion in [14]) noted that the angular momentum with respect to a point P can be defined in two ways.[6]

[6]The angular momenta $\mathbf{L}_P$ and $\mathbf{L}'_P$ are called absolute and relative, respectively, in [7].

First Definition: $\mathbf{L}_P$

Jensen defined $\mathbf{L}_P$ to be the angular momentum with respect to P, ignoring possible motion of P,[7,8]

$$\mathbf{L}_P = \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times m_i\,\mathbf{v}_i = \mathbf{L} - \mathbf{x}_P \times m\,\mathbf{v}_{\rm cm} = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_{\rm cm}. \tag{37}$$

However, when considering $d\mathbf{L}_P/dt$, one should take the possible velocity $\mathbf{v}_P$ into account,[9]

$$\frac{d\mathbf{L}_P}{dt} = \frac{d\mathbf{L}}{dt} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm} - \mathbf{v}_P \times m\,\mathbf{v}_{\rm cm} = \boldsymbol{\tau}_P + m\,\mathbf{v}_{\rm cm} \times \mathbf{v}_P. \tag{38}$$

Second Definition: $\mathbf{L}'_P$

If point P is moving in the lab frame, one can also define the angular momentum with respect to P similarly to eq. (34) for that with respect to the center of mass,[10,11]

$$\begin{aligned}\mathbf{L}'_P &= \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times m_i\,(\mathbf{v}_i - \mathbf{v}_P) = \mathbf{L} - \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_P - \mathbf{x}_P \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_P) \\ &= \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_P) = \mathbf{L}_P - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P. \end{aligned} \tag{39}$$

Of course, if $\mathbf{v}_P = 0$, then $\mathbf{L}'_P = \mathbf{L}_P$.

The time derivative of $\mathbf{L}'_P$ is, recalling eq. (38),

$$\frac{d\mathbf{L}'_P}{dt} = \frac{d\mathbf{L}_P}{dt} - \mathbf{v}_{\rm cm} \times m\,\mathbf{v}_P - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_P = \boldsymbol{\tau}_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times (-m\,\mathbf{a}_P) \equiv \boldsymbol{\tau}'_P. \tag{40}$$

Since the moving point P defines a (nonrotating) accelerated frame, the equation of motion of the angular momentum $\mathbf{L}'_P$ relative to this moving point includes the "fictitious" torque associated with the "fictitious" (coordinate) force $-m\,\mathbf{a}_P$ that appears in this frame to act on the center of mass of the system.

Point P is the Center of Mass of the System

In eqs. (37)-(40), the point P could be any point, moving or not. In particular, it could be at the center of mass. In this case, eqs. (37)-(40) become,

$$\mathbf{L}_P = \mathbf{L}'_P = \mathbf{L}_{\rm cm}, \qquad \frac{d\mathbf{L}_P}{dt} = \frac{d\mathbf{L}'_P}{dt} = \frac{d\mathbf{L}_{\rm cm}}{dt} = \boldsymbol{\tau}_{\rm cm}, \tag{41}$$

which is the well-known torque analysis with respect to the center of mass.

---

[7] Our notation differs from that of Jensen [13].

[8] An early use in English of $\mathbf{L}_P$, for a fixed point P, is in art. 134 of [1]. A use of $\mathbf{L}_P$ in an introductory text is implicit on p. 249 of [15] in the statement: *we shall require the moment of force about P to equal the rate of change of angular momentum about P (P being the point of contact).*

[9] Equation (38) is also deduced in sec. 11.8, p. 256 of [16] and sec. 2 of [17].

[10] An early use of $\mathbf{L}'_P$ is in sec. 4.2 of [18].

[11] Even when point P is moving, one can consider use of angular momentum $\mathbf{L}_P$ rather than $\mathbf{L}'_P$, so if the angular momentum with respect to a point is not clearly defined, confusion can result.

Of course, computation of the torque $\boldsymbol{\tau}_{\rm cm}$ requires knowledge of the forces that do not act on the center of mass, such as possible contact forces between the moving body and a surface at rest (or with known motion) in the (inertial) lab frame.

P is the Point of Contact between the Moving Body and a "Known" Surface

Following Jensen [13], we now consider point P to be at the point of contact between a moving (rigid) body of constant mass m and a surface in the lab frame that is either at rest or has known motion.[12,13]

A motivation for consideration of the point of contact is that computation the torque $\boldsymbol{\tau}_P$ does not require knowledge of the contact force $\mathbf{F}_P$.

