“Hidden” Momentum of a Steady Current Distribution in a System at “Rest”

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(July 9, 2012): This note is largely superceded by [1].

1 Problem

Can a steady current distribution contain a nonzero total mechanical momentum, often called hidden momentum [2]-[17] due to the motion of the charge carriers?[1]

A steady current distribution $\mathbf{J}_{\rm steady}(\mathbf{r})$ is one with no time dependence. If the current density $\mathbf{J}_{\rm steady}$ flows in conductors, they must be at rest.

The divergence of a steady current distribution has no time dependence, so the electric charge density $\rho_{\rm steady}(\mathbf{r},t)$ could have at most a linear time dependence, according to the continuity equation

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}. \tag{1}$$

However, a charge distribution $\rho_{\rm steady}(\mathbf{r},t) = \rho_0(\mathbf{r}) + \dot\rho_0(\mathbf{r})t$ is unbounded at large and small times, so a we take a physical, steady current distribution to be associated with a steady charge distribution $\rho_{\rm steady}(\mathbf{r})$ as well. Hence,

$$\boldsymbol{\nabla} \cdot \mathbf{J}_{\rm steady} = 0, \tag{2}$$

and lines of the vector field $\mathbf{J}_{\rm steady}$ form closed loops. As a consequence,[2]

$$\int \mathbf{J}_{\rm steady}\ d\mathrm{Vol} = \sum_i e_i \mathbf{v}_i = 0, \tag{3}$$

where, in the microscopic view, the current distribution consists of a set of charges e_i with rest masses m_i and velocities $\mathbf{v}_i$. You may assume that all charge carriers have the same ratio e/m of charge to rest mass.

Then, eq. (3) indicates that the total, nonrelativistic, mechanical momentum of the steady current distribution is zero,

$$\mathbf{P}_{\rm mech,nonrel,J_{steady}} = \sum_i m \mathbf{v}_i = 0. \tag{4}$$

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[1] “Hidden” momentum as considered here is to be separated from the problematic issue of a classical description of electron spin and of permanent magnetism. Hence, we do not consider examples such as a charge plus a permanent magnet [7, 9].

[2] Noting that $\boldsymbol{\nabla} \cdot (x_i \mathbf{J}) = x_i \boldsymbol{\nabla} \cdot \mathbf{J} + \mathbf{J} \cdot \boldsymbol{\nabla} x_i = J_i$, we have that $\int J_i\ d\mathrm{Vol} = \int \boldsymbol{\nabla} \cdot (x_i \mathbf{J})\ d\mathrm{Vol} = \oint (x_i \mathbf{J}) \cdot d\mathbf{Area} = 0$ for any steady current distribution that is bounded in space.

However, it is not required that all charge carriers have the same speed, so the relativistic, mechanical momentum,

$$\begin{aligned}
\mathbf{P}_{\mathrm{mech,rel,J_{steady}}} &= \sum_i m\gamma_i \mathbf{v}_i = \sum_i \frac{m_i \mathbf{v}_i}{\sqrt{1 - v_i^2/c^2}} \approx \mathbf{P}_{\mathrm{mech,nonrel,J_{steady}}} + \sum_i \frac{m_i v_i^2}{2c^2}\mathbf{v}_i \\
&= \sum_i \frac{m_i v_i^2}{2c^2}\mathbf{v}_i \equiv \mathbf{P}_{\mathrm{hidden}}, \qquad (5)
\end{aligned}$$

of the steady current distribution can in principle be nonzero.[3] Here, c is the speed of light in vacuum, and it suffices to consider only media for which the relative permittivity and relative permeability are unity: $\epsilon = 1 = \mu$.

Give examples of isolated, spatially bounded systems with nonzero "hidden" mechanical momentum in steady current distributions that flow in conductors of uniform cross section. Comment on the physical origin of the "hidden" momentum.

2 Solution

The fundamental forces that govern most behavior of human-scale systems are electromagnetic, with gravity acting as a background force that we will ignore in the rest of this note. Then, we can consider the total momentum of a system to be the sum of its (macroscopic) mechanical and electrical momenta,

$$\mathbf{P}_{\mathrm{total}} = \mathbf{P}_{\mathrm{mech}} + \mathbf{P}_{\mathrm{EM}}. \qquad (6)$$

If the system contains bulk matter that is both under stress and in motion, the macroscopic mechanical momentum includes the volume integral of the momentum density associated with the stress, which is of electromagnetic origin in a microscopic view. See, for example, [18]. We avoid discussion of this complexity (and of the interesting issue of the relativity of steady energy flow [19]) by restricting our attention to systems with no bulk motion other than that of the steady electrical currents.[4] Then, according to eq. (5),

$$\mathbf{P}_{\mathrm{total}} = \mathbf{P}_{\mathrm{hidden}} + \mathbf{P}_{\mathrm{EM}}. \qquad (7)$$

The system may also involve steady energy flow, such that the center of mass/energy (see, Appendix A.1) varies with time. In this case, the system is not at "rest", and its total momentum need not be zero.

