Electromagnetic Fields of a Small Helical Toroidal Antenna

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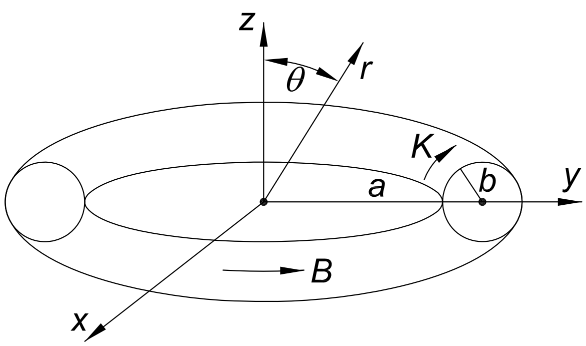
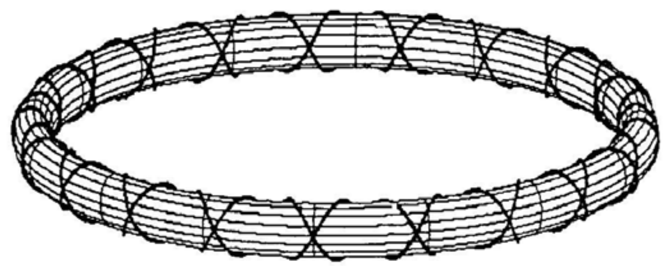
(December 8, 2008)

1 Problem

Discuss the electromagnetic fields of a helical toroidal winding with major radius a and minor radius b, as shown below, that carries an oscillatory surface current $\mathbf{K}e^{-i\omega t}$ that has no azimuthal component. There is no accumulation of charge anywhere, *i.e.*, $\boldsymbol{\nabla}\cdot\mathbf{K} = 0$. It suffices to consider the limit that $ka \ll 1$ where $k = \omega/c$ is the wave number and c is the speed of light.





A physical realization of this winding is shown on the right above, as perhaps first proposed by Corum [1], where two counter windings of $N/2$ turns are each driven with current $I = 2\pi a K(\rho = a)e^{-i\omega t}/N$ such that the magnetic dipole moment is zero. A complication in this case is that at the N close crossings of the two windings, there is localized, oscillating charge accumulation due to the capacitance between the nearby wires which leads to azimuthal variation of the currents in the windings.

2 Solution

2.1 Sheet Currents with No Azimuthal Variation

This solution follows [2], which assumed sheet currents as stated above. See also [3, 4, 5, 6, 7]. A discussion that emphasizes the fields close to the toroid is givn in [8]. The possibility of an elementary particle, called an "anapole," with a helical toroidal field was considered by Zel'dovitch in 1958 [9].

We first consider the static limit of sheet currents represented by N turns of steady current I_0 with no azimuthal component. Then, the electric field is zero everywhere, the magnetic field is zero outside the torus, while inside the torus we use Ampère's law (in

Gaussian units) to find,

$$\mathbf{B}_{\rm static} = \begin{cases} 2NI_0\,\hat{\boldsymbol{\phi}}/c\rho & \text{(inside torus)}, \\ 0 & \text{(outside torus)}, \end{cases} \tag{1}$$

where $\rho = \sqrt{x^2 + y^2}$.

The fact that both the static electric and magnetic vanish outside the torus implies that all multiple moments vanish for a conventional multipole expansion in this region. However, the vector potential is certainly nonzero inside the torus, so that continuity of the vector potential at its surface implies that the vector potential is nonzero outside the torus as well.[1]

To find the vector potential, and the electric and magnetic field, in the case of oscillating currents in the windings, we follow a method due to Hertz. The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be derived from the scalar and vector potentials V and $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{2}$$

in Gaussian units. We work in the Lorentz gauge where the potentials satisfy the auxiliary condition,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t}\,. \tag{3}$$

The potentials then obey the wave equations,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}, \qquad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\varrho, \tag{4}$$

where ϱ and $\mathbf{J}$ are the charge and current densities of the sources of the waves. Formal solutions for the (retarded) vector potential have been given by Lorenz,

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}', t' = t - R/c)}{R} d{\rm Vol}', \qquad V(\mathbf{r}, t) = \int \frac{\varrho(\mathbf{r}', t' = t - R/c)}{R} d{\rm Vol}', \tag{5}$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