In general, the moving body slides with respect to the "known" surface (as in the example of the sliding ladder), but if the motion also includes rolling (with a nonzero radius of curvature of rolling) then different points on the body are in contact with the "known" surface at different times. *That is, the following methods are not restricted to rolling (as implied in [13]).*

If the moving (rigid) body rolls without slipping about a point P of contact on a "known" surface, then the instantaneous motion is rigid-body rotation about the axis through P parallel to the angular velocity vector $\boldsymbol{\omega}$, and we can write the angular momenta as,

$$\mathbf{L}_P = I_P\,\boldsymbol{\omega}, \qquad \mathbf{L}'_P = I_P\,\boldsymbol{\omega} - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P. \tag{42}$$

If slipping occurs at the point of contact, these simpler forms do not hold.[14]

A.1 Method 1. Use $\mathbf{L}_P$ for P on a "Known" Surface

The first method suggested by Jensen [13] considers point P on the surface whose motion is known in the lab frame, and uses the angular momentum $\mathbf{L}_P$ of eq. (37) even though point P moves on that surface.

We now apply this method to the sliding-ladder problem for point C in the figure on p. 1. The coordinates and velocities of point C, and of the center of mass of the ladder, are,

$$\mathbf{x}_{\rm cm} = (l\sin\theta, l\cos\theta), \qquad \mathbf{v}_{\rm cm} = \left(l\,\dot\theta\cos\theta, -l\,\dot\theta\sin\theta\right), \tag{43}$$

$$\mathbf{x}_C = (0, 2l\cos\theta), \qquad \mathbf{v}_C = \left(0, -2l\,\dot\theta\sin\theta\right), \tag{44}$$

$$\mathbf{x}_{\rm cm} - \mathbf{x}_C = (l\sin\theta, -l\cos\theta), \qquad \mathbf{v}_{\rm cm} - \mathbf{x}_C = \left(l\,\dot\theta\cos\theta, l\,\dot\theta\sin\theta\right), \tag{45}$$

$$(\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times \mathbf{v}_{\rm cm} = l^2\,\dot\theta\left(\cos^2\theta - \sin^2\theta\right)\hat{\mathbf{z}}, \tag{46}$$

$$(\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times (\mathbf{v}_{\rm cm} - \mathbf{v}_C) = l^2\,\dot\theta\,\hat{\mathbf{z}}, \tag{47}$$

$$\mathbf{v}_{\rm cm} \times \mathbf{v}_C = -2l^2\,\dot\theta^2\cos\theta\sin\theta\,\hat{\mathbf{z}}. \tag{48}$$

---

[12]The formalism of this Appendix so far applies to, for example, "variable-mass" problems in which the system includes a rigid body whose mass varies with time.

[13]Jensen did not consider examples such as [19, 20, 21] in which a second rigid body moves in contact with the first, and the motion of both bodies is to be determined.

[14]The notes in Table I of [13] appear to be for the case of rolling without slipping.

According to eqs. (37) and (46), the angular momentum $\mathbf{L}_C$ is,[15]

$$\mathbf{L}_C = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_{\rm cm} = \frac{m\,l^2\,\dot\theta}{3}\,\hat{\mathbf{z}} +, l^2\,\dot\theta\left(\cos^2\theta - \sin^2\theta\right)\,\hat{\mathbf{z}}, \tag{49}$$

which does not equal,

$$I_C\,\boldsymbol{\omega} = I_C\,\dot\theta\,\hat{\mathbf{z}} = \frac{4m\,l^2\,\dot\theta}{3}\,\hat{\mathbf{z}}, \tag{50}$$

recalling eq. (13).

The torque $\boldsymbol{\tau}_C = \tau_C\,\hat{\mathbf{z}}$ about point C was given in eq. (22),

$$\tau_C = -m\,g\,l\sin\theta + 2F_f\,l\sin\theta = m\,g\,l\sin\theta - 2m\,l^2\,\ddot\theta\sin^2\theta - 2m\,l^2\,\dot\theta^2\cos\theta\sin\theta, \tag{51}$$

so the z-component of $\tilde{\boldsymbol{\tau}}_C$ of eq. (38) is, recalling eq. (48),

$$\tilde\tau_C = \tau_C - 2ml^2\cos\theta\sin\theta\dot\theta^2 \quad = \quad m\,g\,l\sin\theta - 2m\,l^2\,\ddot\theta\sin^2\theta - 4m\,l^2\,\dot\theta^2\cos\theta\sin\theta. \tag{52}$$

