The Rolling Motion of a Half-Full Beer Can

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1 Problem

Discuss the motion of a half-full (or half-empty) beer can as it rolls down an incline of angle α to the horizontal.

You may simplify your discussion by supposing that the can itself is a massless, thin cylindrical shell that rolls without slipping, while the "beer" is a solid half cylinder that slides without friction inside the can, with the axis of the can always horizontal (and perpendicular to the gradient of the incline). In effect, the half cylinder slides without friction down the incline.

2 Solution

Consider the line through the center of mass of the half cylinder that is parallel to the axis of the can. This line is also horizontal, and perpendicular to the gradient of the incline.

The general motion of the half cylinder can be described as a translation of this line, plus a rotation of the half cylinder about it, such that the half cylinder always maintains contact with the incline.[1]

The forces on the half cylinder are gravity and the normal force of the incline. The force parallel to the incline is just $mg \sin\alpha$, where g is the acceleration due to gravity, and hence the component parallel to the incline of the acceleration of the center of mass is $g \sin\alpha$. Defining s to be the distance the center of mass of the half cylinder has moved down the incline, we have that,

$$s = \frac{1}{2} g \sin\alpha t^2, \tag{1}$$

for motion that starts from rest at time $t = 0$.[2]

In general, the half cylinder rotates as its center of mass obeys eq. (1). We consider that case of no oscillations, of small oscillations, and finally motion in which the half cylinder "loops the loop" as the can rolls down the incline.

[1]Strictly, the half cylinder remains in contact with the can, which latter remains in contact with the incline. To keep the can rotating without slipping with respect to the incline, there must a friction between the can and the incline. However, in the limit of a massless can, the force of friction is zero, so we neglect friction in our analysis.

[2]In the case of no friction between the half cylinder and the can, the motion of the center of mass is the same as for a mass that slides down the incline with no friction. In contrast, it is well known that, for rolling without slipping, a uniform cylinder with moment of inertia $I = kma^2$ about its axis has acceleration $g \sin\alpha/(1 + k)$ down an incline. For a cylindrical shell, $k = 1$, while for a solid cylinder, $k = 1/2$.

Experiments reported in [1] for a can partially filled with water indicated that the acceleration is $\approx g \sin\alpha$ for a nearly full can, while less than $(g/2) \sin\alpha$ for a nearly empty can. The latter result suggests that if the water "sloshes" around inside the can, substantial energy is thereby dissipated, resulting in considerably reduced kinetic energy of rolling. Related studies include [2]-[5].

2.1 The "Trivial" Solution

The "trivial" solution is the case of no rotation (while the half cylinder slides down the incline with acceleration $g \sin \alpha$).

The (flat) surface of the half cylinder is not, however, horizontal but is tilted at some angle. To deduce this quickly, consider the frame that moves parallel to the incline with acceleration $g \sin \alpha$. In this accelerated frame there is an additional, apparent "gravitational" force $mg \sin \alpha$ on any mass m, which force points up the incline at angle α. With the aid of Fig. 1, we infer that $g_{\rm eff} = g \cos \alpha$, and this makes angle α to the vertical. That is, $\mathbf{g}_{\rm eff}$ is perpendicular to the incline.

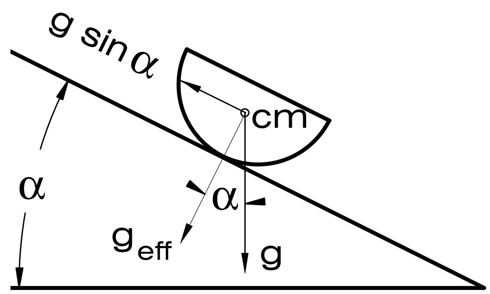


Figure 1: A half cylinder will slide down a frictionless incline of angle α without oscillation if its flat side is parallel to the incline. In the accelerated frame, the effective gravity vector, $\mathbf{g}_{\rm eff}$ is perpendicular to the incline.

