Radiation from the Open End of a Coaxial Cable

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1 Problem

A coaxial cable of inner radius a and outer radius b has the volume between the (perfect) conductors filled with a dielectric of relative permittivity ϵ and unit relative permeability, such that its transmission-line impedance is

$$Z = \frac{Z_0 \ln(b/a)}{2\pi\sqrt{\epsilon}}\,, \quad \text{where} \quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0 c} = 120\pi = 377\ \Omega \ \left(= \frac{4\pi}{c} \text{ in Gaussian units}\right), \tag{1}$$

and c is the speed of light in vacuum. Estimate the time-average power P (*i.e.*, the radiation) that escapes from the end of the cable using vector (or scalar) Huygens-Kirchhoff diffraction theory for the case that the cable is excited with a TEM wave with $\lambda \gg b$.

Compare the cases of a coaxial cable by itself, and one with its outer conductor attached to a ground plane such that the radiation is into a half space. You may assume that the medium outside the cable has unit relative permittivity and unit relative permeability.

2 Solution

The coaxial cable lies along the negative z-axis, with its open end in the plane $z = 0$. It supports a TEM wave with electric and magnetic fields of amplitude $E_+ = V_+/\rho\ln(b/a)$ and $H_+ = \sqrt{\epsilon}E_+/Z_0$ [where (ρ, ϕ, z) are cylindrical coordinates with the z-axis being the axis of the cable] that propagates in the $+z$-direction (from a source at the end of the cable at negative z) towards the open end of the cable. The time-average power that propagates inside the cable in the $+z$-direction follows from the Poynting vector $\mathbf{S}_+ = \mathbf{E}_+ \times \mathbf{H}_+$ as

$$\begin{aligned} P_+ &= \int \langle S_{+z}\rangle \, d\text{Area} = \int \frac{E_+ H_+}{2}\, d\text{Area} = \int_a^b \rho\, d\rho \int_0^{2\pi} d\phi\, \frac{\sqrt{\epsilon}\, V_+^2}{2 Z_0 \rho^2 \ln^2(b/a)} \\ &= \frac{\pi\sqrt{\epsilon}}{\ln(b/a)} \frac{V_+^2}{Z_0}\,. \end{aligned} \tag{2}$$

When this TEM wave reaches the end of the cable, some energy is transmitted into the space beyond the end, some is reflected into a TEM wave in the negative-z direction, and some into higher-order (TM) modes that are damped as they propagate in the negative-z direction.[1]

The "source" currents in the TEM wave that is incident on the end of the cable are longitudinal and azimuthally symmetric. As a consequence, all magnetic fields (and currents) in this problem are azimuthally symmetric.

[1]The role of higher-order modes in transmission lines was perhaps first well explored in [1]. These modes die out over a distance roughly $b - a$ from the end of the cable when the wavelength exceeds this quantity.

We solve this problem in various approximations, first for a cable by itself using vector diffraction theory,[2] following Stratton and Chu [3],[3] and then for a cable connected to a ground plane using scalar diffraction theory.

2.1 Cable by Itself

The surface S' used in the diffraction calculation for a cable by itself lies just outside the surface of the cable, including its open end, as well as the surface at "infinity." If the cable is attached to a ground plane the surface S' consists of that plane, including the end of the cable, and the hemisphere at "infinity." It is more convenient to calculate the magnetic field (whose time dependence is $e^{-i\omega t}$) according to eq. (15) of [3],

$$\mathbf{H}(\mathbf{x}) = \frac{1}{4\pi}\oint_{S'}\left\{i\omega\epsilon_0[\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')]\frac{e^{ikr}}{r} - [\hat{\mathbf{n}}' \times \mathbf{H}(\mathbf{x}')] \times \boldsymbol{\nabla}'\frac{e^{ikr}}{r} - [\hat{\mathbf{n}}' \cdot \mathbf{H}(\mathbf{x}')]\boldsymbol{\nabla}'\frac{e^{ikr}}{r}\right\} d\mathrm{Area}', \quad (3)$$

where $k = \omega/c = 2\pi/\lambda$, c is the speed of light in vacuum, $\hat{\mathbf{n}}'$ is the outward unit vector normal to the surface S' (so $\hat{\mathbf{n}}' = -\hat{\mathbf{z}}$ on the end surface), $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$ (so $\hat{\mathbf{n}}' = -\hat{\mathbf{r}}$ when $\mathbf{x}'$ is on the surface at "infinity"), and

$$\boldsymbol{\nabla}'\frac{e^{ikr}}{r} = -\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}\,\hat{\mathbf{r}}. \quad (4)$$