The torque analysis about point C according to eq. (38) is then,

$$\begin{aligned} \frac{dL_C}{dt} &= \frac{m\,l^2\,\ddot\theta}{3} + m\,l^2\,\ddot\theta\left(\cos^2\theta - \sin^2\theta\right) - 4m\,l^2\,\dot\theta^2\cos\theta\sin\theta \\ &= m\,g\,l\sin\theta - 2m\,l^2\,\ddot\theta\sin^2\theta - 4m\,l^2\,\dot\theta^2\cos\theta\sin\theta \qquad (53) \\ \frac{4m\,l^2\,\ddot\theta}{3} &= m\,g\,l\sin\theta, \qquad (54) \end{aligned}$$

which agrees with the equation of motion (5).[16]

A.2 Method 2. Use $\mathbf{L}_P$ for P on the Moving Body

This author cannot see any difference at the moment, say time t, when the body and the surface are in contact at point P if one considers P to be on the surface or on the body, separated by an infinitesimal difference. It is true that after a finite time interval the location of point P is different in these two cases, but the analysis of the motion of the system is based only on infinitesimal time differences dt from t.

Line 2 of Table I of [13] claims that the angular momentum $\mathbf{L}_P$ is different for Methods 1 and 2, but this author cannot understand why this could be so.

In examples like the sliding ladder, where the radius of curvature of "rolling" is zero, the history of the point of contact on the moving body and on the "known" surface are identical, and there can be no difference between methods 1 and 2.

[15]The $\mathbf{L}_C$ of sec. 2.4 above is $\mathbf{L}'_C$ in the notation of this Appendix, as discussed further in sec. A.3 below.

[16]The author does not agree with the claim in [13] that this method provides the simplest derivation of the equation of motion.

A.3 Method 3. Use $\mathbf{L}'_P$ for P on a "Known" Surface

Jensen's method 3 considers the angular momentum $\mathbf{L}'_P$ of eq. (39), taken relative to the moving point P of contact of the moving body with the "known" surface, taking point P to lie on the "known" surface rather than on the moving body.

Applying method 3 to point C in the sliding-ladder problem, we have that, according to eq. (39), and recalling eqs. (13) and (47),[17]

$$\mathbf{L}'_P = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_P) = \frac{m\,l^2\,\dot\theta}{3}\,\hat{\mathbf{z}} + m\,l^2\,\dot\theta\,\hat{\mathbf{z}} = \frac{4m\,l^2\,\dot\theta}{3}\,\hat{\mathbf{z}} = I_C\,\boldsymbol{\omega}. \quad (55)$$

Then, recalling equation of motion (40) for $\mathbf{L}'_P$, we see that Jensen's method 3 is the same as that used in sec. 2.4 above, and so leads to the equation of motion (5) for the sliding ladder.

A.4 Method 4. Use $\mathbf{L}'_P$ for P on the Moving Body

In case the motion of the moving body involves rolling (with nonzero radius of curvature) on the "known" surface, the acceleration $\mathbf{a}_P$ of the point of contact is, in general, different for the point of contact on the "known" surface and for that on the rolling body. Then, method 4 is different from method 3.

However, in examples like the sliding ladder, where the radius of curvature of "rolling" is zero, the history of the point of contact on the moving body and on the "known" surface are identical, and there can be no difference between methods 3 and 4.

Hence, method 4, when applied to the sliding ladder, also yields the equation of motion (5).

A.5 Method 5. Nominal Use of $\mathbf{L}_P$ for P at Rest in the Lab Frame

Jensen's method 5 follows discussion by Hu in [22] that considered the point of contact P at some time between a rigid body that rolls without slipping and a "known" surface at rest in the (inertial) lab frame, and regarded that point as fixed in the lab frame.

Hu considered the angular momentum $\mathbf{L}_P$ of our eq. (37) relative to the fixed point P.[18] However, it is not, in general, correct to suppose that since $\mathbf{L}_P = I_P\,\boldsymbol{\omega}$, eq. (38) reduces to $I_P\,\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}_P$, although this happens to be valid if the body has a symmetry axis.

Instead, Hu [22] took the time derivative of eq. (37) with $\mathbf{v}_P = 0$, and used eq. (36),

$$\frac{d\mathbf{L}_P}{dt} = \frac{d\mathbf{L}}{dt} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm} = \boldsymbol{\tau} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm} = \boldsymbol{\tau}_P \quad (56)$$

$$= \frac{d\mathbf{L}_{\rm cm}}{dt} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_{\rm cm}, \quad (57)$$

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = I_{\rm cm}\,\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times (-m\,\mathbf{a}_{\rm cm}) = \boldsymbol{\tau}_{\rm cm}, \quad (58)$$

[17] The result (55) is not consistent with the Note in line 3 of Table I of [13]), which Notes were presented without justification.

[18] For a fixed point P, $\mathbf{L}_P$ equals $\mathbf{L}'_P$ of eq. (39).