However, if the center of mass/energy of the system is at rest, such that $\mathbf{P}_{\mathrm{total}} = 0$, then

$$\mathbf{P}_{\mathrm{hidden}} = -\mathbf{P}_{\mathrm{EM}} \qquad \text{(center of mass/energy at rest)}. \qquad (8)$$

The (macroscopic) electromagnetic momentum $\mathbf{P}_{\mathrm{EM}}$ of a system is most generally identified as the volume integral of $1/c^2$ times the Poynting energy-flux vector field [20],

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \qquad (9)$$

[3]The author does not advocate the definition of "hidden momentum" given in eq. (5), but not prefers that given in [1].

[4]That is, we don't consider examples in which the steady current is due to charges fixed to rotating disks.

where we use Gaussian units, and $\mathbf{E}$ and $\mathbf{B}$ are the (macroscopic) electric and magnetic fields of the system. That is,

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{S}}{c^2}\, d\mathrm{Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol}, \tag{10}$$

as first noted by Abraham [21]. The magnetic field is of order $1/c$ (or higher), so the electromagnetic momentum (10) is of order $1/c^2$ (or higher), and is a "hidden" effect is systems where all particle velocities are small compared to c.

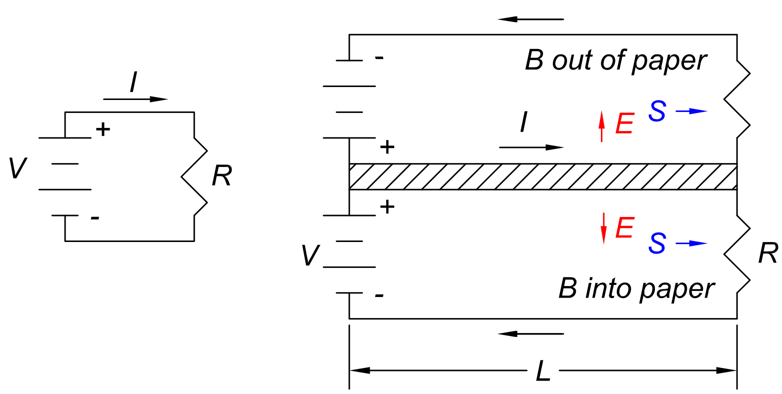
In the following we calculate only to order $1/c^2$, in which case the electromagnetic momentum (10) can be written two other ways (see, for example, [7, 22, 23] and Appendix A.2),

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} \approx \int \frac{\rho \mathbf{A}^{(\rm C)}}{c}\, d\mathrm{Vol} \approx \int \frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d\mathrm{Vol}, \tag{11}$$

where ρ is the electric charge density, $\mathbf{A}^{(\rm C)}$ is the vector potential in the Coulomb gauge,[5] and $V^{(\rm C)}$ is the instantaneous Coulomb potential.[6] The second form is the electromagnetic part of the canonical momentum of the charges of the system as introduced by Faraday [24] and by Maxwell [25] (see also Appendix A.3), and the third form appears to have been introduced by Furry [7].

2.1 Battery Connected to a Resistor via a Superconducting Transmission Line

Consider the case illustrated in the figure below, in which a battery of voltage V is connected to a resistor R via a superconducting coaxial transmission line of length L.



The current in this circuit is $I = V/R$, which produces an azimuthal magnetic field of strength $B_\phi = 2I/cr$ for $a < r < b$, where a is the outer radius of the inner conductor, and b is the inner radius of the outer conductor. The electric field is radial, with magnitude $E_r = V/r \ln b/a$. The Poynting vector points to the right (along the positive z-axis), and has magnitude $S_z = IV/2\pi r^2 \ln b/a$. The power delivered to the resistor is $\int_a^b S_z\, 2\pi r\, dr = IV =$

[5]The second form of eq. (11) is actually gauge invariant if it is written in terms of the (gauge-invariant) rotational part, $\mathbf{A}_{\rm rot}$, of the vector potential, which equals the vector potential in the Coulomb gauge.

[6]$V^{(\rm C)}$ is the electric scalar potential in the Coulomb gauge.

I^2R, which is the power dissipated in the resistor. The electromagnetic field momentum is

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{S}}{c^2}\, d{\rm Vol} = \frac{L}{c^2} \int_a^b S_z\, 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{I^2RL}{c^2}\, \hat{\mathbf{z}}. \tag{12}$$

Because energy flows from the battery to the resistor, the "macroscopic" center of mass/energy is not at rest in the rest frame of the circuit. Indeed, the velocity v_U of the center of energy is, on taking the time derivative of eq. (43),

$$\mathbf{v}_U = \frac{I^2RL}{U_{\rm total}}\, \hat{\mathbf{z}} \approx \frac{I^2RL}{M_{\rm total}c^2}\, \hat{\mathbf{z}}, \tag{13}$$

where $U_{\rm total}$ is the constant, total energy and $M_{\rm total}$ is the total mass of the system in the laboratory frame.[7] Hence, we can not immediately conclude whether the system in which the battery + resistor + cable are at rest in the frame in which the total center of energy is rest (which is the frame in which "hidden" momentum is defined).

