The Greek Eccentricity

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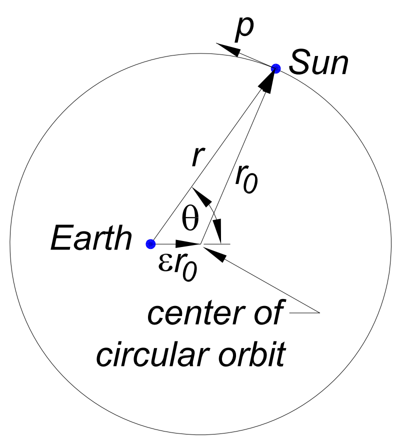
1 Problem

The ancient Greeks considered the orbit of the Sun around the fixed Earth to be a circle (say, of radius r_0), but the center of the circle must be offset from the Earth (say, by distance ϵr_0 where $\epsilon \approx 1/24$ is the eccentricity) to fit the facts reasonably well. Assuming the motion is as described, and that the corresponding force is a central force, deduce the force law (or equivalently, the potential).



Show that the form of the energy in the present problem is the same as that for the bound Coulomb problem if coordinates and momenta are exchanged. [Consequently the orbits in configuration space for one problem have the same form as the orbits in momentum space for the other.]

2 Solution

2.1 Via Use of Conservation of Energy and Angular Momentum

Conservation of energy E for a central potential $V(r)$ with conserved angular momentum $L = mr^2\dot\theta$ can be written as

$$E = \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 + V(r) = \frac{1}{2}m\dot r^2 + \frac{L^2}{2mr^2} + V(r), \tag{1}$$

where m is the mass of the orbiting object.

We know the form of the orbit, $r(\theta)$, so we can write

$$\dot r = \frac{dr}{d\theta}\dot\theta = \frac{L}{mr^2}\frac{dr}{d\theta}. \tag{2}$$

Hence, the potential is related by

$$V(r) = E - \frac{L^2}{2mr^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]. \tag{3}$$

For a circular orbit of radius r_0 whose center is offset from the origin by distance ϵr_0 along the $+x$-axis we have

$$r_0^2 = r^2 + \epsilon^2 r_0^2 - 2\epsilon r_0 r\cos\theta, \tag{4}$$

leading to

$$r = r_0\left(\epsilon\cos\theta + \sqrt{1 - \epsilon^2\sin^2\theta}\right), \tag{5}$$

and

$$\frac{dr}{d\theta} = -\frac{\epsilon r\sin\theta}{\sqrt{1 - \epsilon^2\sin^2\theta}}. \tag{6}$$

Inserting this into the expression (3) for $V(r)$, we find

$$V(r) = E - \frac{L^2}{2mr^2(1 - \epsilon^2\sin^2\theta)}. \tag{7}$$

To complete the solution we note that from eq. (5),

$$\sqrt{1 - \epsilon^2\sin^2\theta} = \frac{r}{r_0} - \epsilon\cos\theta, \tag{8}$$

while from eq. (4),

$$\cos\theta = \frac{r^2 - r_0^2(1 - \epsilon^2)}{2\epsilon r_0 r}, \tag{9}$$

so

$$\sqrt{1 - \epsilon^2\sin^2\theta} = \frac{r^2 + r_0^2(1 - \epsilon^2)}{2r_0 r}. \tag{10}$$

Finally, the potential is

$$V(r) = E - \frac{2L^2 r_0^2}{m[r^2 + r_0^2(1 - \epsilon^2)]^2}, \tag{11}$$

so the (central) force is

$$F(r) = -\frac{dV}{dr} = -\frac{8L^2 r_0^2 r}{m[r^2 + r_0^2(1 - \epsilon^2)]^3}. \tag{12}$$

2.2 Solution Via the Orbit Equation

This is the "textbook" method to deduce the potential given the orbit.

First, I sketch a derivation of this trick, which is based on the substitution $u(\theta) = 1/r$, *where the form of the orbit is* $r(\theta)$ *for distance* r *from the force center, and* θ *is the azimuthal angle in the plane of the orbit. Then, conservation of energy can be written as*

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r), \tag{13}$$

where m *is the mass of the orbiting object,* $L = mr^2\dot{\theta}$ *is the conserved angular momentum and* E *is the total energy. The radial motion can be discussed in terms of an effective potential,*

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + V(r). \tag{14}$$

Then,

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{L}{m}\frac{du}{d\theta}\,. \tag{15}$$

Also,

$$\ddot{r} = -\frac{L}{m}\frac{d^2u}{dt\,d\theta} = -\frac{L}{m}\frac{d^u}{d\theta^2}\dot{\theta} = -\frac{Lu^2}{m}\frac{d^2u}{d\theta^2}\,. \tag{16}$$

The radial equation of motion is

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr} = F(r) + \frac{L^2}{mr^3}\,. \tag{17}$$

Combining, we find the orbit equation*:*

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F\left(\frac{1}{u}\right). \tag{18}$$

We now begin solution 2. Again we use the cosine law to deduce the form $r(\theta)$ as in eq. (5), so

$$u = \frac{1}{r} = \frac{1}{r_0\left(\epsilon\cos\theta + \sqrt{1-\epsilon^2\sin^2\theta}\right)}\,. \tag{19}$$