Our eq. (58) corresponds to eq. (2) of [22], and indicates the close relation between Hu's method (in the form (56), $d\mathbf{L}_P/dt = \boldsymbol{\tau}_P$) and the torque analysis (41) with respect to the (moving) center of mass. That is, Hu's method in the form (58) requires knowledge of the contact force $\mathbf{F}_P$, and so is not entirely in the spirit of Jensen's catalog of torque analyses.[19]

For the sliding-ladder problem, Jensen's Method 5, in the form (58) nominally about point C, is actually the analysis about point B, the center of mass, given in sec. 2.3 above, which of course led to the equation of motion (5).

We can also use eq. (56), noting that point A is the origin, so $\mathbf{L} = \mathbf{L}_A = -\frac{2}{3} m\, l\, \dot{\theta}$,

$$\frac{d\mathbf{L}_C}{dt} = \frac{d\mathbf{L}_A}{dt} - \mathbf{x}_C \times m\, \mathbf{a}_{\rm cm} = \boldsymbol{\tau}_C, \tag{59}$$

$$-\frac{2m\, l^2\, \ddot{\theta}}{3} - \left.(0, 2l\, \cos\theta, 0) \times (m\, a_x, m\, a_y, 0)\right|_z = 2l \sin\theta\, F_f - m\, g\, l \sin\theta, \tag{60}$$

$$-\frac{2m\, l^2\, \ddot{\theta}}{3} + 2m\, l\, a_x \cos\theta = 2m\, l\, \sin\theta \left(g - l \sin\theta\, \ddot{\theta} - l\cos\theta\, \dot{\theta}^2\right) - m\, g\, l \sin\theta, \tag{61}$$

$$-\frac{2m\, l^2\, \ddot{\theta}}{3} + 2m\, l^2 \cos^2\theta\, \ddot{\theta} - 2m\, l^2 \cos\theta \sin\theta\, \dot{\theta}^2 = m\, l\, g \sin\theta - m\, l\, \sin^2\theta\, \ddot{\theta} - m\, l\, \cos\theta \sin\theta\, \dot{\theta}^2, \tag{62}$$

$$\frac{4m\, l^2\, \ddot{\theta}}{3} = m\, l\, g \sin\theta, \tag{63}$$

as previously found in eq. (5). However, this analysis is only a small variant of that given about point A in sec. 2.2 above, as $\boldsymbol{\tau}_A = \boldsymbol{\tau}_C + \mathbf{x}_C \times m\, \mathbf{a}_{\rm cm}$.

A.6 Comments

It appears to this author that only three of the five methods in [13] are distinct for problems with zero radius of curvature for rolling. Only Method 1 provides an alternative torque analysis to those given in sec. 2, should such be desired.

B Appendix: Chap. 8 of Milne's *Vectorial Mechanics*

In this Appendix we transcribe Chap. 8 of [23] (1948), which showed an early awareness of the two definitions (37) and (39) of angular momentum with respect to a point P, into the notation of Appendix A above.

A difficulty in reading chap. 8 of [23] is that Milne used the symbol O to mean either the fixed origin or a moving point.

[19] A possible merit of eq. (58) is that it shows the torque $\boldsymbol{\tau}_{\rm cm}$ about the center of mass to be the sum of the torque $\boldsymbol{\tau}_P$ about the point of contact and an "effective" torque about this point in which an "effective" force $-m\, \mathbf{a}_{\rm cm}$ acts on the center of mass.

B.1 Sec. 295. The Momentum of a System of Particles

While Milne used the symbol $\mathbf{L}$ to mean the total momentum of a system of particles, we use this symbol for angular momentum. And, we write $\mathbf{P}$ for the total momentum of a system,

$$m = \sum_i m_i, \qquad \text{Milne}: \quad \sum m, \tag{64}$$

$$\mathbf{P} = \sum_i m_i \frac{d\mathbf{x}_i}{dt} = \sum_i m_i \, \dot{\mathbf{x}}_i = \sum_i m_i \, \mathbf{v}_i \qquad \text{Milne}: \quad \mathbf{L} = \sum m \, \dot{\mathbf{r}}, \tag{65}$$

$$\mathbf{L} = \sum_i \mathbf{x}_i \times m_i \, \mathbf{v}_i, \qquad \text{Milne}: \quad \mathbf{H}(O) = \sum \mathbf{r} \wedge m \, \dot{\mathbf{r}}. \tag{66}$$

That is, in sec. 295 of [23], symbol O refers to the origin of coordinates in an inertial frame. Note also that Milne used the wedge symbol $\wedge$ for the vector cross product, whereas we use the symbol $\times$.