The electromagnetic field is static in the frame in which the battery + resistor + cable are at rest, so the velocity of the center of electromagnetic energy is zero in this frame, $\mathbf{v}_{\rm EM} = 0$.

It remains to consider the "microscopic" velocity of the charges the participate in the electric current.

However, the charge carriers change their energy and velocity as they pass through the resistor and the battery. In the convention of this example, the inner conductor is at the higher potential, so the electric field inside the resistor points radially outward. The delicate argument is that the charges e gain energy $eV = eIR$ as they move outwards through the resistor, even though they undergo collisions every few Angstroms along the way. As as result, there is a difference in the relativistic masses of the carriers in the inner and outer conductors,[8]

$$m(\gamma_o - \gamma_i)c^2 = \Delta U = eIR. \tag{14}$$

Labeling the carrier number density per unit length in the inner and outer conductors as n_i and n_o, we have

$$I = en_iv_i = en_ov_o, \tag{15}$$

and the momentum of the carriers (and hence the momentum of the matter in the rest frame of battery + resistor + cable) is

$$\mathbf{P}_{\rm charges} = \mathbf{P}_{\rm matter} = (n_iLm\gamma_iv_i - n_oLm\gamma_ov_o)\, \hat{\mathbf{z}} = -\frac{LI\Delta U}{c^2e}\, \hat{\mathbf{z}} = -\frac{I^2RL}{c^2}\, \hat{\mathbf{z}} = -\mathbf{P}_{\rm EM}. \tag{16}$$

Furthermore, there is microscopic motion of the center of mass/energy of the matter, associated with the net mechanical momentum of the charges in the negative-z direction,

$$M_{\rm charges}\mathbf{v}_{\rm ce,charges} = (n_iLm\gamma_iv_i^\star - n_oLm\gamma_ov_o^\star)\, \hat{\mathbf{z}} = -\mathbf{P}_{\rm charges} = -\mathbf{P}_{\rm EM} = -M_{\rm matter}\mathbf{v}_U, \tag{17}$$

[7]See [26] for discussion of a system in which energy flows in sound waves, such that in its center of mass/energy frame the system has mechanical momentum of order $1/c^2$, which is "hidden" by the standards of the acoustical community, and which is due to the tiny overall velocity of the system in that frame, as in the example of sec. 2.1.

[8]A version of this argument was first given on p. 215 of [27].

which is equal and opposite to the macroscopic term $M_{\rm matter}\mathbf{v}_U$ found in eq. (13). Hence, the microscopic velocity of the center of mass/energy is at rest in the rest frame of the battery + resistor + cable,

$$M\mathbf{v}_{\rm ce} = M_{\rm charges}\mathbf{v}_{\rm ce,charges} + M_{\rm matter}\mathbf{v}_U + M_{\rm EM}\mathbf{v}_{\rm EM} = 0, \tag{18}$$

which shows that the rest frame of the battery + resistor + cable is the also the frame in which the center of total mass/energy is at rest.

The nonzero value of $\mathbf{v}_{\rm ce,charges}$ *is observable only at the microscopic level, in that when charges enter/leave the inner or outer conductor the microscopic center of mass/energy of the charges is instantaneously shifted by amount* d/nl *where* d *is the spacing between charges. The center of mass/energy of the charges executes a Zitterbewegung such that the microscopic velocity of the center of mass/energy is nonzero, eq. (17), but on average the center of mass/energy is at rest. Likewise, in a microscopic description the electrical current* I *(charge passing some point per unit time) is not constant if the unit time is very small. In a macroscopic view in which quantities are averaged over times longer than the time for a charge to move one gap length, the macroscopic current* I *is constant and the macroscopic velocity of the center of mass/energy of the charges is zero.*

Using the definition (5) for "hidden" mechanical momentum, we have that[9]

$$\mathbf{P}_{\rm hidden} = \mathbf{P}_{\rm charges} = \mathbf{P}_{\rm matter} = -\frac{I^2RL}{c^2}\,\hat{\mathbf{z}} = -\mathbf{P}_{\rm EM} = -M_{\rm matter}\mathbf{v}_U. \tag{19}$$

The nonzero electromagnetic momentum (12) can be thought of as the potential momentum *(see Appendix A.3),*

$$\mathbf{P}_{\rm EM} = \mathbf{P}_{\rm EM,potenital} = \int \frac{\rho\mathbf{A}^{(\rm C)}}{c}\,d{\rm Vol}, \tag{20}$$

of the charges in the system, which would be converted into ordinary mechanical momentum if the current (and the vector potential) were to drop to zero.