There is nowhere any accumulation of charge in the present problem, so that $\varrho = 0$ and hence the scalar potential V is zero as well. In this case, the Lorentz gauge condition (3) tells us that,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0, \tag{6}$$

so that the vector potential can be written as the curl of another vector, which we will call $\mathbf{Z}_{\rm M}$, the magnetic Hertz vector,[2]

$$\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{Z}_{\rm M}. \tag{7}$$

From the wave equation (4) for the vector potential, we have,

$$\nabla^2 \mathbf{A} = \nabla^2(\boldsymbol{\nabla} \times \mathbf{Z}_{\rm M}) = \boldsymbol{\nabla} \times \nabla^2 \mathbf{Z}_{\rm M} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{4\pi}{c}\mathbf{J} = \boldsymbol{\nabla} \times \frac{1}{c^2}\frac{\partial^2 \mathbf{Z}_M}{\partial t^2} - \frac{4\pi}{c}\mathbf{J} \tag{8}$$

[1]Aharonov and Bohm [10] have discussed quantum phenomena in the regions of zero magnetic field but nonzero vector potential.

[2]For additional discussion of electric and magnetic Hertz vectors and scalars, see the Appendix of [11].

If we write the current density as,

$$\mathbf{J} = c\boldsymbol{\nabla} \times \mathbf{M}, \tag{9}$$

in terms of a magnetization density $\mathbf{M}$, the magnetic Hertz vector satisfies the wave equation,

$$\nabla^2 \mathbf{Z}_{\rm M} - \frac{1}{c^2}\frac{\partial^2 \mathbf{Z}_{\rm M}}{\partial t^2} = -4\pi\mathbf{M}. \tag{10}$$

This justifies the alternative terminology that the magnetic Hertz vector is a polarization potential, with the formal solution,

$$\mathbf{Z}_{\rm M}(\mathbf{r}, t) = \int \frac{\mathbf{M}(\mathbf{r}', t' = t - R/c)}{R} d\mathrm{Vol}'. \tag{11}$$

If we regard the static magnetic field (1) inside the torus as due to a magnetization density $\mathbf{M}$ rather than a conduction current density $\mathbf{J}$ or $\mathbf{K}$, then $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = 0$, so the required magnetization is,

$$\mathbf{M}_{\rm static} = \frac{\mathbf{B}_{\rm static}}{4\pi} = \begin{cases} NI_0\,\hat{\boldsymbol{\phi}}/2\pi c\rho & \text{(inside torus)}, \\ 0 & \text{(outside torus)}. \end{cases} \tag{12}$$

In the case of an oscillating current $I_0 e^{-i\omega t}$, the equivalent oscillating magnetization is,

$$\mathbf{M}(\rho, t) = \mathbf{M}_{\rm static}(\rho) e^{-i\omega t}. \tag{13}$$

Using this in eq. (11), the magnetic Hertz vector is given by,

$$\mathbf{Z}_{\rm M}(\mathbf{r}, t) = \int \frac{\mathbf{M}_{\rm static}(\rho') e^{i(kR-\omega t)}}{R} d\mathrm{Vol}'. \tag{14}$$

For an approximate solution, we use the relations,

$$R \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}', \qquad \frac{1}{R} \approx \frac{1}{r}\left(1 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right), \tag{15}$$

where the term in $1/r^2$ improves the accuracy at small r. Then,[3]

$$\begin{aligned} \mathbf{Z}_{\rm M}(\mathbf{r}, t) &\approx \frac{e^{i(kr-\omega t)}}{r} \int \mathbf{M}_{\rm static}(\rho') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \left(1 + \frac{\mathbf{r}'\cdot\hat{\mathbf{r}}}{r}\right) d\mathrm{Vol}' \\ &\approx \frac{NI_0}{2\pi c}\frac{e^{i(kr-\omega t)}}{r} \int \frac{\boldsymbol{\phi}'}{\rho'} \left[1 + \left(\frac{1}{r} - ik\right)\hat{\mathbf{r}}\cdot\mathbf{r}'\right] d\mathrm{Vol}' \end{aligned} \tag{16}$$

We evaluate integral (16) for $\mathbf{r}$ in spherical coordinates (r, θ, ϕ) but $\mathbf{r}'$ in cylindrical coordinates (ρ, ϕ, z), and for an observer at distance $r \gg a$ from the origin. In rectangular coordinates, $\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\mathbf{r}' = (\rho'\cos\phi', \rho'\sin\phi', z')$ and $\hat{\boldsymbol{\phi}}' =$

[3]Expanding the factor $e^{i\mathbf{k}\cdot\mathbf{r}'}$ to first order in $\mathbf{r}'$ in the Hertz vector $\mathbf{Z}_{\rm M}$ corresponds to expanding this factor to second order in the vector potential $\mathbf{A}$, which corresponds to keeping (magnetic) quadrupole terms in the current density distribution.