B.2 Sec. 296. Equations of Motion

Milne used the symbol $\mathbf{P}$ for the force vector that we write as $\mathbf{F}$, and used $\mathbf{R}$ to mean the total force on a system. Milne used the symbol $\mathbf{\Gamma}$ for torque, whereas we use the symbol $\boldsymbol{\tau}$,

$$\mathbf{F} = \sum_i \mathbf{F}_i = \frac{d\mathbf{P}}{dt} = \sum_i m_i \, \ddot{\mathbf{x}}_i = m \, \mathbf{a}_{\text{cm}}, \qquad \text{Milne}: \quad \mathbf{R} = \sum \mathbf{P} = \frac{d\mathbf{L}}{dt} = \sum m \, \ddot{\mathbf{r}}, \tag{67}$$

$$\boldsymbol{\tau} = \sum_i \mathbf{x}_i \times \mathbf{F}_i, \qquad \text{Milne}: \quad \mathbf{\Gamma}(O) = \sum \mathbf{r} \wedge \ddot{\mathbf{r}}, \tag{68}$$

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}, \qquad \text{Milne}: \quad \frac{d\mathbf{H}(O)}{dt} = \mathbf{\Gamma}(O). \tag{69}$$

In sec. 296, Milne described O as "any fixed point", but the above transcription interprets O as the (fixed) origin.

B.3 Sec. 297. Principles of Linear and Angular Momentum

$$\mathbf{F} = 0 \Rightarrow \mathbf{P} = \text{const}, \qquad \text{Milne}: \quad \mathbf{R} = 0 \Rightarrow \mathbf{L} = \text{const}, \tag{70}$$

$$\boldsymbol{\tau} = 0 \Rightarrow \mathbf{L} = \text{const}, \qquad \text{Milne}: \quad \mathbf{\Gamma}(O) = 0 \Rightarrow \mathbf{H}(O) = \text{const}. \tag{71}$$

B.4 Sec. 298. Motion of the Centre of Mass

In sec. 298 Milne defined the position of the center of mass with respect to the origin as $\bar{\mathbf{r}}$,

$$m \, \mathbf{x}_{\text{cm}} = \sum_i m_i \, \mathbf{x}_i, \qquad \text{Milne}: \quad M \, \bar{\mathbf{r}} = \sum m \, \mathbf{r}, \tag{72}$$

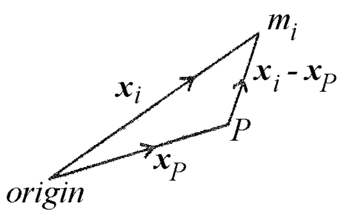
$$\mathbf{P} = m \, \dot{\mathrm{r}}_{\text{cm}} = m \, \mathbf{v}_{\text{cm}}, \qquad \text{Milne}: \quad \mathbf{L} = M \, \dot{\bar{\mathbf{r}}}, \tag{73}$$

$$\mathbf{F} = m \, \ddot{\mathrm{r}}_{\text{cm}} = m \, \mathbf{a}_{\text{cm}}, \qquad \text{Milne}: \quad \mathbf{R} = M \, \ddot{\bar{\mathbf{r}}}. \tag{74}$$

B.5 Sec. 300. Determination of the Angular Momentum about the Origin in Terms of That about Some Other Point P

This section can be hard to follow in that Milne abruptly redefined the origin to be at O', and used symbol O to represent some other point. Then, he considered a particle that he called P at some other point.

In our notation, Milne's O is written as P, and the particle is at position $\mathbf{x}_i$ with respect to the origin.



Milne:

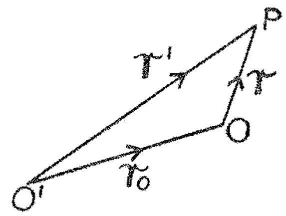


Fig. 74

$$\mathbf{x}_i = \mathbf{x}_P + (\mathbf{x}_i - \mathbf{x}_P), \qquad \text{Milne}: \quad \mathbf{r}' = \mathbf{r}_0 + \mathbf{r}, \tag{75}$$

$$\mathbf{L} = \sum_i \mathbf{x}_i \times m_i\,\mathbf{v}_i, \qquad \text{Milne}: \quad \mathbf{H}(O') = \sum \mathbf{r}' \wedge m\,\mathbf{v}, \tag{76}$$

$$\mathbf{L}_P = \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times m_i\,\mathbf{v}_i, \qquad \text{Milne}: \quad \mathbf{H}(O) = \sum \mathbf{r} \wedge m\,\mathbf{v}, \tag{77}$$

$$\mathbf{L} = \mathbf{L}_P + \mathbf{x}_P \times \sum_i m_i\,\mathbf{v}_i = \mathbf{L}_P + \mathbf{x}_P \times m\,\mathbf{v}_{\rm cm}, \qquad \text{Milne}: \quad \mathbf{H}(O') = \mathbf{H}(O) + \mathbf{r}_0 \wedge \mathbf{L}, \tag{78}$$

$$\mathbf{L} = \mathbf{L}_{\rm cm} + \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_{\rm cm}, \qquad \text{Milne}: \quad \mathbf{H}(O') = \mathbf{H}(C) + \bar{\mathbf{r}} \wedge \mathbf{L}. \tag{79}$$

Note that eq. (77) is the same as eq. (37) of Appendix A above.