We evaluate eq. (19) by writing the vector potential as

$$\mathbf{A}^{(\rm C)}(a \le r \le b) = -\frac{2I}{c}\ln\frac{r}{a}\,\hat{\mathbf{z}}, \tag{21}$$

and noting that the linear charge density on the conductor at $r = b$ *is*

$$\lambda_b = -\frac{E_r}{2b} = -\frac{V}{2b\ln b/a}\,, \tag{22}$$

so that

$$\mathbf{P}_{\rm EM} = \int \frac{\rho\mathbf{A}^{(\rm C)}}{c}\,d{\rm Vol} = \frac{\lambda_b L\mathbf{A}^{(\rm C)}(b)}{c} = \frac{I^2RL}{c^2}\,\hat{\mathbf{z}} \approx M_{\rm total}\mathbf{v}_U, \tag{23}$$

in agreement with the previous estimate (12).

[9]The result agrees with example 8.3 of [15], is discussed in [28] using a different definition of "hidden" momentum.

For completeness, we can evaluate the third form of eq. (11) by defining the electric potential $V^{(\mathrm{C})}$ to be V at $r = a$ and zero at $r = b$, such that

$$\mathbf{P}_{\mathrm{EM}} = \int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\mathrm{Vol}, = \frac{VIL}{c^2}\,\hat{\mathbf{z}} = \frac{I^2RL}{c^2}\,\hat{\mathbf{z}}. \tag{24}$$

A discussion of the forces in this example is given in [28].

2.2 Charges at Rest inside a Superconducting Solenoid

If charges are held at rest inside a superconducting solenoid (also at rest) with a steady current distribution, then a surface charge density will be induced on the solenoid such that the electric scalar potential is constant over its (conducting) surface. Hence, according to the third form of eq. (11) the electromagnetic momentum of the system vanishes,

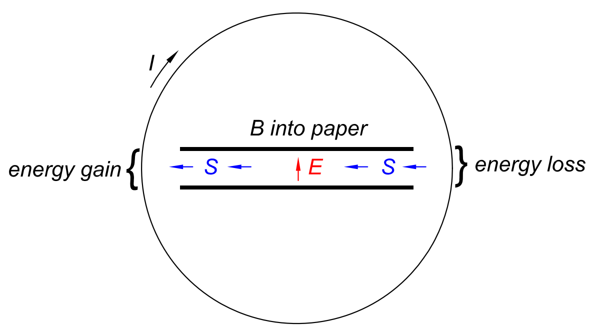
$$\mathbf{P}_{\mathrm{EM}} = \int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\mathrm{Vol} = \frac{V^{(\mathrm{C})}}{c^2}\int \mathbf{J}\, d\mathrm{Vol} = 0. \tag{25}$$

To use the first form of eq. (11), we note that the electric field must be perpendicular to the surface of a superconductor, so that no Poynting flux $\mathbf{S}$ *enters or leaves this surface. Then, lines of the steady, divergence-free field* $\mathbf{S}$ *form closed loops, so that* $\int \mathbf{S}\, d\mathrm{Vol} = 0$ *and again* $\mathbf{P}_{\mathrm{EM}} = 0$*. Confirmation of this result by the second form of eq. (11) is more subtle, and depends on details of the nonuniform surface charge density on the solenoid.*

The center of mass/energy is at rest in these examples, so the total momentum is zero, and hence by eq. (8) the "hidden" mechanical momentum is also zero.

2.3 Capacitor inside a Neutral Solenoid

Consider a parallel-plate capacitor inside a long solenoid magnet, as shown below.



A model of the magnet as an electrically neutral structure, in which the only matter in motion is the charge carriers, is that it consists of neutral, nonconducting tubes at rest that contain a circulating incompressible liquid of charged molecules, and that adjacent tubes have oppositely charged molecules whose flow has opposite senses of rotation.[10] Then, the

[10] Another model is that the magnet consists of a set of thin, nonconducting disks that rotate about the axis of the solenoid. Each disk has a uniform charge distribution about its rim, and adjacent disks are oppositely charge and rotate in opposite senses. However, in this model there is matter other than the steady currents which is in motion in the laboratory frame, which requires consideration of momentum related to the mechanical stresses in the rotating disks. So we do not consider this model further.

electric field **E** is only due to the charges on the capacitor, and is unaffected by the structure of the magnet.

The electric field **E** between the plates is vertical, while the magnetic field **B** points into the paper. The Poynting vector **S** points to the left between the plates, and is small above and below the plates. Because the fields are static, the density,

$$u_{\rm EM} = \frac{E^2 + B^2}{8\pi}, \tag{26}$$

of energy stored in the electromagnetic field is time independent. Hence, the nonzero Poynting vector **S** describes a flow of energy from a portion of the magnet on the right to a portion on the left.

In this example there is no battery to provide this energy, and no resistor to absorb it. For a steady-state system, it must be that energy delivered to the left side of the magnet is transported back to the right side by the electrical current. That is, the charge carriers have higher energy when on the left side of the magnet compared to that when they are on the right side. This implies that charged molecules have higher momentum when they flow from left to right than when they flow from right to left, and that the total mechanical momentum points to the right. Hence, this system contains "hidden" mechanical momentum of a steady current.