The equilibrium surface of the half cylinder is perpendicular to $\mathbf{g}_{\rm eff}$, and hence parallel to the incline.

2.2 Small Oscillations in the Accelerated Frame

If the half cylinder oscillates about equilibrium, there is both rotation of the half cylinder and motion of the c.m. perpendicular to the incline, as viewed in the accelerated frame.

The solution to the present problem can be found using a little-known trick described by Tiersten [6]. If a rigid body has an instantaneous center of motion, then its instantaneous kinetic energy can be written as $I_C \omega^2/2$ where I_C is the moment of inertia about the instantaneous center. The rate of change of energy is $\tau_C \omega$ where τ_C is the torque about the instantaneous center. (If it's not obvious, this follows from $\mathbf{F} \cdot \mathbf{v} = \mathbf{F} \cdot \boldsymbol{\omega} \times \mathbf{r} = \mathbf{r} \times \mathbf{F} \cdot \boldsymbol{\omega}$ where $\mathbf{r}$ runs from the instantaneous center to the point of application of force $\mathbf{F}$.) Taking the time derivative of the instantaneous kinetic energy, we have,

$$\tau_C = \frac{1}{\omega} \frac{d(I_C \omega^2/2}{dt} = I_C \dot{\omega} + \frac{\dot{I}_C \omega}{2}. \tag{2}$$

as the equation of motion.

In the accelerated frame, the half cylinder has an instantaneous center of motion. It can be located by noting that the velocity of a point is at right angles to the line joining it to the center of motion. In particular, the center of mass moves only perpendicular to the incline. Hence, the instantaneous center is on a line parallel to the incline through the center of mass.

Also, the points on the half cylinder that are instantaneously in contact with the incline have instantaneous velocity parallel to the incline. Hence, the instantaneous center of rotation is on the line perpendicular to the incline through the point of contact. The instantaneous center of rotation is at the intersection of these two lines, shown as point C in Fig. 2.

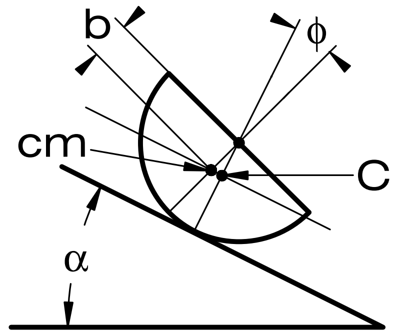


Figure 2: The center of mass of the half cylinder is distance b from the flat surface. When the half cylinder has tilted by angle ϕ from equilibrium in the accelerated frame, the center of rotation C is at the intersection of the normal to the incline through the point of contact with the line parallel to the flat surface that passes through the center of mass.

We define ϕ as the angle between the flat surface of the cylinder and the incline, which is also the angle between the perpendicular to the incline through the point of contact and the line joining the center of the cylinder to the c.m. Then, by the parallel axis theorem,

$$I_C = I_{\rm cm} + mb^2 \sin^2 \phi \approx I_{\rm cm} \tag{3}$$

for small ϕ. In the same approximation, $\dot{I}_C = 0$. The torque about the instantaneous center (calculated in the accelerated frame) is,

$$\tau_C = -mg_{\rm eff} b \sin \phi \approx mbg \cos \alpha \, \phi. \tag{4}$$

Since $\dot{\phi} = \omega$, the equation of motion becomes,

$$I_{\rm cm} \ddot{\phi} \approx -mbg \cos \alpha \, \phi, \tag{5}$$

so the frequency of small oscillations is related by,

$$\omega^2 = \frac{mbg \cos \alpha}{I_{\rm cm}}. \tag{6}$$

To calculate b and $I_{\rm cm}$, we use polar coordinates in which the half cylinder occupies the region $r < a$ and $-\pi/2 < \theta < \pi/2$. The cross-sectional area of the half cylinder is, of course, $\pi a^2/2$. Then,

$$b \cdot \text{Area} = \int x \cdot d\text{Area} = \int_0^a dr \int_{-\pi/2}^{\pi/2} r d\theta \, r \cos \theta = \frac{2a^3}{3}, \qquad \text{so} \qquad b = \frac{4a}{3\pi}\,. \tag{7}$$

