Analysis of TEM Waves in a Coaxial Cable via the Scalar Potential

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1 Problem

As an alternative to more typical approaches, start from the scalar potential to deduce the character of electromagnetic waves of angular frequency ω in a coaxial cable.

2 Solution

Analyses of waves on two-conductor transmission lines are often done in terms the currents in the conductors and the transverse voltage drop between the conductors. These analyses have the flavor of circuit analysis, and typically don't discuss the electromagnetic fields. Another common approach emphasizes the electromagnetic fields, from which the associated currents and transverse voltage drops can be deduced. Analyses based on the scalar potential V and/or the vector potential $\mathbf{A}$ are rare.[1] This situation seems to have left some people with the impression that the potentials are undefined for transmission lines, or that the conductors of the transmission line are equipotentials of V as would hold for the time-independent case.

We consider here a coaxial transmission line centered on the z-axis in a cylindrical coordinate system (r, ϕ, z). The conductors are taken to have zero resistance, and the gap between them has inner radius a and outer radius b. The gap is filled with a medium of (relative) permittivity ϵ and (relative) permeability μ. Consequently, the speed of propagation of waves in such a medium of infinite extent is $v = c/\sqrt{\epsilon\mu} = 1/\sqrt{\epsilon\mu\epsilon_0\mu_0}$, where c is the speed of light in vacuum.

We seek plane-wave solutions to Maxwell's equations where any relevant scalar field component has the form

$$f(r, \phi)\, e^{i(kz-\omega t)}. \tag{1}$$

As usual, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from the scalar potential V and the vector potential $\mathbf{A}$ according to (in SI units)

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{2}$$

where we work in the Lorenz gauge [2], such that

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{v^2}\frac{\partial V}{\partial t} \qquad \text{(Lorenz)}. \tag{3}$$

[1] The author is not aware of any such, although sec. 9.2 of [1] goes part way towards the present analysis.

The wave equation for the scalar potential in the gap of the cable, away from either sources or sinks, is

$$\nabla^2 V = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}\,. \tag{4}$$

Taking the scalar potential to have the form (1), the wave equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} - k^2 f = -\frac{\omega^2 f}{v^2}\,. \tag{5}$$

We restrict the analysis to the simplest mode, which has no azimuthal dependence, and for which the wave number is related to the angular frequency by

$$k = \frac{\omega}{v}\,. \tag{6}$$

Then, f is a function of r only, and we have

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) = 0, \qquad \frac{df}{dr} = \frac{K}{r}\,, \qquad f = K\ln\frac{r}{r_0}\,. \tag{7}$$

To determine the constants K and r_0 we suppose that the scalar potential vanishes at $r = b$, and that the peak value of the potential at $r = a$ is V_0. Then,

$$f = \begin{cases} 0 & (r > b), \\ V_0 \frac{\ln(r/b)}{\ln(a/b)} = -V_0 \frac{\ln(r/b)}{\ln(b/a)} & (a < r < b), \\ V_0 & (r < a), \end{cases} \tag{8}$$

and the scalar potential is

$$V = \begin{cases} 0 & (r > b), \\ -V_0 \frac{\ln(r/b)}{\ln(b/a)}\, e^{i(kz-\omega t)} & (a < r < b), \\ V_0\, e^{i(kz-\omega t)} & (r < a). \end{cases} \tag{9}$$

Because of the azimuthal symmetry, the current is only in the z-direction, and only nonzero component of the vector potential is $A_z = g(r)\, e^{i(kz-\omega t)}$. This can be obtained from the gauge condition (3), which tells us that

$$ikg = \frac{i\epsilon\mu\omega f}{c^2} = \frac{ik^2 f}{\omega}\,. \tag{10}$$

Thus, the vector potential is proportional to the scalar potential,

$$A_z = \frac{k}{\omega}V. \tag{11}$$

The electric and magnetic fields follow from (2) as

$$E_r = -\frac{\partial V}{\partial r} = \begin{cases} 0 & (r > b), \\ \frac{V_0}{r \ln(b/a)}\, e^{i(kz-\omega t)} & (a < r < b), \\ 0 & (r < a), \end{cases} \tag{12}$$

$$E_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = 0, \tag{13}$$

$$E_z = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t} = -ikV + i\omega A_z = 0, \tag{14}$$

and

$$B_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} = 0, \tag{15}$$

$$B_\phi = -\frac{\partial A_z}{\partial r} = -\frac{k}{\omega}\frac{\partial V}{\partial r} = \frac{k}{\omega} E_r = \frac{E_r}{v}\,, \tag{16}$$

$$B_z = 0. \tag{17}$$

Since both E_z and B_z vanish, this solution is called a transverse electromagnetic (TEM) wave.

It is noteworthy that the electric field is zero inside the inner conductor ($r < a$) even though both the scalar and vector potential are nonzero there.[2,3]

To make a connection with circuit analysis, we note that the potential difference between the inner and outer conductors is

$$V_a - V_b = V_0\, e^{i(kz-\omega t)}, \tag{18}$$

and the current I on the inner conductor follows from an application of the fourth Maxwell equation to an azimuthal loop that spans the inner conductor,

$$I(r = a) = \frac{2\pi a B_\phi(r = a)}{\mu\mu_0} = \frac{2\pi V(r = a)}{\ln(b/a)} \sqrt{\frac{\epsilon\epsilon_0}{\mu\mu_0}}\,. \tag{19}$$

As usual, we write $V = IZ$, where

$$Z = \frac{1}{2\pi}\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} \ln(b/a) = \frac{377}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln(b/a) \text{ Ohms} \tag{20}$$

is the characteristic impedance of the coaxial transmission line.