In the present example, the speed v of the charged molecules in the incompressible liquid is constant, so the mechanical momentum $\gamma m\mathbf{v}$ of the charged molecules has constant magnitude, IF the rest mass m is constant. So, we are led to conclude that the rest mass of the molecules varies as they circulate.[11]

As a molecule of charge e circulates it experiences a spatially varying electrical potential V due to the charges on the capacitor. We expect that the change in mechanical energy of the charge is opposite to the change in its electrical potential energy, $\Delta U_{\rm mech} = \Delta\gamma mc^2 = -e\Delta V$. Since the Lorentz factor γ cannot change for incompressible liquid flow, we must conclude that

$$\Delta m = -\frac{e\Delta V}{\gamma c^2}, \tag{27}$$

and the variable rest mass of the charged molecules can be written

$$m = m_0 - \frac{eV}{\gamma c^2}, \tag{28}$$

where m_0 is constant. That is, we are led to consider a kind of "molecular mass renormalization".[12,13]

[11]Such effects appear to have been first suggested in footnote 9 of [6].

[12]The relation (28) does not hold in general, but only for charged molecules that are somehow constrained to move at constant speed in a varying electrical scalar potential. These moving molecules are subject to internal stresses, which contribute to the momentum of the molecule in a manner summarized by eq. (28) but not explained in detail. See [9, 11] for a discussion of the stress-momentum of the charged incompressible fluid. If we insist that the notion of "hidden" momentum does not apply to examples in which "bulk matter" is in motion, then this concept would not apply to the present example of an incompressible fluid of charged molecules.

[13]Equation (28) is a reminder of lengthy efforts to interpret classical charge carriers as extended objects. See [32, 33] for reviews.

The effect of the variable rest mass (28) on the mechanical momentum of the electrical current is

$$\begin{aligned} \mathbf{P}_{\rm hidden} &= \mathbf{P}_{\rm mech,J} = \int \gamma \rho_{\rm mass} \mathbf{v}\, d{\rm Vol} = \int \gamma \frac{\rho}{e} m \mathbf{v}\, d{\rm Vol} = \int \gamma \frac{\rho}{e} \left(m_0 - \frac{eV}{\gamma c^2} \right) \mathbf{v}\, d{\rm Vol} \\ &= -\int \frac{\rho V \mathbf{v}}{c^2}\, d{\rm Vol} = -\int \frac{V \mathbf{J}}{c^2}\, d{\rm Vol} = -\mathbf{P}_{\rm EM}\,, \end{aligned} \tag{29}$$

recalling the third form of eq. (11).[14] Thus, the total momentum is zero for this system, as expected since its center of mass/energy is at rest.

2.4 A Bunch of Free Electrons

In particle accelerators one often considers bunches of electrons and protons that are accelerated or deflected by external electric and magnetic fields. The laboratory frame is almost never the center of mass/energy frame for such bunches, whose particles have speeds close to that of light. Nonetheless, we consider a bunch of free electrons with velocities small compared to c in their center of mass/energy frame. Is the concept of "hidden" momentum relevant to this idealized example?

Darwin [35] has given a systematic discussion of electrodynamics to order $1/c^2$ in a microscopic view, which is readily applied to a bunch of electrons of charge $-e$ and rest mass m. See also sec. 65 of [36], sec. 12.6 of [37], and [23, 38].

The (constant) total momentum of a bunch of free electrons, in any frame in which all speeds v_i are small compared to c, is given to order $1/c^2$ by

$$\mathbf{P}_{\rm total} \approx \sum_i m \left(1 + \frac{v_i^2}{2c^2} \right) \mathbf{v}_i + \sum_{i \neq j} e^2 \frac{\mathbf{v}_j + (\mathbf{v}_j \cdot \hat{\mathbf{r}}_{ij}) \hat{\mathbf{r}}_{ij}}{2c^2 r_{ij}} = \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM}, \tag{30}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$.

In the center of mass/energy frame of the bunch, $\mathbf{P}_{\rm total} = 0$ and so $\mathbf{P}_{\rm mech} = -\mathbf{P}_{\rm EM}$ in this frame. That is, the mechanical momentum of the bunch in the center of mass/energy frame is of order $1/c^2$. But, is this mechanical momentum "hidden"?