Similarly,

$$I_{\rm axis} = \frac{m}{\rm Area}\int_0^a dr \int_{-\pi/2}^{\pi/2} r d\theta\, r^2 = \frac{ma^2}{2}\,, \tag{8}$$

as might have been "guessed". The moment of inertia $I_{\rm cm}$ of the half cylinder about its center of mass is then (by the parallel-axis theorem),

$$I_{\rm cm} = I_{\rm axis} - mb^2 = m(a^2/2 - b^2) \approx 0.32\, ma^2. \tag{9}$$

Returning to the oscillations of the half cylinder, the frequency is now expected to obey,

$$\omega^2 = \frac{2bg\cos\alpha}{a^2 - 2b^2} = \frac{24\pi g\cos\alpha}{9\pi^2 - 32} \approx \frac{1.33 g\cos\alpha}{a}\,. \tag{10}$$

Since the trick using the instantaneous center of motion is obscure, we Lagrange's method.

2.3 Solution via Lagrange's Method in the Lab Frame

We use Lagrange's method with generalized coordinates s and ϕ in the lab frame, where s is the distance parallel to the incline that the c.m. has moved, and ϕ is the angle between the flat surface of the half cylinder and the incline.

The perpendicular distance between the incline and the c.m. is then $h = a - b\cos\phi$. The kinetic energy is the sum of the kinetic energy of the motion of the c.m. plus the energy of rotation about the c.m.,

$$T = \frac{m}{2}(\dot{s}^2 + \dot{h}^2) + \frac{I_{\rm cm}}{2}\dot{\phi}^2 = \frac{m}{2}(\dot{s}^2 + b^2\sin^2\phi\,\dot{\phi}^2) + \frac{I_{\rm cm}}{2}\dot{\phi}^2. \tag{11}$$

The potential is just $V = mgz$ where z is the vertical coordinate of the c.m. Some care is required to relate z to s and ϕ. I found it useful to introduce another set of coordinates for this: u = distance that the axis of the cylinder has moved parallel to the incline, and θ = angle between the vertical and the line joining the axis of the half cylinder to its c.m.[3]

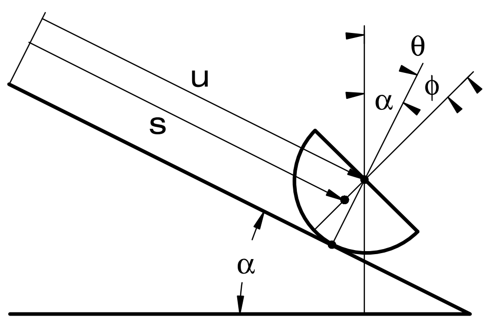


Figure 3: Illustrating the coordinates s, u, ϕ and θ.

We see from Fig. 3 that,

$$u = s + b\sin\phi, \qquad \text{and} \qquad \theta = \alpha + \phi. \tag{12}$$

[3]These coordinate were used in [1], where u was called x.

Then,

$$z = -u \sin\alpha - b\cos\theta = -s\sin\alpha - b\sin\alpha\sin\phi - b\cos(\alpha+\phi) = -s\sin\alpha - b\cos\alpha\cos\phi. \tag{13}$$

Finally, the potential energy is,

$$V = -mgz = -mg(s\sin\alpha + b\cos\alpha\cos\phi). \tag{14}$$

The Lagrange equation of motion deduced from coordinate s is,

$$\ddot{s} = g\sin\alpha, \tag{15}$$

as was expected. The equation from coordinate ϕ is,

$$(mb^2\sin^2\phi + I_{\rm cm})\,\ddot{\phi} + mb^2\sin\phi\cos\phi\,\dot{\phi}^2 = -mbg\cos\alpha\sin\phi. \tag{16}$$