At large distances the radiation fields fall off as $1/r$ and are related by the 4th Maxwell equation according to

$$ik\,\hat{\mathbf{n}}' \times \mathbf{H} = i\omega\epsilon_0\mathbf{E}, \quad (5)$$

such that the contribution to the integral (3) from surface at "infinity" vanishes. The magnetic field just outside the surface of a good conductor is parallel to the surface,

$$\mathbf{H} = \hat{\mathbf{n}}' \times \mathbf{K}, \qquad \mathbf{K} = -\hat{\mathbf{n}}' \times \mathbf{H}, \quad (6)$$

where $\mathbf{K}$ is the surface current density and $\hat{\mathbf{n}}'$ is the inward normal to the surface, such that $\hat{\mathbf{n}}' \cdot \mathbf{H} = 0$ there. In addition, $\hat{\mathbf{n}}' \cdot \mathbf{H} = 0$ at the open end of the cable (and also on the (spherical) surface at "infinity"), because this field has only an azimuthal component. Hence, the third term in eq. (3) is zero. Since the electric field is normal to the surface of the coaxial cable, eq. (3) reduces to

$$\mathbf{H}(\mathbf{x}) = \frac{1}{4\pi}\oint_{\text{end of cable}} i\omega\epsilon_0[\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')]\frac{e^{ikr}}{r}\,d\mathrm{Area}' + \frac{1}{4\pi}\oint_{\text{cable}} \mathbf{K}(\mathbf{x}') \times \boldsymbol{\nabla}'\frac{e^{ikr}}{r}\,d\mathrm{Area}'. \quad (7)$$

2.1.1 A First Approximation

As a first approximation (Schelkunoff [4], 1936), we ignore the excitation of higher-order modes, and suppose that the reflected TEM wave has the same amplitude as the incident

[2] For a review, see the Appendix of [2].

[3] Vector diffraction theory for electromagnetic fields is equivalent to Schelkunoff's "equivalence method" [4, 5] (that is a curious mixture of scalar and vector diffraction theory for the "electric vector potential."

TEM wave.[4] This implies that the outer radius b of the cable is small compared to a wavelength λ (*i.e.*, $kb \ll 1$). Then, the magnetic field is negligible at the open end of the cable, and the electric field is twice that of the incident wave,

$$E_\rho(a < \rho < b, 0) \approx 2E_+ = \frac{2V_+}{\rho \ln(b/a)}, \qquad H_\phi(\rho < b, 0) \approx 0. \tag{8}$$

In this approximation it also seems reasonable that the currents on the outer surface of the outer conductor of the cable can be neglected, such that the 2nd term in the diffraction integral (7) vanishes, and the magnetic field outside the cable is given by

$$\mathbf{H}(\mathbf{x}) \approx -\frac{i\omega\epsilon_0}{4\pi} \oint_{\text{end of cable}} d\text{Area}'\, E_\rho(\mathbf{x}') \frac{e^{ikr}}{r}\, \hat{\boldsymbol{\phi}}'. \tag{9}$$

The azimuthal component of eq. (9) is

$$\begin{aligned}
H_\phi(\mathbf{x}) &\approx -\frac{i\omega\epsilon_0}{4\pi} \oint_{\text{end of cable}} d\text{Area}'\, E_\rho(\mathbf{x}') \frac{e^{ikr}}{r} \cos(\phi' - \phi) \\
&= -\frac{i\omega\epsilon_0}{4\pi} \oint_{\text{end of cable}} d\text{Area}'\, E_\rho(\mathbf{x}') \frac{e^{ikr}}{r} \cos\phi' \\
&\approx -\frac{i\omega\epsilon_0 V_+}{2\pi \ln(b/a)} \int_a^b \rho'\, d\rho' \int_0^{2\pi} d\phi'\, \frac{e^{ikr}}{\rho' r} \cos\phi'.
\end{aligned} \tag{10}$$

It suffices to consider an observer at $\mathbf{x} = (r_0, \theta, 0)$ in spherical coordinates $[(r_0 \sin\theta, 0, r_0 \cos\theta)$ in rectangular coordinates], where $r_0 \gg b$. A source point in the integrand of eq. (5) is at $\mathbf{x}' = (\rho', \phi', 0)$ in cylindrical coordinates $[(\rho' \cos\phi', \rho' \sin\phi', 0)$ in rectangular coordinates]. As usual in diffraction calculations, we approximate the distance r in the exponential factor as

$$r \approx r_0 - \hat{\mathbf{x}} \cdot \mathbf{x}' = r_0 - \rho' \sin\theta \cos\phi', \tag{11}$$

while in the denominator we approximate r as r_0. Then, since $kb \ll 1$,

$$e^{ikr} \approx e^{ikr_0}(1 - ik\rho' \sin\theta \cos\phi'), \tag{12}$$

and eq. (10) becomes

$$\begin{aligned}
H_\phi(r_0, \theta) &\approx -\frac{i\omega\epsilon_0 V_+}{2\pi \ln(b/a)} \frac{e^{ikr_0}}{r_0} \int_a^b \rho'\, d\rho' \int_0^{2\pi} d\phi' \frac{(1 - ik\rho' \sin\theta \cos\phi') \cos\phi'}{\rho'} \\
&= -\frac{\pi^2 \epsilon_0 c V_+ (b^2 - a^2)}{\lambda^2 \ln(b/a)} \frac{\sin\theta\, e^{ikr_0}}{r_0} = -\frac{V_+}{4Z_0 \ln(b/a)} \frac{4\pi A}{\lambda^2} \frac{\sin\theta\, e^{ikr_0}}{r_0},
\end{aligned} \tag{13}$$