$-\cos\phi'\,\hat{\mathbf{x}} + \sin\phi'\,\hat{\mathbf{y}}$, so that,

$$
\begin{aligned}
\mathbf{Z}_{\rm M}(r \gg a, t) &\approx \frac{NI_0}{2\pi c}\frac{e^{i(kr-\omega t)}}{r}\int \frac{-\cos\phi'\,\hat{\mathbf{x}} + \sin\phi'\,\hat{\mathbf{y}}}{\rho'} \times \\
&\quad \times \left[1 + \left(\frac{1}{r} - ik\right)[\rho'(\cos\phi'\sin\theta\cos\phi + \sin\phi'\sin\theta\sin\phi) + z'\cos\theta]\right] d{\rm Vol}' \\
&= \frac{NI_0 V}{4\pi c}\frac{e^{i(kr-\omega t)}}{r}\left(\frac{1}{r} - ik\right)\sin\theta\,(-\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) \\
&= \frac{NI_0 V}{4\pi c}\frac{e^{i(kr-\omega t)}}{r}\left(\frac{1}{r} - ik\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad (17)
\end{aligned}
$$

where $V = 2\pi^2 ab^2$ is the volume of the torus.

The vector potential is,

$$
\mathbf{A}(r \gg a, t) = \boldsymbol{\nabla} \times \mathbf{Z}_{\rm M} \approx \frac{NI_0 V}{4\pi c}\frac{e^{i(kr-\omega t)}}{r}\left[2\left(\frac{1}{r^2} - \frac{ik}{r}\right)\cos\theta\,\hat{\mathbf{r}} + \left(\frac{1}{r^2} - \frac{ik}{r} - k^2\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right]. \qquad (18)
$$

The static vector potential is obtained by setting ω and k to zero,

$$
\mathbf{A}_{\rm static}(r \gg a) \approx \frac{NI_0 V}{4\pi c r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{3(\mathbf{m}_T \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_T}{r^3}, \qquad (19)
$$

where,

$$
\mathbf{m}_T = \frac{NI_0 V}{4\pi c}\,\hat{\mathbf{z}}. \qquad (20)
$$

That is, the static vector potential outside the torus has the form of the magnetic field of a magnetic dipole $\mathbf{m}_T$ given by eq. (20), although the actual magnetic dipole moment of the toroidal winding is zero. The quantity $\mathbf{m}_T$ of eq. (20) has been called the toroid dipole moment [5] because if this moment oscillates it generates an electric field formally similar to that of an oscillating electric dipole (see eq. (21)). However, the moment (20) is the result of weighting the current density $\mathbf{J}$ by two powers of distance, rather than by one, and so is an aspect of the magnetic quadrupole moment of the current density.

The electric and magnetic fields outside the torus are,

$$
\begin{aligned}
\mathbf{E}(r \gg a, t) &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = ik\mathbf{A} \\
&\approx ik\frac{NI_0 V}{4\pi c}\frac{e^{i(kr-\omega t)}}{r}\left[2\left(\frac{1}{r^2} - \frac{ik}{r}\right)\cos\theta\,\hat{\mathbf{r}} + \left(\frac{1}{r^2} - \frac{ik}{r} - k^2\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] \\
&= ik^3\frac{e^{i(kr-\omega t)}}{r}(\hat{\mathbf{r}} \times \mathbf{m}_T) \times \hat{\mathbf{r}} + (ik + k^2 r)e^{i(kr-\omega t)}\frac{3(\mathbf{m}_T \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_T}{r^3}, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}(r \gg a, t) &= \boldsymbol{\nabla} \times \mathbf{A} \approx -ik\frac{NI_0 V}{4\pi c}\frac{e^{i(kr-\omega t)}}{r}\left(\frac{1}{r^2} + \frac{ik}{r} + k^2\right)\sin\theta\,\hat{\boldsymbol{\phi}} \\
&= ik^3\left(1 - \frac{1}{ikr} + \frac{1}{k^2 r^2}\right)\frac{e^{i(kr-\omega t)}}{r}\hat{\mathbf{r}} \times \mathbf{m}_T. \qquad (22)
\end{aligned}
$$