For completeness, we note that in terms of coordinates $u = s + b\sin\phi$ and $\theta = \alpha + \phi$, for which $\phi = \theta - \alpha$ and $s = u - b\sin(\theta - \alpha)$, the Lagrangian is, recalling that $mb^2 + I_{\rm cm} = ma^2/2$,

$$\mathcal{L}(u,\phi) = \frac{m}{2}\left[\dot{u}^2 - 2b\cos\phi\,\dot{u}\,\dot{\phi} + \frac{a^2}{2}\,\dot{\phi}^2\right] + mg[u\sin\alpha + b\cos(\alpha+\phi)], \tag{17}$$

for which the equations of motion are,

$$\ddot{u} - b\cos\phi\,\ddot{\phi} + b\sin\phi\,\dot{\phi}^2 = g\sin\alpha, \tag{18}$$

$$\frac{a^2}{2}\ddot{\phi} - b\cos\phi\,\ddot{u} = -b\sin(\alpha+\phi). \tag{19}$$

And, in terms of coordinates u and $\theta = \alpha + \phi$, the Lagrangian is,

$$\mathcal{L}(u,\theta) = \frac{m}{2}\left[\dot{u}^2 - 2b\cos(\theta-\alpha)\dot{u}\,\dot{\theta} + \frac{a^2}{2}\,\dot{\theta}^2\right] + mg(u\sin\alpha + b\cos\theta), \tag{20}$$

for which the equations of motion are,

$$\ddot{u} - b\cos(\theta-\alpha)\,\ddot{\theta} + b\sin(\theta-\alpha)\,\dot{\theta}^2 = g\sin\alpha, \tag{21}$$

$$\frac{a^2}{2}\ddot{\theta} - b\cos(\theta-\alpha)\,\ddot{u} = -b\sin\theta. \tag{22}$$

2.3.1 Small Oscillations

For small oscillations about the equilibrium $\phi = 0$, say of the form $\phi = \phi_0\sin\omega t$, eq. (16) simplifies to,

$$I_{\rm cm}\ddot{\phi} = -mbg\cos\alpha\,\phi, \tag{23}$$

as found previously in eq. (5), sec. 2.2. Again, the frequency ω of small oscillations is related by,

$$\omega^2 = \frac{mbg\cos\alpha}{I_{\rm cm}} = \frac{2bg\cos\alpha}{a^2 - 2b^2}\,, \tag{24}$$

where $b = 4a/3\pi$.

The center of the can is at,

$$\begin{aligned} u &= s + b\cos\phi = \frac{g\sin\alpha t^2}{2} + b\cos(\phi_0\sin\omega t) \approx \frac{g\sin\alpha t^2}{2} + b\left(1 - \frac{\phi_0^2\sin^2\omega t}{2}\right) \\ &= \frac{g\sin\alpha t^2}{2} + b\left(1 - \frac{\phi_0^2}{4} + \frac{\phi_0^2\cos 2\omega t}{2}\right). \end{aligned} \tag{25}$$

This is like the case of sliding without friction down the incline, but with a small modulation at angular frequency 2ω, which would appear as a small, periodic "hesitation" in the position u.

If the half cylinder started from rest with its surface horizontal, then $\phi_0 = \alpha$ and the oscillations would not be small unless α were small.

2.3.2 Motion Where ϕ Becomes Large

Another type of possible motion is that where the half cylinder "loops the loop," and rolls down the incline essentially as a rigid body (inside the outer can). This motion is not possible starting from rest, but could occur if the half cylinder somehow had a large initial angular velocity $\dot{\phi}(t = 0) = \Omega$.

Here, we consider that case that angle ϕ varies with time as,

$$\phi = \Omega t + \epsilon\sin\omega t. \tag{26}$$

That is, the half cylinder does not quite rotate uniformly (at angular velocity Ω, but undergoes small oscillations about this.