where $A = \pi(b^2 - a^2)$ is the cross-sectional area between the conductors of the coaxial cable.[5] The electric field $\mathbf{E}$ in the far zone is equal to $Z_0\, \hat{\mathbf{x}} \times \mathbf{H}$, which has only a θ component,

$$E_\theta = Z_0 H_\phi, \tag{14}$$

[4]This assumption contrasts with that for radiation from the open end of a hollow waveguide, whose transverse dimensions are necessarily of order λ, where the reflected wave is ignored [6].

[5]The field (13) agrees with that in eq. (33) of [4] noting that Schelkunoff's V is $2V_+$.

and the time-average radiated power into 4π solid angle is

$$
\begin{aligned}
P_1 &= \frac{r_0^2}{2}\int Re(\mathbf{E}\times\mathbf{H}^\star)\,d\Omega \approx \frac{V_+^2}{32Z_0\ln^2(b/a)}\left(\frac{4\pi A}{\lambda^2}\right)^2\int \sin^2\theta\,d\Omega = \frac{\pi V_+^2}{12Z_0\ln^2(b/a)}\left(\frac{4\pi A}{\lambda^2}\right)^2 \\
&= \frac{1}{12\sqrt{\epsilon}\ln(b/a)}\left(\frac{4\pi A}{\lambda^2}\right)^2 P_+ = \frac{k^4(b^2-a^2)^2}{12\sqrt{\epsilon}\ln(b/a)}P_+, \qquad (15)
\end{aligned}
$$

recalling eq. (2) for the power P_+ incident on the open end. The fraction of the incident power that is radiated is very small for $A \ll \lambda^2$, as is usual for the operation of coaxial cables. Example: For a cable with $Z = 50\ \Omega$ and outer radius $b = 2$ mm, $P_1/P_+ \approx -90$ dB for $\lambda = 1$ m.

If the outer radius b were greater than $(\lambda/2\pi)[12\sqrt{\epsilon}\ln(b/a)]^{1/4}$ the radiated power according to eq. (15) would exceed the incident power. An open-ended coaxial cable can be a good radiator, if the outer radius b is of order of a wavelength, for which case the present approximation is poor.

The power radiated into the forward hemisphere is, in the present approximation, 1/2 that given by eq. (15). This suggests that if the cable were connected to a ground plane at $z = 0$ (with the end of the cable still open), the radiated power would be 1/2 that given by eq. (15). For this to be so, the currents on the conductors must be different from those assumed above, which casts doubt on the assumption that the currents are negligible on the outside of the outer conductor of the coaxial cable. Indeed, when coaxial cables are used an antennas, typically by extending the inner conductor some distance beyond the outer, the performance is greatly influenced by the currents on the outside of the outer conductor [7].[6]

Radiation from the Open End of a Stripline

As a special case, we consider a coaxial cable with limited radial extent h, for which $b = a + h = a(1+\delta)$ with $\delta = h/a \ll 1$. Then, $b^2 - a^2 \approx 2ah$, $\ln b/a \approx \delta = h/a$, and the radiated power is

$$P_1 \approx \frac{k^4a^3h}{3\sqrt{\epsilon}}P_+. \qquad (16)$$

A possible application of eq. (16) is to radiation at the open end of a two-conductor stripline (that supports TEM waves) of width w with gap $h \ll w$ between the two parallel conducting strips [8]. The width w corresponds to the circumference $2\pi a$ in eq. (16), so we estimate the power radiated at the end of a stripline as

$$P_{\rm stripline} \approx \frac{k^4w^3h}{24\pi^3\sqrt{\epsilon}}P_+, \qquad (17)$$

which approximation is for widths w small compared to a wavelength.

2.1.2 A Second Approximation

In a second approximation we still ignore the surface currents on the outside of the outer conductor of the cable, but we take into account that the reflected TEM wave inside the

[6]See also the comments at the end of sec. 2.2.2.

cable has amplitude R times that of the incident wave,

$$E_\rho(z<0) \approx \frac{V_+}{\rho \ln(b/a)} \left(e^{i(k_\epsilon z - \omega t)} + R\, e^{-i(k_\epsilon z + \omega t)} \right), \tag{18}$$

where $k_\epsilon = \omega\sqrt{\epsilon}/c$, and R is the (complex) amplitude reflection coefficient. The excitation of higher-order modes is also neglected in this approximation. The magnetic field inside the cable follows from Faraday's law as

$$H_\phi(z<0) \approx \frac{\sqrt{\epsilon} V_+}{\rho Z_0 \ln(b/a)} \left(e^{i(k_\epsilon z - \omega t)} - R\, e^{-i(k_\epsilon z + \omega t)} \right). \tag{19}$$

