Green's Function for a Conducting Plane with a Hemispherical Boss

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(April 23, 2002; updated July 8, 2016)

1 Problem

What is the electric potential in rectangular coordinates (x, y, z) when a charge q is located at $(x_0, y_0, 0,)$ and there is a grounded conducting plane at $y = 0$ that has a (conducting) hemispherical boss of radius $a < b = \sqrt{x_0^2 + y_0^2}$ whose center is at the origin? What is the electrostatic force on the charge q for the case that $x_0 = 0$?

Consider also the case of a grounded conducting plane with a half-circular, conducting ridge of radius a.

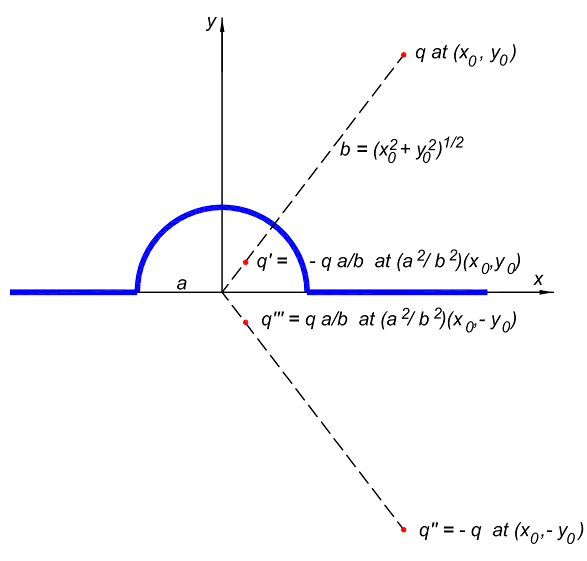
2 Solution

2.1 Hemispherical Boss

This example is posed as prob. 23, p. 284 of [2], prob. 13, p. 224 of [3], and as prob. 17 p. 232 of [4].

We use the image method [1].

First, we bring the hemispherical boss to zero potential by imagining that a charge $q' = -qa/b$ is placed at distance a^2/b along the line from the origin to charge q. The rectangular coordinates of charge q' are $(a^2/b^2)(x_0, y_0, 0)$. Next, to bring the plane $y = 0$ to zero potential, we add images charges for both q and q'. Namely, we imagine charge $q'' = -q$ at $(x_0, -y_0, 0)$, and charge $q''' = -q' = qa/b$ at $(a^2/b^2)(x_0, -y_0, 0)$. Then, both the plane $y = 0$ and the spherical shell of radius a about the origin are at zero potential.



The electric scalar potential V at an arbitrary point (x, y, z) outside the conductor is therefore

$$V = \frac{q}{r_1} - \frac{q}{r_2} - \frac{qa}{br_3} + \frac{qa}{br_4}\,, \tag{1}$$

where

$$r_{1,2} = \sqrt{(x - x_0)^2 + (y \mp y_0)^2 + z^2}, \qquad r_{3,4} = \sqrt{(x - a^2x_0/b^2)^2 + (y \mp a^2y_0/b^2)^2 + z^2}. \tag{2}$$

When $x_0 = 0$, then $y_0 = b$ and the force on charge q is in the $-y$ direction, with magnitude

$$F = \frac{q^2}{4b^2} + \frac{q^2a/b}{(b - a^2/b)^2} - \frac{q^2a/b}{(b + a^2/b)^2} = \frac{q^2}{4b^2} + \frac{4q^2a^3b^3}{(b^4 - a^4)^2}\,. \tag{3}$$

The electric field at the origin in the absence of the boss would be $E_0 = 2q/y_0^2 = 2q/b^2$. With the boss present, the electric potential along the y-axis is

$$V(0, y > a, 0) = \frac{q}{|b - y|} - \frac{q}{b + y} - \frac{qab}{|by - a^2|} + \frac{qab}{by + a^2}\,, \tag{4}$$

so the electric field at the pole of the boss, $(0, a, 0)$ has magnitude

$$|E_y(0, a, 0)| = \left| -\frac{dV(0, a, 0)}{dy} \right| = \frac{2q(2b^2 + a^2)}{(b^2 - a^2)^2} \approx \frac{4q}{b^2} = 2E_0, \tag{5}$$

where the approximation holds for $b \gg a$. The field at the pole of the boss is roughly twice that at the origin in its absence.