B.6 Sec. 301. Angular Momentum about a Moving Point

In sec. 301, Milne's O became a moving reference point, called P in the notation of Appendix A above. Also, vector $\bar{\mathbf{r}}$ of sec. 298 was redefined to be the position of the center of mass with respect to his point O = our point P,

$$\mathbf{x}_{\rm cm} - \mathbf{x}_P, \qquad \text{Milne}: \quad \bar{\mathbf{r}}, \tag{80}$$

$$\mathbf{v}_P, \qquad \text{Milne}: \quad \mathbf{V}, \tag{81}$$

$$\mathbf{v}_i = \mathbf{v}_P + (\mathbf{v}_i - \mathbf{v}_P), \qquad \text{Milne}: \quad \mathbf{v} = \mathbf{V} + \mathbf{v}', \tag{82}$$

$$\mathbf{L}_P = \mathbf{L}'_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P, \qquad \text{Milne}: \quad \mathbf{H}(O) = \mathbf{H}_r(O) + \bar{\mathbf{r}} \wedge M\mathbf{V}, \tag{83}$$

That is, Milne's $\mathbf{H}_r(O)$ corresponds to $\mathbf{L}'_P$ of our eq. (39), the angular momentum with respect to a moving reference point.

Milne then noted that if the moving reference point is the center of mass, which he called point G, the two definitions of relative angular momentum, his $\mathbf{H}(O)$ and $\mathbf{H}_r(O)$, our $\mathbf{L}_P$ and $\mathbf{L}'_P$, become the same,

$$\mathbf{L}_{\rm cm} = \mathbf{L}'_{\rm cm}, \qquad \text{Milne}: \quad \mathbf{H}(G) = \mathbf{H}_r(G). \tag{84}$$

B.7 Sec. 303. Rate of Change of $\mathbf{H}(O) = \mathbf{L}_P$

Milne took the time derivative of our eq. (78) to find,

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}_P}{dt} + \mathbf{v}_P \times m\,\mathbf{v}_{\rm cm} + \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm},$$

$$\text{Milne :} \quad \frac{d\mathbf{H}(O')}{dt} = \frac{d\mathbf{H}(O)}{dt} + \dot{\mathbf{r}}_0 \wedge \mathbf{L} + \mathbf{r}_0 \wedge \frac{d\mathbf{L}}{dt}\,, \tag{85}$$

which is the same as the first form of our eq. (38).

Perhaps because of his redefinition of the origin O of secs. 295-299 to the O' of later sections, Milne did not relate $d\mathbf{H}(O')/dt$ of sec. 303 to the torque about the origin, nor did he consider the torque τ_P of our eq. (36) about the point P (= point O of sec. 303).

Instead, Milne remarked on the special case that his moving point O instantaneously coincides with the origin, O', at which time $\mathbf{r}_0 = 0$ and $\dot{\mathbf{r}}_0 = \mathbf{V}$, such that eq. (85) becomes,

$$\frac{d\mathbf{L}}{dt} = \left.\frac{d\mathbf{L}_P}{dt}\right|_{\mathbf{x}_P=0} + \mathbf{v}_P \times m\,\mathbf{v}_{\rm cm}, \quad \text{Milne :} \quad \frac{d\mathbf{H}(O')}{dt} = \left.\frac{d\mathbf{H}(O)}{dt}\right|_{O=O'} + \mathbf{V} \wedge \mathbf{L}. \tag{86}$$

He then specialized this result even further, supposing that point O is the center of mass G,

$$\left.\frac{d\mathbf{L}}{dt}\right|_{\mathbf{x}_P=\mathbf{x}_{\rm cm}=0} = \left.\frac{d\mathbf{L}_{\rm cm}}{dt}\right|_{\mathbf{x}_P=\mathbf{x}_{\rm cm}=0}, \quad \text{Milne :} \quad \frac{d\mathbf{H}(G')}{dt} = \left.\frac{d\mathbf{H}(G)}{dt}\right|_{G=G'}. \tag{87}$$

Milne seemed to consider this rather trivial result so important that he devoted sec. 304 to further discussion of it.