The electric and magnetic fields just inside the end of the cable are

$$E_\rho(z=0^-) \approx \frac{V_+(1+R)}{\rho \ln(b/a)}, \qquad H_\phi(z=0^-) \approx \frac{\sqrt{\epsilon} V_+(1-R)}{\rho Z_0 \ln(b/a)}. \tag{20}$$

By matching the magnetic field to that just outside the end of the cable according to eq. (10), using eq. (20) for the electric field, the reflection coefficient R can be determined [9]. Thus,

$$\begin{aligned} H_\phi(\rho, z=0^+) &\approx -\frac{i\omega\epsilon_0 V_+(1+R)}{4\pi \ln(b/a)} \int_a^b d\rho' \int_0^{2\pi} d\phi' \, \frac{e^{ikr}}{r} \cos\phi' \\ &= -\frac{i\omega\epsilon_0 V_+(1+R)}{2 \ln(b/a)} \int_a^b d\rho' \int_0^\infty \frac{k'\, dk'}{\sqrt{k'^2 - k^2}} J_1(k'\rho) J_1(k'\rho'), \end{aligned} \tag{21}$$

using an impressive result from Appendix A of [9]. Comparing with eq. (20), we have

$$\frac{1}{\rho}\frac{1-R}{1+R} \approx -\frac{ik}{2\sqrt{\epsilon}} \int_0^\infty \frac{k'\, dk'}{\sqrt{k'^2-k^2}} J_1(k'\rho) \int_a^b d\rho' \; J_1(k'\rho'), \tag{22}$$

Integrating this from a to b with respect to ρ yields (see Appendix C of [9])

$$\begin{aligned} \frac{1-R}{1+R} &\approx -\frac{ik}{2\sqrt{\epsilon}\ln(b/a)} \int_0^\infty \frac{k'\, dk'}{\sqrt{k'^2-k^2}} \left[\int_a^b J_1(k'\rho)\, d\rho \right]^2 \qquad (23) \\ &= -\frac{ik}{2\sqrt{\epsilon}\ln(b/a)} \left\{ \int_0^k + \int_k^\infty \right\} \frac{dk'}{k'\sqrt{k'^2-k^2}} \left[J_0(k'a) - J_0(k'b) \right]^2 \\ &= \frac{1}{2\sqrt{\epsilon}\ln(b/a)} \left\{ \int_0^{\pi/2} \frac{du}{\sin u} \left[J_0(ka\sin u) - J_0(kb\sin u) \right]^2 \qquad (\text{using } k' = k\sin u) \right. \\ &\quad \left. -\frac{i}{\pi} \int_0^\pi dv \, \left[2Si\left(k\sqrt{a^2+b^2-2ab\cos v}\right) - Si(2ka\sin v/2) - Si(2kb\sin v/2) \right] \right\}. \end{aligned}$$

For $kb \ll 1$ the integrals can be approximated, noting that for small x, $J_0(x) \approx 1 - x^2/4$ and $Si(x) \approx x$,

$$\frac{1-R}{1+R} \approx \frac{1}{2\sqrt{\epsilon}\ln(b/a)} \left\{ \frac{k^4(b^2-a^2)^2}{16} \int_0^{\pi/2} du \, \sin^3 u \right.$$

$$
-\frac{2ik}{\pi}\int_0^{\pi} dv\,\left[\sqrt{a^2+b^2-2ab\cos v}-(a+b)\sin v/2\right]\Bigg\}
$$

$$
= \frac{1}{2\sqrt{\epsilon}\ln(b/a)}\left\{\frac{2}{3}\left(\frac{\pi A}{\lambda^2}\right)^2-\frac{4ik(a+b)}{\pi}\left(\int_0^{\pi/2} dw\,\sqrt{1-\frac{4ab}{(a+b)^2}\sin^2 w}-1\right)\right\},
$$

$$
= \frac{1}{2\sqrt{\epsilon}\ln(b/a)}\left\{\frac{2}{3}\left(\frac{\pi A}{\lambda^2}\right)^2-\frac{4ik(a+b)}{\pi}\left[E\left(\frac{4ab}{(a+b)^2}\right)-1\right]\right\}, \tag{24}
$$

where again $A = \pi(b^2-a^2)$ is the area of the open end of the cable, and $E(x) = \int_0^{\pi/2} dw\,\sqrt{1-x\sin^2 w}$ is the complete elliptic integral of the second kind (Abramowitz and Stegun, 17.3.3. 17.3.35), which varies between $\pi/2$ and 1 as x increases from 0 to 1; $E(1-x)-1 \approx 0.71x$ for $x \ll 1$.