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[2]Since a scalar potential of the form $C\, e^{i(kz-\omega t)}$ results in zero electric and magnetic field, this term could be added to the potential (9) for any value of C. For example, use of $C = -V_0$ leads to the convention that the inner conductor is at zero scalar potential, and the outer conductor has potential $-V_0\, e^{i(kz-\omega t)}$. Similarly, use of $C = -V_0/2$ leads to the convention that the potentials on the outer and inner conductors are $\pm V_0\, e^{i(kz-\omega t)}/2$. *Thus, there is no intrinsic requirement that the outer conductor of a coaxial cable be "grounded."*

[3]Note also that all four fields V, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$ have the form of a static solution times the wavefunction $e^{i(kz-\omega t)}$, as holds for TEM waves on any two-conductor transmission line in the limit of perfect conductors.

Appendix A: Use of the Coulomb Gauge

Instead of working in the Lorenz gauge (3), we could work in the Coulomb gauge, for which the gauge condition is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \qquad \text{(Coulomb).} \tag{21}$$

In the Coulomb gauge the potentials are related to the "free" charge and current densities, ρ and $\mathbf{J}$, by the differential equations (see, for example, Appendix A of [3])

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon\epsilon_0}\,, \tag{22}$$

and

$$\boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mu_0 \mathbf{J} + \frac{1}{v^2}\frac{\partial \boldsymbol{\nabla} V}{\partial t}\,. \tag{23}$$

The charge and current densities have time dependence $e^{-i\omega t}$, and are periodic in z with wave number $k = \omega/v$. We again consider only solutions with azimuthal symmetry, so the scalar potential has the form

$$V(r, z, t) = f(r)\cos(kz)e^{-i\omega t}. \tag{24}$$

Using this form in the gap between the conductors, where $\rho = 0$, eq. (22) becomes

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) - k^2 f = 0. \tag{25}$$

This is a form of Bessel's equation, whose solutions are the modified Bessel functions $I_0(kr)$ and $K_0(kr)$. As before, we take the boundary conditions to be $V_{\rm max}(r = a) = V_0$ and $V(r = b) = 0$. Then, we have

$$f(r) = \alpha I_0(kr) + \beta K_0(kr), \tag{26}$$

where

$$V_0 = \alpha I_0(ka) + \beta K_0(ka), \qquad \text{and} \qquad 0 = \alpha I_0(ka) + \beta K_0(ka). \tag{27}$$

Solving this for α and β we find

$$f(r) = V_0 \frac{K_0(kb)I_0(kr) - I_0(ka)K_0(kr)}{I_0(ka)K_0(kb) - I_0(kb)K_0(ka)}\,, \tag{28}$$

and so the scalar potential has the complicated form

$$V(r, z, t) = \begin{cases} 0 & (r > b), \\ V_0 \frac{K_0(kb)I_0(kr) - I_0(ka)K_0(kr)}{I_0(ka)K_0(kb) - I_0(kb)K_0(ka)} \cos(kz)e^{-i\omega t} & (a < r < b), \\ V_0\, \cos(kz)e^{-i\omega t} & (r < a). \end{cases} \tag{29}$$

In principle, we can use this in eq. (23) to obtain the vector potential in the gap, and then we could calculate the fields from the potentials according to eq. (2). However, the details in the Coulomb gauge are much more complicated than in the Lorenz gauge, as is typically the case for examples involving wave propagation.

Appendix B: Use of the Gibbs Gauge

In the Gibbs gauge [4],[4] the scalar potential V is everywhere zero. Since the electric field is related by $\mathbf{E} = -\boldsymbol{\nabla}V - \partial\mathbf{A}/\partial t = -\partial\mathbf{A}/\partial t = i\omega\mathbf{A}$, the vector potential in this gauge can be taken as

$$\mathbf{A} = \frac{\mathbf{E}}{i\omega} = \begin{cases} 0 & (r > b), \\ -\frac{iV_0}{\omega r \ln(b/a)}\, e^{i(kz-\omega t)}\,\hat{\mathbf{r}} & (a < r < b) \qquad \text{(Gibbs)}, \\ 0 & (r < a), \end{cases} \tag{30}$$

recalling eq. (12). Then, the magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{\partial A_r}{\partial z}\,\hat{\boldsymbol{\phi}} = \frac{1}{i\omega}\frac{\partial E_r}{\partial z}\,\hat{\boldsymbol{\phi}} = \frac{k}{\omega}E_r\,\hat{\boldsymbol{\phi}} = \frac{E_r}{v}\,\hat{\boldsymbol{\phi}}, \tag{31}$$

as previously found in eq. (16).

Of course, in the Gibbs gauge one deduces the potentials from the fields, rather than the fields from the potentials. But, the simplicity of the Gibbs-gauge vector potential can make it convenient for use in Hamiltonian dynamics, in which gauge the Hamiltonian for a charge e of rest mass m is $H = \sqrt{m^2c^4 + (\mathbf{p} - e\mathbf{A})^2}$, where $\mathbf{p}$ is the canonical momentum of the charge.

References

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[3] K.T. McDonald, *The Helmholtz Decomposition and the Coulomb Gauge* (April 17, 2008), http://physics.princeton.edu/~mcdonald/examples/helmholtz.pdf

[4] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, Nature **53**, 509 (1896),
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[5] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), http://physics.princeton.edu/~mcdonald/examples/gauge.pdf

[4] For a survey of various gauges, see [5].