A Mechanical Model
That Exhibits a Gravitational Critical Radius

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1 Problem

A popular model at science museums (and also a science toy [1]) that illustrates how curvature can be associated with gravity consists of a surface of revolution $r = -k/z$ with $z < 0$ about a vertical axis z. The curvature of the surface, combined with the vertical force of Earth's gravity, leads to an inward horizontal acceleration of kg/r^2 for a particle that slides freely on the surface in a circular, horizontal orbit.

Consider the motion of a particle that slides freely on an arbitrary surface of revolution, $r = r(z) \geq 0$, defined by a continuous and differentiable function on some interval of z. The surface may have a nonzero minimum radius R at which the slope dr/dz is infinite. Discuss the character of oscillations of the particle about circular orbits to deduce a condition that there be a critical radius $r_{\rm crit} > R$, below which the orbits are unstable. That is, the motion of a particle with $r < r_{\rm crit}$ rapidly leads to excursions to the minimum radius R, after which the particle falls off the surface.

Give one or more examples of analytic functions $r(z)$ that exhibit a critical radius as defined above. These examples provide a mechanical analogy as to how departures of gravitational curvature from that associated with a $1/r^2$ force can lead to a characteristic radius inside which all motion tends toward a singularity.

2 Solution

We work in a cylindrical coordinate system (r, θ, z) with the z axis vertical. It suffices to consider a particle of unit mass.

In the absence of friction, there is no torque on a particle about the z axis, so the angular momentum component $J = r^2\dot\theta$ about that axis is a constant of the motion, where $\dot{}$ indicates differentiation with respect to time.

For motion on a surface of revolution $r = r(z)$, we have $\dot r = r'\dot z$, where $'$ indicates differentiation with respect to z. Hence, the kinetic energy can be written

$$T = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2 + \dot z^2) = \frac{1}{2}[\dot z^2(1 + r'^2) + r^2\dot\theta^2]. \tag{1}$$

The potential energy is $V = gz$. Using Lagrange's method, the equation of motion associated with the z coordinate is

$$\ddot z(1 + r'^2) + \dot z^2 r' r'' = -g + \frac{J^2 r'}{r^3}. \tag{2}$$

For a circular orbit at radius r_0, we have

$$r_0^3 = \frac{J^2 r_0'}{g}. \tag{3}$$

We write $\dot{\theta}_0 = \Omega$, so that $J = r_0^2\Omega$.

For a perturbation about this orbit of the form

$$z = z_0 + \epsilon \sin \omega t, \tag{4}$$

we have, to order ϵ,

$$\begin{aligned} r(z) &\approx r(z_0) + r'(z_0)(z - z_0) \\ &= r_0 + \epsilon r_0' \sin \omega t, \qquad (5) \\ r' &\approx r_0' + \epsilon r_0'' \sin \omega t, \qquad (6) \\ \frac{1}{r^3} &\approx \frac{1}{r_0^3}\left(1 - 3\epsilon \sin \omega t \frac{r_0'}{r_0}\right). \qquad (7) \end{aligned}$$

Inserting (4-7) into (2) and keeping terms only to order ϵ, we obtain

$$-\epsilon\omega^2 \sin \omega t (1 + r_0'^2) \approx -g + \frac{J^2}{r_0^3}\left(r_0' - 3\epsilon \sin \omega t \frac{r_0'^2}{r_0} + \epsilon \sin \omega t \, r_0''\right). \tag{8}$$

From the zeroeth-order terms we recover (3), and from the order-ϵ terms we find that

$$\omega^2 = \Omega^2 \frac{3r_0'^2 - r_0 r_0''}{1 + r_0'^2}. \tag{9}$$

The orbit is unstable when $\omega^2 < 0$, *i.e.*, when

$$r_0 r_0'' > 3r_0'^2. \tag{10}$$

This condition has the interesting geometrical interpretation (noted by a referee) that the orbit is unstable wherever

$$(1/r^2)'' < 0, \tag{11}$$

i.e., where the function $1/r^2$ is concave inwards.

For example, if $r = -k/z$, then $1/r^2 = z^2/k^2$ is concave outwards, $\omega^2 = J^2/(k^2 + r_0^4)$, and there is no regime of instability.

We give three examples of surfaces of revolution that satisfy condition (11).

First, the hyperboloid of revolution defined by

$$r^2 - z^2 = R^2, \tag{12}$$

where R is a constant. Here, $r_0' = z_0/r_0$, $r_0'' = R^2/r_0^3$, and

$$\omega^2 = \Omega^2 \frac{3z_0^2 - R^2}{2z_0^2 + R^2} = \Omega^2 \frac{3r_0^2 - 4R^2}{2r_0^2 - R^2}. \tag{13}$$

The orbits are unstable for

$$z_0 < \sqrt{3}R, \tag{14}$$

or equivalently, for

$$r_0 < \frac{2\sqrt{3}}{3}R = 1.1547R \equiv r_{\text{crit}}. \tag{15}$$

As r_0 approaches R, the instability growth time approaches an orbital period.

Another example is the Gaussian surface of revolution,

$$r^2 = R^2 e^{z^2}, \tag{16}$$

which has a minimum radius R, and a critical radius $r_{\rm crit} = R\sqrt[4]{e} = 1.28R$.

Our final example is the surface

$$r = -\frac{k}{z\sqrt{1-z^2}}\ , \qquad (-1 < z < 0), \tag{17}$$

which has a minimum radius of $R = 2k$, approaches the surface $r = -k/z$ at large r (small z), and has a critical radius of $r_{\rm crit} = 6k/\sqrt{5} = 1.34R$.

These examples arise in a $2+1$ geometry with curved space but flat time. As such, they are not fully analagous to black holes in $3+1$ geometry with both curved space and curved time. Still, they provide a glimpse as to how a particle in curved spacetime can undergo considerably more complex motion than in flat spacetime.

3 Acknowledgement

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References

[1] The Vortx(tm) Miniature Wishing Well, Divnick International, Inc., 321 S. Alexander Road, Miamisburg, OH 45342, http://www.divnick.com/