The voltage and current across the end of the transmission line are related to the line impedance Z, eq. (1), by

$$
V = V_+ + V_- = (1+R)V_+, \qquad \text{and} \qquad I = I_+ + I_- = \frac{(1-R)V_+}{Z} = \frac{2\pi\sqrt{\epsilon}(1-R)V_+}{Z_0\ln(b/a)}. \tag{25}
$$

The ratio of the current to the voltage at the end of the line is the admittance Y',[7]

$$
\begin{aligned}
Y' &= \frac{I(z=0)}{V(z=0)} = \frac{1}{Z}\frac{1-R}{1+R} = \frac{2\pi\sqrt{\epsilon}}{Z_0\ln(b/a)}\frac{1-R}{1+R} \equiv G' + iB' \\
&\approx \frac{\pi}{Z_0\ln^2(b/a)}\left\{\frac{1}{24}\left(\frac{4\pi A}{\lambda^2}\right)^2-\frac{4ik(a+b)}{\pi}\left[E\left(\frac{4ab}{(a+b)^2}\right)-1\right]\right\}.
\end{aligned} \tag{26}
$$

The power radiated from the open end of the coaxial cable is the time-average power delivered into this admittance,[8]

$$
\begin{aligned}
P_2 &= \frac{1}{2}Re(VI^\star) = \frac{1}{2}Re(Y')\,|V|^2 = \frac{1}{2}Re(Y')\,|1+R|^2\,V_+^2 = \frac{2Re(Y')V_+^2}{|1+ZY'|^2} \\
&\approx \frac{\pi V_+^2}{12Z_0\ln^2(b/a)}\left(\frac{4\pi A}{\lambda^2}\right)^2\frac{1}{|1+ZY'|^2} = \frac{P_1}{|1+ZY'|^2},
\end{aligned} \tag{27}
$$

in agreement with eq. (15) when $ZY' \ll 1$, *i.e.*, when the outer radius b of the cable is small compared to a wavelength.

2.2 Cable Connected to a Ground Plane

The case of an open-ended coaxial cable connected to a ground plane was considered in 1946 by Pistolkors [10] using scalar diffraction theory and a Dirichlet Green's function for a plane. Readers in the USA may be more familiar with a similar work by Levine and Papas

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[7] Numerical calculation [9] of $G' = Re(Y')$ using the integral of Bessel functions in eq. (23) shows that the approximation (26) is quite accurate even for $kb = 1$. This suggests that eq. (17) is still reasonably good for stripline width $w \approx \lambda$.

[8] While the radiated power depends on the square of the far-zone magnetic field for a given incident field strength E_+, it is linear in the (small) magnetic field at the end of the cable since the Poynting vector there is $|\mathbf{E}\times\mathbf{H}| \approx 2E_+H_\phi(z=0)$.

[9] (written in 1950) that gives two derivations, using a Neumann and a Dirichlet Green's function, respectively.[9]

Open-ended coaxial cables find application in bioscience as probes of the dielectric constant of the medium just beyond the open end. See, for example, [14], where some analytic analysis of the near fields is presented.

2.2.1 Review of Scalar Diffraction Theory

Recall that Green noted how Gauss' divergence theorem for the function $\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi$ of any two well-behaved scalar fields ϕ and ψ implies,[10]

$$\int_{V'} (\phi\nabla'^2\psi - \psi\nabla'^2\phi)\, d\mathrm{Vol}' = \oint_{S'} (\phi\boldsymbol{\nabla}'\psi - \psi\boldsymbol{\nabla}'\phi)\cdot\hat{\mathbf{n}}'\, d\mathrm{Area}', \tag{28}$$

where unit vector $\hat{\mathbf{n}}'$ is the outward normal from the closed surface S' bounds volume V'.

Helmholtz [15], and later Kirchhoff [16], solved Helmholtz' wave equation for a scalar function ψ in a source-free region bounded by surface S',

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0, \tag{29}$$

by use of a Green's function,

$$\phi = G(\mathbf{x},\mathbf{x}') = \frac{e^{ikr}}{r} \qquad \text{where} \qquad r = |\mathbf{r}| = |\mathbf{x}-\mathbf{x}'|\,, \tag{30}$$

that obeys

$$(\nabla^2 + k^2)G(\mathbf{x},\mathbf{x}') = (\nabla'^2 + k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\,\delta^3(\mathbf{x}-\mathbf{x}'), \tag{31}$$

such that eq. (28) leads to

$$\begin{aligned}\psi(\mathbf{x}) &= \frac{1}{4\pi}\int_{S'} d\mathrm{Area}'\,\{\psi(\mathbf{x}')[\hat{\mathbf{n}}'\cdot\boldsymbol{\nabla}'G(\mathbf{x},\mathbf{x}')] - G(\mathbf{x},\mathbf{x}')[\hat{\mathbf{n}}'\cdot\boldsymbol{\nabla}'\psi(\mathbf{x}')]\}\\ &= -\frac{1}{4\pi}\int_{S'} d\mathrm{Area}'\,\frac{e^{ikr}}{r}\left[\psi(\mathbf{x}')\left(ik - \frac{1}{r}\right)\hat{\mathbf{n}}'\cdot\hat{\mathbf{r}} + \frac{\partial\phi(\mathbf{x}')}{\partial n'}\right],\end{aligned} \tag{32}$$

where the normal derivative $\hat{\mathbf{n}}'\cdot\boldsymbol{\nabla}'\psi$ is often written as $\partial\phi/\partial n'$.