Then, the equation of motion (16) averages to the form,

$$-\epsilon\omega^2(mb^2/2 + I_{\rm cm})\sin\omega' t \approx -mbg\cos\alpha\sin\Omega t, \tag{27}$$

which implies that,

$$\omega = \Omega, \qquad \text{and} \qquad \epsilon = \frac{mbg\cos\alpha}{\Omega^2(mb^2/2 + I_{\rm cm})}. \tag{28}$$

The angular velocity Ω is arbitrary, but must be large enough that ϵ is small for eq. (26) to hold.

The position u of the center of the can along the incline is,

$$u = s + b\sin\phi \approx g\sin\alpha\frac{t^2}{2} + b\sin(\Omega t + \epsilon\sin\Omega t) \approx g\sin\alpha\frac{t^2}{2} + b\sin\Omega t. \tag{29}$$

This motion would appear "hesitant" to the eye, but not "chaotic".

For cases in between the limits of small oscillation and the motion of eq. (26), the motion is complex, and the approximation of the half cylinder of beer as a rigid body would fail, resulting in "sloshing" of the beer that would have to be described by the (nonlinear) Navier-Stokes equation. Other aspects of a rolling can of liquid have been discussed in [1, 7].

3 Massive Can with Partial Filling

In this section we extend the analysis to include the case of a can of length l and radius a made from sheet metal of mass density $\rho_{\rm can}$, that is filled with "beer" of density $\rho_{\rm beer}$ to height $h < 2a$. We again suppose that the "beer" slides without friction inside the can while keeping the same shape as when the can is at rest.

The cylindrical wall of the can has mass $m_1 = 2\pi alt\rho_{\rm can}$, where $t \ll a$ is the thickness of the sheet metal, and moment of inertia $I_1 = m_1a^2$ about its axis. Each end of the can has mass $m_2 = \pi a^2t\rho_{\rm can}$ and moment of inertial $I_2 = m_2a^2/2$. The total mass of the can is,

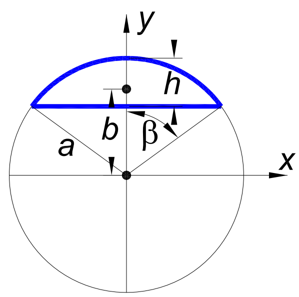
$$m_{\rm can} = m_1 + 2m_2 = 2\pi at(l + a)\rho_{\rm can}, \tag{30}$$

and its total moment of inertia (about its axis) is,

$$I_{\rm can} = I_1 + 2I_2 = (m_1 + m_2)a^2 = \pi a^3t(2l + a)\rho_{\rm can} = \frac{2l + a}{2(l + a)} m_{\rm can}a^2 \approx 0.89\, m_{\rm can}a^2, \tag{31}$$

for typical 12-oz. cans of radius $a \approx 3.3$ cm and length $l \approx 12$ cm.

The cross section of the "beer" is a circular segment that subtends angle 2β with respect to the axis of the can, as illustrated below.



The height h of the filling of the can is, in terms of angle β ($0 \le \beta \le \pi$),

$$h = a(1 - \cos\beta). \tag{32}$$

A lamina at coordinate y has area $dA = 2\sqrt{a^2 - y^2}\, dy$, so the area of the cross section, and the mass of the "beer", are, using Dwight 350.01,

$$A = 2\int_{a\cos\beta}^{a} dy\,\sqrt{a^2 - y^2} = a^2\left(\frac{\pi}{2} - \cos\beta\sin\beta - \sin^{-1}\cos\beta\right) = \frac{a^2}{2}\left(2\beta - \sin 2\beta\right), \tag{33}$$

$$m = Al\rho_{\rm beer} = \frac{a^2 l\rho_{\rm beer}}{2}\left(2\beta - \sin 2\beta\right). \tag{34}$$