If the conducting plane with the hemispherical boss of radius a were part of a parallel-plate capacitor, with separation $b \gg a$ between the plates, the above results indicate that the peak electric field at the pole of the boss would be $\approx 2E_0$, where E_0 is the field inside the capacitor in the absence of the boss.[1]

2.2 Half-Cylindrical Ridge

We now consider the case of a conducting plane $y = 0$ with a conducting, half-cylindrical ridge of radius a and axis $(0, 0, z)$, together with a line charge q per unit length in the z-direction, located at (x_0, y_0, z). Again, we use an image method, now for 2-dimensional conductors.[2]

Here, the image of the line charge at distance $b = \sqrt{x_0^2 + y_0^2}$ from the z-axis is a line charge $q' = -q$ per unit length at distance a^2/b from that axis, with coordinates $(a^2/b^2)(x_0, y_0, z)$. The solution is completed by the image line charges $q'' = -q$ and $q''' = q$ at coordinates

[1]The potential difference between the capacitor plates is $V \approx E_0 b$. In contrast, an isolated conducting sphere of radius a at potential $V = E_0 b$ has electric field of strength $V/a = E_0 b/a \gg E_0$ at its surface.

Note that for large b, the potential takes the form $V = E_0(r - a^3/r^2)\cos\theta = E_0 y(1 - a^3/r^3)$, where angle θ is measured with respect to the y-axis, and $r = \sqrt{x^2 + y^2 + z^2}$.

Compare also to the case of a conducting sphere in an otherwise uniform external field $\mathbf{E}_0$, where the peak field at the surface of the sphere is $3\mathbf{E}_0$. See, for example, sec. 2.3 of [5].

[2]See, for example, prob. 11(a) of [6].

$(x_0, -y_0, z)$ and $(a^2/b^2)(x_0, -y_0, z)$, respectively. The electric scalar potential V at an arbitrary point (x, y, z) outside the conductor is therefore (to within a constant),

$$V = -2q(\ln r_1 - \ln r_2 - \ln r_3 + \ln r_4), \tag{6}$$

where

$$r_{1,2} = \sqrt{(x - x_0)^2 + (y \mp y_0)^2}, \qquad r_{3,4} = \sqrt{(x - a^2 x_0/b^2)^2 + (y \mp a^2 y_0/b^2)^2}. \tag{7}$$

When $x_0 = 0$, then $y_0 = b$ and the force per unit length on charge q (per unit length) is in the $-y$ direction, with magnitude

$$F = \frac{q^2}{b} + \frac{2q^2 b}{b^4 - a^4}\,. \tag{8}$$

The electric field strength at the origin in the absence of the boss would be $E_0 = 4q/y_0 = 4q/b$. With the boss present, the electric potential in the plane $x = 0$ is (to within a constant),

$$V(0, y > a, z) = -2q\left[\ln|b - y| - \ln|b + y| - \ln\left|by - a^2\right| + \ln\left|by + a^2\right|\right], \tag{9}$$

so the electric field long the peak of the ridge, $(0, a, z)$ has magnitude

$$|E_y(0, a, 0)| = \left|-\frac{dV(0, a, z)}{dy}\right| = \frac{8qb}{b^2 - a^2} \approx \frac{8q}{b} = 2E_0, \tag{10}$$

where the approximation holds for $b \gg a$. The peak field along the ridge is roughly twice that at the origin in its absence.

If the conducting plane with the half-cylindrical ridge of radius a were part of a parallel-plate capacitor, with separation $b \gg a$ between the plates, the above results indicate that the peak electric field at the pole of the boss would be $\approx 2E_0$, where E_0 is the field inside the capacitor in the absence of the boss.[3]

References

[1] W. Thomson, *Geometrical Investigations with Reference to the Distribution of Electricity on Spherical Conductors*, Camb. Dublin Math. J. **3**, 141 (1848), http://physics.princeton.edu/~mcdonald/examples/EM/thomson_cdmj_3_131_48.pdf

[2] J.H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge U. Press, 1908), http://physics.princeton.edu/~mcdonald/examples/EM/jeans_electricity.pdf

[3]The potential difference between the capacitor plates is $V \approx E_0 b$. In contrast, an isolated conducting cylinder of radius a at potential $V = E_0 b$ (with $V = 0$ at distance b from its axis) has charge $Q = E_0 b/(2 \ln b/a)$ per unit length, and electric field of strength $2Q/a = E_0 b/(a \ln b/a) \gg E_0$ at its surface.

Note that for large b, the potential takes the form $V = E_0(r - a^2/r)\cos\theta = E_0 y(1 - a^2/r^2)$, where angle θ is measured with respect to the y-axis, and $r = \sqrt{x^2 + y^2}$. See prob. 5, p. 229 of [4].

Compare also to the case of a conducting cylinder in an otherwise uniform external field $\mathbf{E}_0$, where the peak field at the surface of the sphere is $2\mathbf{E}_0$. See, for example, sec. 2.2 of [5].

[3] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968).

[4] E. Durand, *Electrostatique, Tome 2. Problèmes Généraux Conducteurs* (Masson, 1966).

[5] K.T. McDonald, *Charging of an Insulator in a Liquid Argon Detector* (May 26, 2016),
http://physics.princeton.edu/~mcdonald/examples/insulator.pdf

[6] K.T. McDonald, *Electrodynamics Problem Set 3* (1999),
http://physics.princeton.edu/~mcdonald/examples/ph501set3.pdf