While this tiny mechanical momentum is undetectable in practice, opinions differ as to whether or not it is "hidden" in principle. By the narrow definition of "hidden" momentum considered thus far in this note, only steady current distributions have "hidden" momentum. The current density of the bunch can be written

$$\mathbf{J}(\mathbf{r}) = e \sum_i \mathbf{v}_i\, \delta^3(\mathbf{r} - \mathbf{r}_i), \tag{31}$$

which is certainly not steady. It is then not required that $\int \mathbf{J}\, d{\rm Vol} = e \sum_i \mathbf{v}_i$ be zero. That is, $\mathbf{P}_{\rm mech,nonrel} = \sum_i m \mathbf{v}_i$ can be nonzero at order $1/c^2$, which occurs because the forces on the

[14] It turns out that explicit evaluation of the electromagnetic momentum in the present example by any the forms of eq. (11) is more subtle than one might naïvely expect [29]. In particular, a substantial fraction of the electromagnetic momentum density $\mathbf{E} \times \mathbf{B}/4\pi c$ lies outside the capacitor.

moving charges differ from the Coulomb forces by terms of order $1/c^2$ due to their magnetic interactions, and due to effects of retardation of the Coulomb forces [39].

This suggests that if we wish to expand the definition of the term "hidden" mechanical momentum, a possibility is to designate this to mean any mechanical momentum at order $1/c^2$. In this broader view, the example of sec. 2.1 does include "hidden" mechanical momentum.

2.5 Towards a Broader Definition of "Hidden" Momentum

The author now (July 12, 2012) prefers the broader definition of "hidden momentum" given in [1], which does not restrict this concept to an aspect of mechanical momentum.

"Hidden" mechanical momentum, as narrowly defined to be nonzero mechanical momentum of a steady electrical current, is an effect at order $1/c^2$ that can exists in a few examples, most of which are highly idealized where the current is due to flow of a charged, incompressible fluid (sec. 2.3). In the examples with charged fluids, "hidden" momentum can be associated with a kind of "renormalization" (28) of the rest mass of (stressed) charged molecules that are in an external electric scalar potential.

In a larger context, "hidden" momentum could be taken to mean any and all momentum-related effects at order $1/c^2$ in systems that are nominally "nonrelativistic". In this sense, all electromagnetic momentum (11) is a kind of "hidden" momentum, as is also the relativistic correction $m(v^2/2c^2)\mathbf{v}$ to the mechanical momentum of a mass m. Matter that is in motion with velocity of order $1/c^2$ carries "hidden" mechanical momentum (secs. 2.1 and 2.4). Furthermore, moving matter that is under stress includes "hidden" contributions at order $1/c^2$ to its macroscopic "mechanical" momentum, which corrections are actually electromagnetic in a microscopic view (sec. 2.3).[15]

We also consider whether "elementary" particles such as electrons, neutrons and protons can carry "hidden" momentum. In the broader classical view, "hidden" mechanical momentum is always associated with motion, so this view could be applied to elementary particles only in models of them as composite objects with moving internal parts. While this is consistent with the quark model of nucleons, even in the quantum realm there is, at present, no viable model of an electron as a composite object. Hence, it may be best to continue to exclude examples involving electron spin (ferromagnetism) from discussion of "hidden" momentum.[16]

[15] A variant of a broader definition is that "hidden" mechanical momentum is that part of the mechanical momentum of a moving body which is associated with internal stresses. However, as discussed in [14, 19], the stress-related part of the momentum-density vector of a body with speed v can be of order v/c. As the general view is that "hidden" momentum is an effect of order v^2/c^2 we do not advocate the definition considered in this footnote.

[16] There is one example in the literature in which "hidden" momentum of neutrons may be a useful concept [40].

A Appendices

A.1 Momentum of a System at "Rest"

The mechanical behavior of a macroscopic system can be described with the aid of the (symmetric) stress-energy-momentum tensor $T^{\mu\nu}$ of the system. The total energy-momentum 4-vector of the system is given by

$$U^\mu = (U_{\rm total}, P^i_{\rm total}\, c) = \int T^{0\mu}\, d{\rm Vol}. \tag{32}$$

As first noted by Abraham [21], at the microscopic level the electromagnetic parts of $T^{\mu\nu}$ are

$$T^{00}_{\rm EM} = \frac{E^2 + B^2}{8\pi} = u_{\rm EM}\,, \tag{33}$$

$$T^{0i}_{\rm EM} = \frac{S^i}{c} = p^i_{\rm EM}\, c, \tag{34}$$

$$T^{ij}_{\rm EM} = \frac{E^i E^j + B^i B^j}{4\pi} - \delta^{ij}\frac{E^2 + B^2}{8\pi}\,, \tag{35}$$

in terms of the microscopic fields $\mathbf{E}$ and $\mathbf{B}$. In particular, the density of electromagnetic momentum stored in the electromagnetic field is