Both **E** and **B** vanish outside the torus in the static limit $k = 0$. The radiation fields are,

$$E_{\mathrm{rad},\theta} = B_{\mathrm{rad},\phi} = -ik^3 \frac{NI_0V}{4\pi c} \frac{e^{i(kr-\omega t)}}{r} \sin\theta. \tag{23}$$

The fields (21)-(22) have the same form as those for a small oscillating electric dipole, except that they are multiplied by an additional factor of k. And indeed, in a systematic multipole expansion of the sources of vector electromagnetic fields (see, for example, sec. 9.10 of [12]) an oscillating magnetization can contribute to the electric dipole moment, but with an additional factor of k compared to the part of the electric dipole moment due to the electric charge distribution.

The time-average radiated power has the angular distribution,

$$\frac{dP_{\mathrm{rad}}}{d\Omega} = \frac{1}{8\pi c}\left(\frac{NI_0Vk^3}{4\pi}\right)^2 \sin^2\theta, \tag{24}$$

and the time-average radiated power is,

$$P_{\mathrm{rad}} = \frac{1}{2} R_{\mathrm{rad}} I_0^2, \tag{25}$$

where the radiation resistance R_{rad} is given by,

$$R_{\mathrm{rad}} = \frac{1}{3c}\left(\frac{NVk^3}{4\pi}\right)^2 = \frac{\pi^2}{12c}\left(Nk^3ab^2\right)^2 = \frac{16\pi^8}{3c}\left(\frac{Nab^2}{\lambda^3}\right)^2 \approx 1.5\times 10^6 \left(\frac{Nab^2}{\lambda^3}\right)^2\ \Omega. \tag{26}$$

For example, if $a = \lambda/10$, $b = \lambda/100$ and $N \approx 580$ turns, then $R_{\mathrm{rad}} = 50\,\Omega$.

Of course, the inductance L of such a winding is high. In SI units, the inductance is,

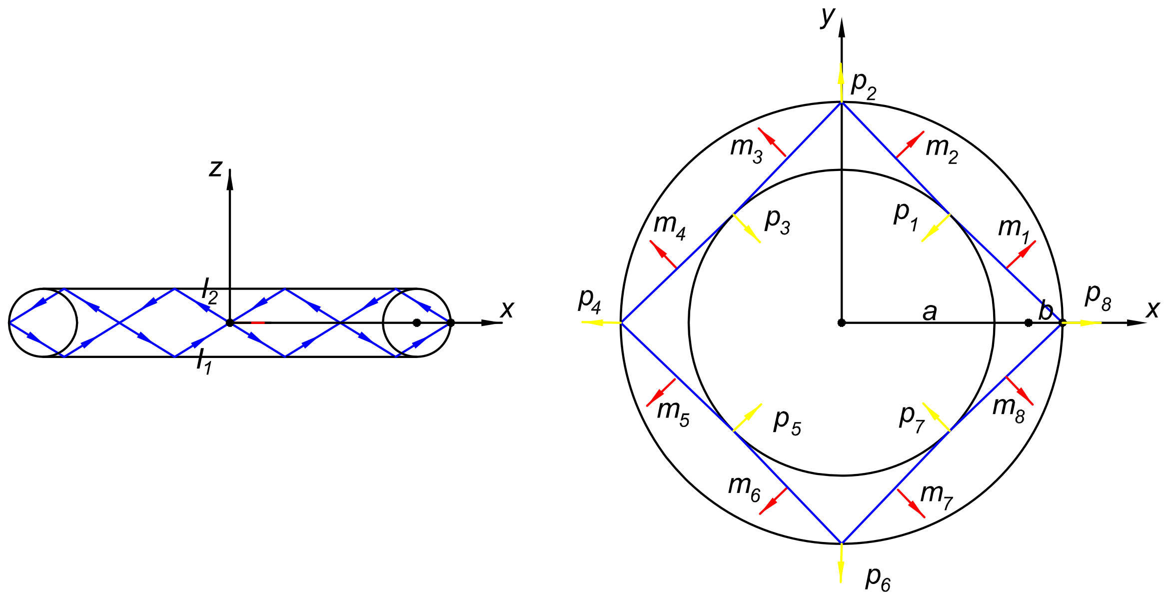
$$L = \frac{\mu_0 N^2 b^2}{2\pi a}\,. \tag{27}$$

For the above example, $L \approx 0.007\,b$ henries for b in meters.