Poincaré [18] noted that both ψ and its normal derivative cannot be independently specified over the surface S', although rather good results are often obtained with eq. (32) despite this inconsistency [19]. Sommerfeld [20] and Rayleigh [21] subsequently deduced that when the surface S' consists of the plane $z = 0$ plus a hemisphere at "infinity," and the function ψ falls off at least as fast as $1/r$ at large r,[11] one can use

$$G_D(\mathbf{x},\mathbf{x}') = \frac{e^{ikr}}{r} - \frac{e^{ikr'}}{r'}\,, \qquad \text{or} \qquad G_N(\mathbf{x},\mathbf{x}') = \frac{e^{ikr}}{r} + \frac{e^{ikr'}}{r'}\,, \tag{33}$$

[9]Levine and Papas do not cite Pistolkors, although the title of their paper is identical to his. Their paper follows the spirit of Schwinger's great 1943 paper [11, 12, 13], and is perhaps therefore somewhat hard to follow.

[10]For a textbook discussion, see sec. 10.5 of [17].

[11]This requirement (that there is no contribution to the diffraction integral from the hemisphere at "infinity") is often called the "radiation condition," following Sommerfeld [22].

where $r' = |\mathbf{x} - \mathbf{x}''|$ and $\mathbf{x}''$ is the image of $\mathbf{x}'$ in the plane $z = 0$ (and thereby outside surface S'), such that the Dirichlet Green's function G_D, and the normal derivative of the Neumann Green's function G_N, vanish on the plane $z = 0$. These functions obey

$$(\nabla'^2 + k^2) G_{D,N}(\mathbf{x}, \mathbf{x}') = -4\pi\, \delta^3(\mathbf{x} - \mathbf{x}') \pm 4\pi\, \delta^3(\mathbf{x} - \mathbf{x}''), \tag{34}$$

so that inside surface S' eqs. (31)-(32) hold for $G = G_D$ or G_N. Then, one can specify either the function ψ or its normal derivative $\partial\psi/\partial n'$ on the plane $z = 0$ to obtain, from eq. (32),

$$\psi(\mathbf{x}) = \frac{1}{4\pi} \int_{z=0} d\text{Area}'\, \psi(\mathbf{x}') [\hat{\mathbf{n}}' \cdot \boldsymbol{\nabla}' G_D(\mathbf{x}, \mathbf{x}')] = -\frac{1}{2\pi} \int_{z=0} d\text{Area}'\, \psi(\mathbf{x}') \left(ik - \frac{1}{r} \right) \frac{e^{ikr}}{r}\, \hat{\mathbf{n}}' \cdot \hat{\mathbf{r}}, \tag{35}$$

and

$$\psi(\mathbf{x}) = -\frac{1}{4\pi} \int_{z=0} d\text{Area}'\, G_N(\mathbf{x}, \mathbf{x}') [\hat{\mathbf{n}}' \cdot \boldsymbol{\nabla}' \psi(\mathbf{x}')] = -\frac{1}{2\pi} \int_{z=0} d\text{Area}'\, \frac{e^{ikr}}{r} \frac{\partial \psi(\mathbf{x}')}{\partial n'}\,. \tag{36}$$

Comparing eqs. (35)-(36) with (32), we see that if ψ and $\partial\psi/\partial n'$ were both correctly specified on the plane $z = 0$, then use of eq. (32) gives ψ as the average of eqs. (35) and (36).

2.2.2 Application to Waveguides That End in an Infinite, Conducting Flange

Electromagnetic waves in circular, coaxial, or rectangular waveguides can be decomposed into modes with transverse electric (TE) and transverse magnetic (TM) fields each with a single independent scalar amplitude. We consider the case that the waveguide is excited in a single mode that propagates in the $+z$-direction for $z < 0$, and that the guide ends in an infinite, conducting flange in the plane $z = 0$. The reflected wave, in general, consists of many modes, while the fields in the half space beyond the end can be related to those just next to the plane of the flange via eqs. (35) or (36). A complete solution can be obtained by matching the two sets fields at the plane $z = 0$ [9].[12]

For examples such as the present case where the incident (TEM) wave has azimuthal symmetry we would like to take the scalar function ψ to be the azimuthal magnetic field H_ϕ, or perhaps the radial electric field E_ρ. However, only in Cartesian coordinates does $(\nabla^2 \mathbf{H})_j = \nabla^2 H_j$ [26], so scalar Kirchhoff diffraction theory applies only to Cartesian components of $\mathbf{E}$ and $\mathbf{H}$. Because of the azimuthal symmetry of the present example, we can obtain general expressions for E_ρ and H_ϕ by considering E_x and H_y for an observer at $\mathbf{x}$ with $x = 0$.[13]

Since $H_z = 0$, the 4th Maxwell equation tells us that,

$$\frac{\partial H_y}{\partial n'} = -\frac{\partial H_y}{\partial z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \epsilon_0 \frac{\partial E_x}{\partial t} = -i\omega\epsilon_0 E_x, \tag{37}$$

on the plane $z = 0$ where $\hat{\mathbf{n}}' = -\hat{\mathbf{z}}$ when the surface S' consists of this plane plus the hemisphere at "infinity" for $z > 0$. Using eq. (36), and noting that $\mathbf{E}_\perp(z = 0)$ is nonzero

[12]This method can also be applied to radiation by a hollow waveguide with an infinite conducting flange. Early studies of this are [23, 24]. For a recent review, see [25].

