Magnetic Damping

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(April 14, 2012)

1 Problem

Discuss the most prominent effect on the vertical oscillations of a copper cube of edge a that is suspended from a spring of constant k when the cube is immersed in a uniform, horizontal magnetic field $\mathbf{B}_0$ which is normal to two of the cube's vertical faces. The cube is electrically neutral in the absence of the magnetic field.

2 Solution

This problem was posed on p. 250 of the April 2012 issue of The Physics Teacher.

When a conductor moves through a nonuniform, external magnetic field, the magnetic flux varies through loops fixed inside the conductor, so an electromotive force is induced around the loops, according to Faraday's law (in the rest frame of the conductor), and eddy currents flow. The Lorentz force on these eddy currents, due to the external magnetic field, opposes the motion, and one speaks of magnetic braking/damping. For an example of this phenomenon, see [1]. This effect is (ultra)relativistic, being of order v^2/c^2, where v is the speed of the conductor and c is the speed of light in vacuum. While such relativistic effects are generally small for "ordinary" velocities, the eddy current density obeys $J = \sigma E$, where the conductivity σ for good conductors approaches c^2/v^2 when measured in Gaussian units, such that eddy-current braking is a rare example of an important (ultra)relativistic correction at low velocities.

In the present problem the magnetic field is spatially uniform, so the magnetic flux through a moving loop does not change, and no eddy currents develop. Yet, there exists a very weak magnetic-damping effect, as discussed below.

The cube has mass $m = \rho a^3$, where ρ is the mass density of copper, so in the absence of the magnetic field it oscillates vertically with angular frequency

$$\omega = \sqrt{\frac{k}{m}}\,. \tag{1}$$

When the oscillations have amplitude A the vertical velocity has the form

$$v(t) = A\omega\, e^{i\omega t}, \tag{2}$$

which is surely small in magnitude compared to the speed of light c in vacuum.

In the instantaneous rest frame of the cube there appears to be a horizontal electric field of (time-dependent) magnitude

$$E_0(t) \approx \frac{v(t)B_0}{c} = \frac{AB_0\,\omega\, e^{i\omega t}}{c} \tag{3}$$

(to order v/c, and in Gaussian units),[1] whose direction is perpendicular to that of $\mathbf{B}_0$, and hence normal to two faces of the cube. Assuming that the frequency ω is low enough that the magnetic field penetrates into the copper cube, the electric field (3) exists throughout its interior, and adds to the electric field $E_1 \approx 4\pi Q/a^2$ associated with surface charge Q (to be determined). The total electric field $E = E_0 + E_1$ gives rise to electrical current density of magnitude $J = \sigma E$ where $\sigma \approx 6 \times 10^{17}$ s is the electrical conductivity of copper.[2] As a consequence, an electrical current,

$$I(t) = a^2 J = a^2 \sigma E \tag{4}$$

flows through the copper cube, along the direction of the electric field. This current leads to an accumulation of charge on the faces of the cube normal to the electric field,

$$Q(t) = \int I\, dt = -\frac{ia^2 \sigma E}{\omega}\,, \tag{5}$$

and this surface-charge distribution leads to electric field

$$E_1(t) \approx \frac{4\pi Q(t)}{a^2} = -\frac{4\pi i \sigma E}{\omega}\,, \tag{6}$$

The total electric field inside the cube is

$$E = E_0 + E_1 \approx E_0 - \frac{4\pi i \sigma E}{\omega}\,, \tag{7}$$

i.e.,

$$E \approx \frac{E_0}{1 + 4\pi i \sigma/\omega} \approx -\frac{i\omega E_0}{4\pi\sigma} = -\frac{i\omega^2 A B_0\, e^{i\omega t}}{4\pi\sigma c}\,. \tag{8}$$

The electric current is

$$I(t) = a^2 \sigma E \approx -\frac{ia^2 \omega^2 A B_0\, e^{i\omega t}}{4\pi c}\,, \tag{9}$$

independent of the conductivity σ.[3]

The copper cube presents electrical resistance

$$R = \frac{1}{a\sigma} \tag{11}$$

[1] A lab-frame argument is that the Lorentz force on the conduction electrons can be thought of as due to an effective electric field, given by eq. (3).

[2] At high frequencies the conductivity σ has a significant imaginary part, which we neglect in the present low-frequency example.

[3] The result (9) could be gotten more quickly by arguing that the electric field in the interior of a good conductor is essentially zero, so that there must be charge $\pm Q \approx \pm a^2 E_0/4\pi$ on the faces of the cubes normal to E_0 to "short out" this field. Then, the current in the interior of the cube is

$$I(t) = \frac{dQ}{dt} \approx \frac{a^2}{4\pi}\frac{dE_0}{dt} = \frac{ia^2 \omega^2 A B_0\, e^{i\omega t}}{4\pi c}\,. \tag{10}$$

However, this argument fails when eddy currents are present, so care is required in using it.

to the current flow, and heat is generated at the time-average rate

$$\langle P \rangle = \frac{|I|^2 R}{2} = \frac{a^3 \omega^4 A^2 B_0^2}{32\pi^2 c^2 \sigma} \equiv C A^2 \,, \tag{12}$$

where

$$C = \frac{a^3 \omega^4 B_0^2}{2c^2 \sigma} = \frac{k^2 B_0^2}{32\pi c^2 \rho^2 \sigma} \,. \tag{13}$$

This is an effect at order v^2/c^2, and will be extremely small due to the additional presence of the large conductivity σ in the denominator.[4]

The energy deposition (12) results in a gradual decrease of the amplitude A of oscillation, as the energy U of the oscillation system is[5]

$$U(t) = kA^2(t). \tag{14}$$

Then, the energy-loss equation $dU/dt = -\langle P \rangle$ leads to

$$\frac{dA^2}{dt} = -\frac{CA^2}{k} \,, \tag{15}$$

whose solution is

$$A = A_0 e^{-t/\tau}, \tag{16}$$

with (large!) damping time constant

$$\tau = 2k/C = \frac{64\pi^2 c^2 \rho^2 \sigma}{k B_0^2} \,. \tag{17}$$

For parameters typical of laboratory experiments, the time constant τ is larger than the age of the Universe.

2.1 Other Effects at Order v^2/c^2

Even for $\mathbf{B} = 0$ the "relativistic mass" increase with velocity implies that the angular frequency of oscillation is smaller than $\sqrt{k/m}$ by a term of order v^2/c^2.

For nonzero magnetic field the angular frequency is also slightly reduced by the vertical Lorentz force on the horizontal current discussed above.

Whether the tiny frequency shift or the very weak magnetic damping is the more prominent effect is a matter of opinion.

References

[1] K.T. McDonald, *Pitching Pennies into a Magnet* (Jan. 15, 1997),
http://physics.princeton.edu/~mcdonald/examples/pennies.pdf

[4]When eddy currents are generated the surface charge is negligible, and a factor of σ appears in the numerator rather than the denominator of the expression for $\langle P \rangle$.

[5]We ignore the tiny change in the magnetic field energy due to the field associated with the current (9).