Then,

$$\frac{du}{d\theta} = \frac{\epsilon u\sin\theta}{\sqrt{1-\epsilon^2\sin^2\theta}}\,, \qquad \frac{d^2u}{d\theta^2} = \frac{\epsilon u\left(\cos\theta + \epsilon\sin^2\theta\sqrt{1-\epsilon^2\sin^2\theta}\right)}{(1-\epsilon^2\sin^2\theta)^{3/2}}\,, \tag{20}$$

and

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{r_0(1-\epsilon^2\sin^2\theta)^{3/2}}\,. \tag{21}$$

Again, we note that

$$\sqrt{1-\epsilon^2\sin^2\theta} = \frac{r}{r_0} - \epsilon\cos\theta = \frac{r^2 + r_0^2(1-\epsilon^2)}{2r_0 r}\,. \tag{22}$$

Then,

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{8L^2 r_0^2 r}{m[r^2 + r_0^2(1-\epsilon^2)]^3}\,, \tag{23}$$

as in eq. (12).

2.3 Vectorial Solution

(*Akin to the Lenz-vector approach*)

We consider the radius vector $\mathbf{r}$, the constant vector $\boldsymbol{\epsilon} r_0$ from the force center (the Earth) to the center of the orbit, and the vector $\mathbf{r}_0$ of constant length that points from the center of the orbit to the Sun. Then, $\mathbf{r} = \mathbf{r_0} + \boldsymbol{\epsilon} r_0$.

The momentum $\mathbf{p}$ lies in the plane of the orbit and is perpendicular to vector $\mathbf{r}_0$. The (constant) angular momentum vector $\mathbf{L}$ (about the Earth) is perpendicular to both $\mathbf{r}_0$ and $\mathbf{p}$.

As for the Lenz-vector approach, it is interesting to consider the vector

$$\mathbf{c} = \mathbf{p} \times \mathbf{L}, \qquad c = pL. \tag{24}$$

This vector is in the same direction as $\mathbf{r}_0$, so we write

$$\mathbf{c} = K\mathbf{r}_0, \qquad c = Kr_0. \tag{25}$$

Then, we can express the scalar product $\mathbf{r}\cdot\mathbf{c}$ two ways. First,

$$\mathbf{r}\cdot\mathbf{c} = \mathbf{r}\cdot\mathbf{p}\times\mathbf{L} = \mathbf{r}\times\mathbf{p}\cdot\mathbf{L} = L^2. \tag{26}$$

But also,

$$\mathbf{r}\cdot\mathbf{c} = K\mathbf{r}\cdot\mathbf{r}_0 = K\mathbf{r}\cdot(\mathbf{r} - \boldsymbol{\epsilon} r_0) = K(r^2 - \epsilon r_0 r\cos\theta) = K\frac{r^2 + r_0^2(1-\epsilon^2)}{2}\,, \tag{27}$$

using the cosine law (4). Hence,

$$K = \frac{2L^2}{r^2 + r_0^2(1-\epsilon^2)}\,. \tag{28}$$

Furthermore, we note that from eqs. (24)-(25), the magnitude of the momentum can be written as

$$p = \frac{Kr_0}{L} = \frac{2Lr_0}{r^2 + r_0^2(1-\epsilon^2)}\,. \tag{29}$$

We insert this into the energy equation,

$$E = \frac{p^2}{2m} + V(r), \tag{30}$$

to find

$$V(r) = E - \frac{2L^2 r_0^2}{m[r^2 + r_0^2(1 - \epsilon^2)]^2}\,, \tag{31}$$

as in eq. (11).

2.4 Relation to the Coulomb Potential

Conservation of energy for the present problem can be written as

$$E = \frac{p^2}{2m} + V(r) = \frac{p^2}{2m} + E - \frac{2L^2 r_0^2}{m[r^2 + r_0^2(1 - \epsilon^2)]^2}\,, \tag{32}$$

and hence,

$$\frac{p^2}{2m} - \frac{2L^2 r_0^2}{m[r^2 + r_0^2(1 - \epsilon^2)]^2} = 0. \tag{33}$$

Compare this to a negative-energy Coulomb problem, for which the energy is

$$E = \frac{p^2}{2m} - \frac{\alpha}{r} \equiv -\frac{p_0^2}{2m}\,. \tag{34}$$

This can be rearranged to yield

$$\frac{r^2}{2m} - \frac{2\alpha^2 m^2}{(p^2 + p_0^2)^2} = 0. \tag{35}$$

Thus, the present problem and the Coulomb problem are related by the exchange transformation $q_i \leftrightarrow P_i$, $p_i \leftrightarrow Q_i$, between coordinates and their conjugate momenta (q_i, p_i) of one system and the (Q_i, P_i) of the other. More formally, the two problems are related by the canonical transformation generated by the function

$$\mathcal{F} = \sum_i q_i Q_i. \tag{36}$$

As a consequence, the orbits in configuration space of one problem have the same form as the orbits in momentum space for the other, *etc.*