[13]Levine and Papas [9] solve the wave equation for H_ϕ (which differs from the scalar Helmholtz equation) using Green's functions that are suitable modifications of those in eq. (33).

only at the open end of the cable, we find for $\mathbf{x} = (\rho, \phi, z)$ inside surface S',

$$\begin{aligned} H_\phi(\rho, \phi, z) &= H_y(x = 0, y = \rho, z) = \int_{\text{end of cable}} d\text{Area}' \, \frac{\partial H_y}{\partial n'} \frac{e^{ikr}}{r} \\ &= -\frac{i\omega\epsilon_0}{2\pi} \int_{\text{end}} d\text{Area}' \, E_x(\mathbf{x}') \frac{e^{ikr}}{r} = -\frac{i\omega\epsilon_0}{2\pi} \int_{\text{end}} d\text{Area}' \, E_\rho(\mathbf{x}') \cos\phi' \, \frac{e^{ikr}}{r} \,, \qquad (38) \end{aligned}$$

since $E_x = E_\rho \cos\phi + E_\phi \sin\phi = E_\rho \cos\phi$.[14,15]

The "exact" result (38) for a coaxial cable connected to a ground plane is twice the approximate result for a cable by itself found in the second line of eq. (10). As a consequence, the analysis of sec. 2.1.2 can be taken over to the present case on multiplying the far-zone magnetic field (13), the admittance (26), and the radiated power (27) by a factor of 2. The latter result is surprising in that one might have supposed that the power radiated when the cable is connected to a conducting plane is 1/2 that radiated when the cable is by itself. This suggests that the power radiated from the end of a cable by itself is actually 4 times larger than that found in sec. 2.1, that the far-zone magnetic field is double, and the magnetic field just outside the cable end is quadruple than that found in sec. 2.1, at least when $b \ll \lambda$.

2.2.3 Effect of Excitation of Higher-Order Modes

All approximations made so far have included neglect of the excitation of higher-order (TM) modes in the coaxial cable by the reflection off its end. See [30] for numerical calculations that include these modes.

[14] Alternatively [10], we can integrate eq. (37) and use eq. (35) to obtain

$$\begin{aligned} H_\phi(\rho, \phi, z) &= H_y(x = 0, y = \rho, z) = i\omega\epsilon_0 \int E_x(x = 0, y = \rho, z') \, dz' \\ &= \frac{i\omega\epsilon_0}{2\pi} \int dz' \int_{\text{end of cable}} d\text{Area}' \, E_x(\mathbf{x}') \frac{\partial}{\partial n'} \frac{e^{ikr}}{r} = -\frac{i\omega\epsilon_0}{2\pi} \int_{\text{end}} d\text{Area}' \, E_x(\mathbf{x}') \int dz' \, \frac{\partial}{\partial z'} \frac{e^{ikr}}{r} \\ &= -\frac{i\omega\epsilon_0}{2\pi} \int_{\text{end}} d\text{Area}' \, E_x(\mathbf{x}') \frac{e^{ikr}}{r} = -\frac{i\omega\epsilon_0}{2\pi} \int_{\text{end}} d\text{Area}' \, E_\rho(\mathbf{x}') \cos\phi' \, \frac{e^{ikr}}{r} \,. \qquad (39) \end{aligned}$$

[15] The result (38) is sometimes expressed [27, 28] in the language of Schelkunoff's "equivalence method" [4] in which the plane $z = 0$ is taken to be a perfect conductor and the radial electric field in the aperture is called an "equivalent magnetic surface current" on the assumed perfect conductor. The factor of two that comes from use of the Green's function (33) when evaluated on the plane $z = 0$ is said to be the result of the "image" of the "magnetic surface current" in the conducting plane. I find this mnemonic procedure misleading as an explanation, since the image with respect to a plane of a current (whether electric or magnetic) parallel to that plane is opposite to the original current. Hence, one ignores all surface currents (and charges) when using the image method. The fact that one can ignore the electric currents on a planar boundary if one doubles the "magnetic currents" follows from scalar diffraction theory and not from image theory. See [29] for use of this method to compute the near fields of the open-ended cable.