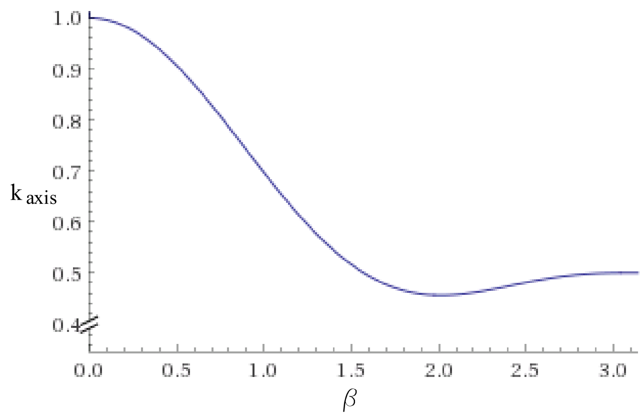
The center of mass of the beer is located at distance b from the axis related by,

$$b = \frac{1}{A}\int_{a\cos\beta}^{a} dy\, 2y\sqrt{a^2 - y^2} = \frac{2a^3\sin^3\beta}{3A} = \frac{4a\sin^3\beta}{3(2\beta - \sin 2\beta)}. \tag{35}$$

The moment of inertia of a lamina at coordinate y is $\rho_{\rm beer}l(2\sqrt{a^2 - y^2})^3\, dy/12$ about its symmetry axis, and so the moment about the axis of the can is, according to the parallel-axis theorem, $2\rho_{\rm beer}l\sqrt{a^2 - y^2})\, dy[y^2 + (a^2 - y^2)/3]$. Hence, the moment of inertia of the beer about the axis of the can is, using Dwight 352.01,

$$
\begin{aligned}
I_{\rm axis} &= \int_{a\cos\beta}^{a} dy\, \frac{2\rho_{\rm beer} l\sqrt{a^2-y^2}}{3}(a^2+2y^2) \\
&= \frac{2\rho_{\rm beer}a^4 l}{3}\left(\frac{\pi}{4} - \frac{\cos\beta\sin\beta}{2} - \frac{\sin^{-1}\cos\beta}{2} + \frac{\pi}{8} + \frac{\cos\beta\sin^3\beta}{2} - \frac{\cos\beta\sin\beta}{4} - \frac{\sin^{-1}\cos\beta}{4}\right) \\
&= \frac{2\rho_{\rm beer}a^4 l}{3}\left(\frac{3\beta}{4} - \frac{3\sin 2\beta}{8} + \frac{\sin 2\beta\sin^2\beta}{8}\right) = \frac{\rho_{\rm beer}a^4 l}{4}\left(2\beta - \sin 2\beta + \frac{2\sin 2\beta \sin^2\beta}{3}\right) \\
&= \frac{ma^2}{2}\left(1 + \frac{2\sin 2\beta\sin^2\beta}{3(2\beta - \sin 2\beta)}\right). \qquad (36)
\end{aligned}
$$

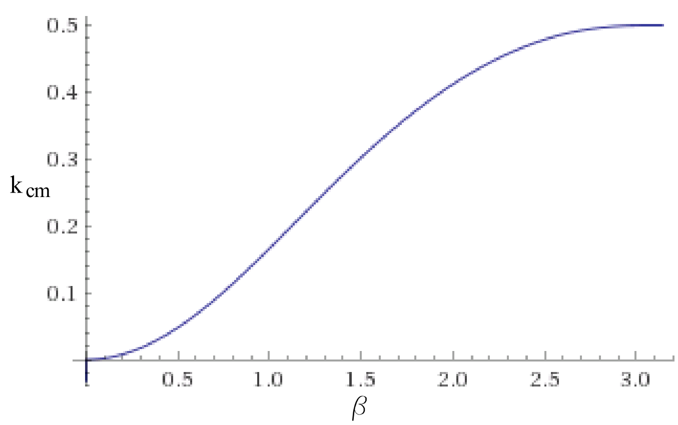
Writing $I_{\rm axis}$ as kma^2, as angle β increases from 0 to π, k decreases from 1 (for a point mass at distance a from the axis) to 1/2 at $\beta = \pi/2$ (as in eq. (8), then dips slightly and returns to $k = 1/2$ at $\beta = \pi$ (as for a solid cylinder), as shown in the figure below.