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\,. \tag{36}$$

The macroscopic stress tensor $T^{\mu\nu}$ also includes the "mechanical" stresses within the system, which are actually electromagnetic at the atomic level. The form (35) still holds in terms of the macroscopic fields $\mathbf{E}$ and $\mathbf{B}$ in media where $\epsilon = 1 = \mu$ such that strictive effects can be neglected. The macroscopic stresses T^{ij} are related the volume density $\mathbf{f}$ of force on the system according to

$$f^i = \frac{\partial T^{ij}}{\partial x^j}\,. \tag{37}$$

The stress tensor $T^{\mu\nu}$ obeys the conservation law

$$\frac{\partial T^{\mu\nu}}{\partial x_\mu} = 0, \tag{38}$$

with $x^\mu = (ct, \mathbf{x})$ and $x_\mu = (ct, -\mathbf{x})$. Once consequence of this is that the total momentum is constant for an isolated, spatially (closed) bounded system, *i.e.*,

$$\int \frac{\partial T^{\mu i}}{\partial x_\mu} = 0 = \frac{\partial}{\partial ct}\int T^{0i}\, d{\rm Vol} - \int \frac{\partial T^{ji}}{\partial x^j}\, d{\rm Vol} = \frac{dP^i_{\rm total}}{dt} - \int T^{ji}\, d{\rm Area}^j = \frac{dP^i_{\rm total}}{dt}\,. \tag{39}$$

A related result is that the total (relativistic) momentum $\mathbf{P}_{\rm total}$ of an isolated system is proportional to the velocity $\mathbf{v}_U = d\mathbf{x}_U/dt$ of the center of mass/energy of the system [6],

$$\mathbf{P}_{\rm total} = \frac{U_{\rm total}}{c^2}\mathbf{v}_U = \frac{U_{\rm total}}{c^2}\frac{d\mathbf{x}_U}{dt}\,, \tag{40}$$

where

$$
\begin{aligned}
U_{\text{total}} &= \int T^{00}\, d\text{Vol}, && (41)\\
P^i_{\text{total}} &= \frac{1}{c}\int T^{0i}\, d\text{Vol}, && (42)\\
\mathbf{x}_U &= \frac{1}{U_{\text{total}}}\int T^{00}\,\mathbf{x}\, d\text{Vol}. && (43)
\end{aligned}
$$

That is, the total momentum of a closed system is zero in that (inertial) frame in which the center of mass/energy is at rest.

A.2 Electromagnetic Momentum to Order $1/c^2$

See, for example, [23] for a discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence.

Since the magnetic field $\mathbf{B}$ is always of order $1/c$ (or higher), we can calculate the electromagnetic momentum (10) to order $1/c^2$ using approximations to the electric field at zeroth order, *i.e.*, $\mathbf{E} \approx -\boldsymbol{\nabla} V^{(\mathrm{C})}$ (the Coulomb electric field), and to the magnetic field at order $1/c$, *i.e.*, $\boldsymbol{\nabla}\times\mathbf{B} \approx 4\pi\mathbf{J}/c$ with the neglect of the displacement current. Then, as argued by Furry [7],

$$
\begin{aligned}
\mathbf{P}_{\text{EM}} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\text{Vol} \approx -\int \frac{\boldsymbol{\nabla} V^{(\mathrm{C})}\times\mathbf{B}}{4\pi c}\, d\text{Vol} = \int \frac{V^{(\mathrm{C})}\boldsymbol{\nabla}\times\mathbf{B}}{4\pi c}\, d\text{Vol} - \int \frac{\boldsymbol{\nabla}\times V^{(\mathrm{C})}\mathbf{B}}{4\pi c}\, d\text{Vol}\\
&= \int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\text{Vol} - \oint \frac{d\mathbf{Area}\times V^{(\mathrm{C})}\mathbf{B}}{4\pi c} = \int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\text{Vol}, \qquad (44)
\end{aligned}
$$

whenever the charges and currents are contained within a finite volume.

The following argument is due to Vladimir Hnizdo.

We can avoid use of the current density $\mathbf{J}$ and instead consider the vector potential $\mathbf{A}^{(\mathrm{C})}$ in the Coulomb gauge, which has zero divergence,

$$
\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{C})}, \qquad \boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})} = 0. \qquad (45)
$$

In addition to well-known vector calculus relations, it is useful to define a combined operation

$$
\boldsymbol{\nabla}\cdot\mathbf{ab} \equiv (\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{b} + (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} = (\boldsymbol{\nabla}\cdot b_x\mathbf{a})\,\hat{\mathbf{x}} + (\boldsymbol{\nabla}\cdot b_y\mathbf{a})\,\hat{\mathbf{y}} + (\boldsymbol{\nabla}\cdot b_z\mathbf{a})\,\hat{\mathbf{z}}. \qquad (46)
$$