3 Effective Area and Effective Height of an Open-Ended Coaxial Cable Used as a Receiver

The radiation pattern of the open-ended coaxial cable varies as $\sin^2\theta$, as for a Hertzian dipole antenna oriented along the z-axis (= axis of cable). The pattern has a null along the z-axis, and is maximal in the plane $z = 0$. Conversely, an open-ended coaxial cable receives no power from a plane wave that propagates towards it along the z-axis, and receives maximal (but very small) power from a plane wave that propagates perpendicular to the z-axis.

If such a latter wave results in an open-circuit voltage $V_{\rm oc}$ at the open end of the cable, then when the cable is terminated in a ("matched") load resistor $R = Z$, the current in that resistor will be $V_{\rm oc}/Z$, and the time-average power received will be $P_{\rm rec} = V_{\rm oc}^2/2Z$.

To determine the open circuit voltage $V_{\rm oc}$, we suppose that the incident wave is from a short linear dipole antenna located at $\mathbf{x} = (r_0, \theta, 0)$ that is driven by unit current. The dipole antenna is oriented perpendicular to the direction $\mathbf{x}$, with the conductors in the x-z plane. Then, according to an antenna reciprocity theorem, $V_{\rm oc} = V'_{\rm oc}$ where $V'_{\rm oc}$ is the open circuit voltage induced in the small dipole antenna when unit current flows in open-ended coaxial cable. See, for example, Appendix B of [32]. Now, for a short dipole antenna of half height h used as a receiver, the open-circuit voltage is simply h times the electric field incident on the antenna. See, for example, [33]. With unit current I_+ in the coaxial cable we have that $V_+ = Z$, and from eqs. (13)-(14) we then have

$$E = \frac{\pi Z A \sin\theta}{2\lambda^2 r_0 \ln(b/a)}\,, \qquad \text{and} \qquad V_{\rm oc} = V'_{\rm oc} = Eh = \frac{\pi Z A h \sin\theta}{2\lambda^2 r_0 \ln(b/a)}\,, \tag{40}$$

where $A = \pi(b^2 - a^2)$ is the cross-sectional area between the conductors of the cable. Thus, the time-average power received in the load resistor $R = Z = Z_0 \ln(b/a)/2\pi\sqrt{\epsilon}$ is

$$P_{\rm rec}(\theta) = \frac{V_{\rm oc}^2}{2Z} = \frac{\pi^2 Z A^2 h^2 \sin^2\theta}{8\lambda^4 r_0^2 \ln^2(b/a)} = \frac{\pi Z_0 A^2 h^2 \sin^2\theta}{16\sqrt{\epsilon}\lambda^4 r_0^2 \ln(b/a)}\,. \tag{41}$$

The time-average power per area (perpendicular to $\mathbf{x}$) incident from the small dipole antenna when it is driven by unit current is

$$\frac{dP_{\rm in}}{dA} = \frac{3}{8\pi r_0^2} P_{\rm dipole} = \frac{3}{8\pi r_0^2}\frac{R_{\rm rad,dipole}}{2} = \frac{3}{8\pi r_0^2}\frac{\pi}{3} Z_0 \left(\frac{h}{\lambda}\right)^2 = \frac{Z_0}{8 r_0^2}\left(\frac{h}{\lambda}\right)^2, \tag{42}$$

and hence the effective area of the open-ended coaxial cable when used as a receiver of radiation that propagates at angle θ to the z-axis is

$$A_{\rm eff}(\theta) = \frac{P(\theta)}{dP_{\rm in}/dA} = \frac{\sin^2\theta}{8\sqrt{\epsilon}\ln(b/a)}\frac{4\pi A}{\lambda^2} A \ll A \ll \frac{\lambda^2}{4\pi}\,. \tag{43}$$

We can also deduce an effective height $h_{\rm coax}$ of the coaxial cable as a receiving antenna according to

$$V_{\rm oc}(\theta) = E_{\rm in}(\theta) h_{\rm coax}(\theta), \tag{44}$$

where $V_{\rm oc}$ is given by eq. (40). The incident electric field $E_{\rm in}$ due to unit current in the small dipole antenna is related to the time-average incident power (42) by

$$\frac{dP_{\rm in}}{dA} = \langle S_{\rm in} \rangle = \frac{E_{\rm in} H_{\rm in}}{2} = \frac{E_{\rm in}^2}{2Z_0}\,, \tag{45}$$

so that

$$E_{\rm in}(\theta) = \frac{Z_0 \sin\theta}{2r_0}\,, \tag{46}$$

and

$$h_{\rm coax}(\theta) = \frac{\pi AZ \sin\theta}{\lambda Z_0 \ln(b/a)} = \frac{A \sin\theta}{2\sqrt{\epsilon}\lambda} \approx \frac{b^2 - a^2}{\lambda} = (b-a)\frac{b+a}{\lambda} \ll b - a. \tag{47}$$

The effective height (47) is much smaller than the gap $b - a$ between the inner and outer conductors of the cable, which is the characteristic length over which higher-order modes fall off from the end of the cable. Hence, it appears to this author that there is no simple relation between the effective height and the structure of those higher-order modes.

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