2.2 Counter-Wound Helical Toroidal Antenna

We give only a qualitative discussion of the counter-wound helical toroidal antenna, using the case that $N = 8$ as an example.

As shown in the sketch below, the $8 = 2\times 4$ turns can be thought of as forming 8 elongated loops, each of which is formed from a half turn of each of the two counter windings, indicated as diamonds on the left part of the figure. We suppose that the wires of the two windings pass so close to one another that the capacitance of the 8 crossing points of the two windings is very large. Then, there exist resonant frequencies at which the direction of the current in each winding reverses at each crossing point, and the toroid dipole moment (20) vanishes.



Each of these loops has a magnetic moment $\mathbf{m}_j$ that is perpendicular to the plane of each loop, and which lies in the x-y plane, as shown on the right part of the figure.

At each of the 8 crossing points of the wires there is a large capacitance, and a corresponding electric dipole moment $\mathbf{p}_j$ that points towards (or away from) the center of the toroid.

All electric and magnetic multipole moments of order less than N vanish, so the radiation from the system is highly suppressed.

For an observer on the z axis, above the center of the antenna, the radiation electric field vectors from the 8 magnetic moments $\mathbf{m}_j$ and from the 8 electric dipole moments $\mathbf{p}_j$ are parallel to the x-y plane. The sum of the radiation electric fields is zero, and hence, there is no radiation along the z axis.

An observer in the x-y plane sees very small electric field from the 8 electric dipole moments, while the electric field from the 8 magnetic moments sums to a nonzero vertical component.

The overall pattern of the radiation electric field is that it is largest, and approximately independent of azimuth, in the x-y plane, while vanishing along the z direction. This is the qualitative character of electric dipole radiation from an oscillating dipole moment directed along the z-axis. However, it is ironic that when a counter-wound helical toroidal antenna is operated at resonance, the toroid dipole moment vanishes, and this exotic antenna configuration reduces to a circular array of "ordinary" electric and magnetic dipoles.

There exist some analytic studies in the literature for the case of small N [14, 15], but some of their conclusion are at odds with the present analysis.

A Appendix: The Chu Limit

One design goal of small antennas is a large bandwidth. In classic papers, Wheeler [17] and Chu [18] noted that there is a limit to the bandwidth of an antenna that fits inside a sphere of radius $R \ll \lambda$ that can be expressed as a lower limit on the Q of the antenna,

$$Q = \frac{\omega \langle U_{\rm near} \rangle}{\langle P_{\rm rad} \rangle} \gtrsim \frac{1}{4(kR)^3}\,, \tag{28}$$

where $\langle U_{\rm near} \rangle$ is the time-averaged energy stored in the non-propagating near electromagnetic fields of the antenna and $\langle P_{\rm rad} \rangle$ is the time-average radiated power. See also [19]. The bound (28) is closely approached by a small linear dipole antenna of half-height R.

This argument is based on a decomposition of the fields of the antenna into multipole moments. Since the counter-wound helical toroidal antenna has no nonzero electric or magnetic multipole moments of the usual sort, it could be that this type of antenna can have a Q smaller than the limit of eq. (28). We saw in sec. 2.1 that the fields outside the windings of a small counter-wound helical toroidal antenna are essentially the same as those of a small electric dipole antenna. Hence, the part of Q of a small counter-wound helical toroidal antenna associated with the fields outside the windings is essentially the same as that of a small electric dipole antenna, and therefore satisfies the bound (28). However, $\langle U_{\rm near} \rangle$ also includes the energy stored inside the windings,

$$\langle U_{\rm inside} \rangle = \frac{1}{2} L I_0^2, \tag{29}$$

where $L = 2\pi N^2 b^2 / c^2 a$ in Gaussian units. This energy is much larger than that stored in the near fields outside the windings. Then, recalling eqs. (25)-(26),

$$Q \approx \frac{\omega L}{R_{\rm rad}} = \omega \frac{2\pi N^2 b^2}{c^2 a} \frac{12c}{\pi^2 N^2 a^2 b^4 k^6} = \frac{24}{\pi k^5 a^2 b^3} \gg \frac{1}{4(ka)^3}\,. \tag{30}$$

Thus, a small counterwound helical toroidal antenna is a high-Q device and does not come close to satisfying the Chu limit.

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