Then,

$$
\begin{aligned}
\mathbf{E}\times\mathbf{B} &= \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{C})}) = \boldsymbol{\nabla}(\mathbf{A}^{(\mathrm{C})}\cdot\mathbf{E}) - (\mathbf{A}^{(\mathrm{C})}\cdot\boldsymbol{\nabla})\mathbf{E} - (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}^{(\mathrm{C})} - \mathbf{A}^{(\mathrm{C})}\times(\boldsymbol{\nabla}\times\mathbf{E})\\
&= (\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A}^{(\mathrm{C})} + \boldsymbol{\nabla}(\mathbf{A}^{(\mathrm{C})}\cdot\mathbf{E})\\
&\quad -[(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A}^{(\mathrm{C})} + (\mathbf{A}^{(\mathrm{C})}\cdot\boldsymbol{\nabla})\mathbf{E}] - [(\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})})\mathbf{E} + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}^{(\mathrm{C})}]\\
&= 4\pi\rho\mathbf{A}^{(\mathrm{C})} + \boldsymbol{\nabla}(\mathbf{A}^{(\mathrm{C})}\cdot\mathbf{E}) - \boldsymbol{\nabla}\cdot\mathbf{E}\mathbf{A}^{(\mathrm{C})} - \boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})}\mathbf{E}, \qquad (47)
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} \\
&= \int \frac{\rho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} + \oint (\mathbf{A}^{(\mathrm{C})}\cdot\mathbf{E})\, d\mathbf{Area} - \oint \mathbf{E}(\mathbf{A}^{(\mathrm{C})}\cdot d\mathbf{Area}) - \oint \mathbf{A}^{(\mathrm{C})}(\mathbf{E}\cdot d\mathbf{Area}) \\
&\approx \int \frac{\rho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol}. \qquad (48)
\end{aligned}
$$

The surface integrals in eq. (48) are negligible when the charges and currents that create the electric field $\mathbf{E}$ and the vector potential $\mathbf{A}^{(\mathrm{C})}$ lie within a finite volume that is small compared to the volume of integration, and when radiation can be neglected. As previously noted, the electromagnetic momentum can be calculated to order $1/c^2$ with the neglect of the displacement current, which implies neglect of radiation, and justifies the approximation in eq. (48).

A.3 Electromagnetic Potential Momentum of a Charge

Faraday [24] and Maxwell [25] noted that a charge e in a vector potential $\mathbf{A}$ has a kind of electromagnetic potential momentum

$$\mathbf{P}_{\rm EM,potential} = \frac{e\mathbf{A}}{c}\,, \qquad (49)$$

in addition to its mechanical momentum, if any.[17] That is, if the vector potential drops to zero from its initial value $\mathbf{A}$, the charge experiences a "kick",

$$\Delta\mathbf{P}_{\rm mech} = \int \mathbf{F}\, dt = \int e\mathbf{E}\, dt = -\frac{e}{c}\int \frac{\partial \mathbf{A}}{\partial t}\, dt = \frac{e\mathbf{A}}{c} = \mathbf{P}_{\rm EM,potential,initial}. \qquad (50)$$

Since the vector potential is of order $1/c$, the potential momentum (49) is of order $1/c^2$, *i.e.*, very small for "nonrelativistic" systems.

The electromagnetic potential momentum (49) is the electromagnetic part of the canonical momentum,

$$\mathbf{P}_{\rm canonical} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} + \frac{e\mathbf{A}}{c}\,, \qquad (51)$$

of a charge e that interacts with an electromagnetic field, as described by the Lagrangian

$$\mathcal{L} = -mc^2\sqrt{1-v^2/c^2} - e\phi + e\frac{\mathbf{v}}{c}\cdot\mathbf{A}. \qquad (52)$$

See, for example, sec. 16 of [36].

The electromagnetic potential momentum (49) can be nonzero for a charge at rest, if there are charges in motion elsewhere in the system that create a nonzero vector potential at the location of the test charge. The momentum (49), which was historically the first

[17]The momentum (49) is actually gauge invariant if one writes it as $e\mathbf{A}_{\rm rot}/c$, where $\mathbf{A}_{\rm rot}$ is the rotational part of the vector potential, *i.e.*, $\boldsymbol{\nabla}\cdot\mathbf{A}_{\rm rot} = 0$. See, for example, sec. 2.3 of [23].

contribution to momentum to be identified at order $1/c^2$, could therefore be described as "hidden". However, this is not commonly done.

The description "electromagnetic potential momentum" is also not common, although perhaps it should be [30, 31]. The electromagnetic potential momentum $e\mathbf{A}/c$ can be combined with the electrical potential energy,

$$U_{\text{EM,potential}} = eV, \tag{53}$$

where V is the electric scalar potential at the position of the charge, to form a potential-energy-momentum 4-vector,

$$U^{\mu}_{\text{EM,potential}} = (U_{\text{EM,potential}}, \mathbf{P}_{\text{EM,potential}}\, c) = e(V, \mathbf{A}) = eA^{\mu}. \tag{54}$$

where

$$A^{\mu} = (V, \mathbf{A}) \tag{55}$$

is the electromagnetic potential 4-vector.

However, in (quasi)static cases like those considered in this note, the quantity $\int \rho A \, d\text{Vol}/c$ is equal and opposite to the "hidden" mechanical momentum of the system, and if the vector potential goes to zero, so does the "hidden" mechanical momentum, while the "overt" mechanical momentum of the system remains unchanged. There is no conversion of a "potential" momentum into an "overt" mechanical momentum, so the designation "potential momentum" is not very apt.[18]

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