

Relativistic Harmonic Oscillator

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1 Problem

Estimate the period τ of a “simple” harmonic oscillator consisting of a zero-rest-length massless spring of constant k that is connected to a rest mass m_0 (with the other end of the spring fixed to the origin), taking in account the relativistic mass.

2 Solution

2.1 Quick Estimates

Ignoring relativistic effects, the angular frequency ω_0 and the period τ_0 of the oscillator are

$$\omega_0 = \sqrt{\frac{k}{m_0}}, \quad \tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m_0}{k}}. \quad (1)$$

In this approximation, the oscillating mass has position and velocity

$$x = A \cos \omega_0 t, \quad v = -A\omega_0 \sin \omega_0 t. \quad (2)$$

In general, the oscillating mass has (time-dependent) relativistic mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0 \left(1 + \frac{v^2}{2c^2}\right), \quad (3)$$

where c is the speed of light in vacuum. We expect that the period τ of oscillation of the relativistic mass can be approximated as

$$\tau \approx \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\langle m \rangle}{k}} > \tau_0, \quad (4)$$

where $\langle m \rangle > m_0$ is an appropriate average of the relativistic mass. This might be the time average,

$$\begin{aligned} \langle m \rangle_t &= \frac{1}{\tau} \int_0^\tau m(t) dt \approx \frac{m_0}{\tau} \int_0^\tau \left(1 + \frac{v^2}{2c^2}\right) dt \approx m_0 \left(1 + \frac{1}{2\tau_0 c^2} \int_0^{\tau_0} A^2 \omega_0^2 \cos^2 \omega_0 t dt\right) \\ &= m_0 \left(1 + \frac{A^2 \omega_0^2}{4c^2}\right) = m_0 \left(1 + \frac{kA^2}{4m_0 c^2}\right), \end{aligned} \quad (5)$$

in which case

$$\tau \approx \tau_0 \sqrt{1 + \frac{kA^2}{4m_0 c^2}} \approx \tau_0 \left(1 + \frac{kA^2}{8m_0 c^2}\right), \quad \langle m \rangle = \langle m \rangle_t. \quad (6)$$

However, it could be that the spatial average is more appropriate,

$$\begin{aligned}\langle m \rangle_x &= \frac{1}{A} \int_0^A m(x) dx \approx \frac{m_0}{A} \int_0^A \left(1 + \frac{v^2}{2c^2} \right) dx \approx m_0 \left[1 + \frac{1}{2Ac^2} \int_0^A A^2 \omega_0^2 \left(1 - \frac{x^2}{A^2} \right) dx \right] \\ &= m_0 \left(1 + \frac{A^2 \omega_0^2}{4c^2} \right) = m_0 \left(1 + \frac{kA^2}{3m_0 c^2} \right),\end{aligned}\quad (7)$$

noting that $\sin \omega t = \sqrt{1 - \cos^2 \omega t} \approx \sqrt{1 - x^2/A^2}$, in the approximation that oscillating mass has x -coordinate $x = A \cos \omega t$. In this case,

$$\tau \approx \tau_0 \sqrt{1 + \frac{kA^2}{3m_0 c^2}} \approx \tau_0 \left(1 + \frac{kA^2}{6m_0 c^2} \right), \quad \langle m \rangle = \langle m \rangle_x. \quad (8)$$

As many other averages of the relativistic mass can be imagined, we seek a method that clarifies what type of approximation is best.

2.2 A Better Estimate

A different approach is to note that the motion is periodic with spatial amplitude A , and so the period τ can be computed as

$$\tau = 4 \int_0^A \frac{dt}{dx} dx = 4 \int_0^A \frac{dx}{v}. \quad (9)$$

Total energy E is conserved in this example,

$$E = mc^2 + \frac{kx^2}{2} = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} + \frac{kx^2}{2} = m_0 c^2 + \frac{kA^2}{2}, \quad (10)$$

where the potential energy of the system is $kx^2/2$, such that¹

$$\frac{1}{v} = \frac{\tau_0}{2\pi} \frac{1 + k(A^2 - x^2)/2m_0 c^2}{\sqrt{A^2 - x^2} \sqrt{1 + k(A^2 - x^2)/4m_0 c^2}} \approx \frac{\tau_0}{2\pi} \left(\frac{1}{\sqrt{A^2 - x^2}} + \frac{3k\sqrt{A^2 - x^2}}{8m_0 c^2} \right). \quad (11)$$

Hence,

$$\tau \approx \frac{2\tau_0}{\pi} \left(\int_0^A \frac{dx}{\sqrt{A^2 - x^2}} + \frac{3k}{8m_0 c^2} \int_0^A \sqrt{A^2 - x^2} dx \right) = \tau_0 \left(1 + \frac{3kA^2}{16m_0 c^2} \right). \quad (12)$$

The correction term in this result is 2% larger than that in the estimate (8) based on the spatial average of the relativistic mass.

References

- [1] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, 1980).

¹There is a sign error in the correction term of eq. (7-150), p. 325 of [1], which corresponds to eq. (11) of the present note. Thanks to Bill Jones for pointing this out.