B.8 Sec. 305. Rate of Change of $\mathbf{H}_r(G) = \mathbf{L}'_{\rm cm}$

Milne next recalled our eq. (84), which when used in eq. (87) implies the somewhat trivial relation,

$$\left.\frac{d\mathbf{L}}{dt}\right|_{\mathbf{x}_{\rm cm}=0} = \frac{d\mathbf{L}'_{\rm cm}}{dt}\,, \quad \text{Milne :} \quad \frac{d\mathbf{H}(G')}{dt} = \frac{d\mathbf{H}_r(G)}{dt}\,. \tag{88}$$

B.9 Sec. 306. Rate of Change of $\mathbf{H}_r(O) = \mathbf{L}'_P$ when $\mathbf{x}_P = 0$

In this section, Milne inserted our eq. (83) into our eq. (86) to obtain, when $O = O'$ ($\mathbf{x}_P = 0$),

$$\frac{d\mathbf{L}}{dt} = \left.\frac{d\mathbf{L}'_P}{dt}\right|_{\mathbf{x}_P=0} + \frac{d}{dt}[(\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P] + \mathbf{v}_P \times m\,\mathbf{v}_{\rm cm} = \left.\frac{d\mathbf{L}'_P}{dt}\right|_{\mathbf{x}_P=0} + \mathbf{x}_{\rm cm} \times m\,\frac{d\mathbf{v}_P}{dt}\,,$$

$$\text{Milne :} \quad \frac{d\mathbf{H}(O')}{dt} = \left.\frac{d\mathbf{H}_r(O)}{dt}\right|_{O=O'} + \mathbf{V} \wedge \mathbf{L} + \frac{d}{dt}[\bar{\mathbf{r}} \wedge M\,\mathbf{V}]_{O=O'} = \left.\frac{d\mathbf{H}_r(O)}{dt}\right|_{O=O'} + \bar{\mathbf{r}} \wedge M\,\frac{d\mathbf{V}}{dt}\,. \tag{89}$$

This is as close as Milne came to our general relation (40), that could also have been written in the less insightful form,

$$\frac{d\mathbf{L}'_P}{dt} = \frac{d\mathbf{L}}{dt} - \mathbf{x}_{\rm cm} \times m\,\mathbf{a}_P + \mathbf{x}_P \times m\,(\mathbf{a}_P - \mathbf{a}_{\rm cm}), \tag{90}$$

which reduces to eq. (89) when $\mathbf{x}_P = 0$.

B.10 Comments

Despite the formal development of the two definitions of angular momentum about an arbitrary, and in general moving, point in Chap. 8 of [23], when Milne turned to specific examples in Chap. 15, he recommended (sec. 373) that the torque analyses be made with respect to the center of mass, or about a fixed point of the motion if that exists.

The author finds Milne's treatments of the specific examples to be noteworthy, and even rather entertaining as in the case of the "golfer's nemesis" in sec. 421 that considers a sphere which rolls without slipping inside a vertical cylinder, displaying a vertical oscillation such that a golf ball could roll down into the cup and then move back up and out![20]

Thanks to Amin Rezaeezadeh for e-discussions of this problem.

References

[1] E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, 1905),
http://physics.princeton.edu/~mcdonald/examples/mechanics/routh_elementary_rigid_dynamics.pdf

[2] G. Faucher, *Fixed points in torque-angular momentum relations*, Am. J. Phys. **51**, 758 (1983), http://physics.princeton.edu/~mcdonald/examples/mechanics/faucher_ajp_51_758_83.pdf

[3] F.C. Crawford, *Problem: Moments to Remember*, Am. J. Phys. **57**, 105, 177 (1989),
http://physics.princeton.edu/~mcdonald/examples/mechanics/crawford_ajp_57_105_89.pdf

[4] C. Kacser, *Are deliberate mistakes a valid teaching tool?* Am. J. Phys. **57**, 583, 1063 (1990), http://physics.princeton.edu/~mcdonald/examples/mechanics/kacser_ajp_57_583_89.pdf

[5] F.R. Zypman, *Moments to remember—The conditions for equating torque and rate of change of angular momentum*, Am. J. Phys. **58**, 41 (1990),
http://physics.princeton.edu/~mcdonald/examples/mechanics/zypman_ajp_58_41_90.pdf

[6] M.S. Tiersten, *Moments not to forget—The conditions for equating torque and rate of change of angular momentum about the instantaneous center*, Am. J. Phys. **59**, 733 (1991), http://physics.princeton.edu/~mcdonald/examples/mechanics/tiersten_ajp_59_733_91.pdf

[7] M.A. Illarramendi and T. del Rio Gaztelurrutia, *Moments to be cautious of—relative versus absolute angular momentum*, Eur. J. Phys. **16**, 249 (1995),
http://physics.princeton.edu/~mcdonald/examples/mechanics/illarramendi_ejp_16_249_95.pdf

[8] L. Rodríguez V, *Torque and the rate of change of angular momentum at an arbitrary point*, Am. J. Phys. **71**, 1201 (2003),
http://physics.princeton.edu/~mcdonald/examples/mechanics/rodriguez_ajp_71_1201_03.pdf

[20]The author has spent considerable effort discussing the "sphere of death" [24], corresponding to the example of secs. 413-415 of Milne [23], but with a disk (motorcycle), rather than a sphere, that rolls without slipping inside the upper hemisphere of a fixed sphere.

[9] W.F.D. Theron, *On using moments around the instantaneous center of rotation*, Am. J. Phys. **77**, 918 (2009),
http://physics.princeton.edu/~mcdonald/examples/mechanics/theron_ajp_77_918_09.pdf

[10] *Sliding Ladder* (Aug. 4, 2003),
http://webdev.physics.harvard.edu/academics/undergrad/probweek/sol47.pdf
http://physics.princeton.edu/~mcdonald/examples/mechanics/ladder_sol47_03.pdf

[11] See, for example, sec. 39 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed., (Pergamon Press, Oxford, 1960).

[12] K.T. McDonald, *Comments on Torque Analyses* (Apr. 28, 2019),
http://physics.princeton.edu/~mcdonald/examples/torque.pdf

[13] J.H. Jensen, *Five ways of deriving the equation of motion for rolling bodies*, Am. J. Phys. **80**, 1073 (2012), http://physics.princeton.edu/~mcdonald/examples/mechanics/jensen_ajp_80_1073_12.pdf

[14] L. Turner and A.M. Turner, *Asymmetric rolling bodies and the phantom torque*, Am. J. Phys. **78**, 905 (2010),
http://physics.princeton.edu/~mcdonald/examples/mechanics/turner_ajp_78_905_10.pdf

[15] C. Kittel, W.D. Knight and M.A. Ruderman, *Mechanics*, $2^{\rm nd}$ ed. (McGraw-Hill, 1973),
http://physics.princeton.edu/~mcdonald/examples/mechanics/kittel_chap8_73.pdf
Chapter 8 was considerably different from that in the 1^{st} (1965) edition.

[16] J.M. Knudsen and P.G. Hjorth, *Elements of Newtonian Mechanics*, $3^{\rm rd}$ ed. (Springer, 2000), http://physics.princeton.edu/~mcdonald/examples/mechanics/knudsen_3ed_00.pdf

[17] J.H. Jensen, *Rules for rolling as a rotation about the instantaneous point of contact*, Eur. J. Phys. **32**, 389 (2011),
http://physics.princeton.edu/~mcdonald/examples/mechanics/jensen_ejp_32_389_11.pdf

[18] K. Symon, *Mechanics* (Addison-Wesley, 1953, 1960, 1971),
http://physics.princeton.edu/~mcdonald/examples/mechanics/symon_chap4_53.pdf
http://physics.princeton.edu/~mcdonald/examples/mechanics/symon_chap4_60.pdf
Equation (4-19) was expressed more clearly in the 1960 edition.

[19] K.T. McDonald, *Cylinder Rolling on Another Rolling Cylinder* (Oct. 2, 2014),
http://physics.princeton.edu/~mcdonald/examples/2cylinders.pdf

[20] K.T. McDonald, *Cylinder Rolling inside Another Rolling Cylinder* (Oct. 23, 2014),
http://physics.princeton.edu/~mcdonald/examples/2cylinders_in.pdf

[21] K.T. McDonald, *Slab Rolling on a Rolling Cylinder* (Mar. 28, 2016),
http://physics.princeton.edu/~mcdonald/examples/slaboncylinder.pdf

[22] B.Y.-K. Hu, *Rolling of asymmetric discs on an inclined plane*, Eur. J. Phys. **32**, L51 (2011), http://physics.princeton.edu/~mcdonald/examples/mechanics/hu_ejp_32_L51_11.pdf

[23] E.A. Milne, *Vectorial Mechanics* (Metheun, London; Interscience Publishers, New York, 1948), http://physics.princeton.edu/~mcdonald/examples/mechanics/milne_mechanics.pdf

[24] K.T. McDonald, *Circular Orbits Inside the Sphere of Death* (Nov. 8, 1993), http://physics.princeton.edu/~mcdonald/examples/sphereofdeath.pdf