As in eq. (9), the moment of inertia of the beer about its own center of mass is given by,

$$
I_{\rm cm} = I_{\rm axis} - mb^2 = \frac{ma^2}{2}\left(1 + \frac{2\sin 2\beta\sin^2\beta}{3(2\beta - \sin 2\beta)} - \frac{32\sin^6\beta}{9(2\beta - \sin 2\beta)^2}\right), \qquad (37)
$$

using eq. (35). Writing $I_{\rm cm}$ as kma^2, as angle β increases from 0 to π, k increases from 0 (as for a point mass) to 0.32 at $\beta = \pi/2$ (as in eq. (9)), and then continues to increase to 0.5 at $\beta = \pi$ (as expected for a solid cylinder).



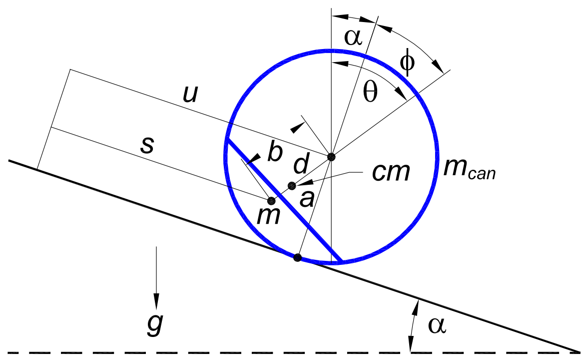
The center of mass of the entire system is at distance,

$$d = \frac{m}{m + m_{\text{can}}} b \tag{38}$$

from the axis of the can, on the radius from the axis of the can to the center of mass of the "beer".

3.1 Equations of Motion

The system has two degrees of freedom, and as before, we can use either s or $u = s + b \sin\phi$, and θ or $\phi = \theta + \alpha$ as the independent coordinates.



The condition for rolling without slipping of the can is that its angular velocity is $\dot{u}/a$. The kinetic energy of the can is,

$$T_{\text{can}} = \frac{1}{2}\left(m_{\text{can}} + \frac{I_{\text{can}}}{a^2}\right)\dot{u}^2 = \frac{1}{2}\left(m_{\text{can}} + \frac{I_{\text{can}}}{a^2}\right)\left(\dot{s}^2 + 2b\cos\phi\,\dot{s}\,\dot{\phi} + b^2\sin^2\phi\,\dot{\phi}^2\right), \tag{39}$$

and its potential energy is,

$$V_{\text{can}} = -m_{\text{can}} g u \sin\alpha = -m_{\text{can}} g (s + b\sin\phi)\sin\alpha. \tag{40}$$

The coordinates and velocities parallel and perpendicular to the incline of the center of mass of the "beer" (labeled by m in the figure above) are,

$$x = s = u - b\sin\phi = u - b\sin(\theta - \alpha), \qquad y = -b\cos\phi = -b\cos(\theta-\alpha), \tag{41}$$

$$\dot{x} = \dot{s}^2 = \dot{u} - b\cos(\theta - \alpha)\,\dot{\theta}, \qquad \dot{y} = b\sin\phi\,\dot{\phi} = b\sin(\theta-\alpha)\,\dot{\theta}. \tag{42}$$

$$v_{\text{beer}}^2 = \dot{x}^2 + \dot{y}^2 = \dot{u}^2 - 2b\cos(\theta-\alpha)\,\dot{u}\,\dot{\theta} + b^2\dot{\theta}^2 \quad = \quad \dot{s}^2 + b^2\sin^2\phi\,\dot{\phi}^2. \tag{43}$$

The kinetic energy of the "beer" (with angular velocity $\omega_{\text{beer}} = \dot{\phi} = \dot{\theta}$) is,

$$\begin{aligned} T_{\text{beer}} = \frac{1}{2} m v_{\text{beer}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{beer}}^2 \quad &= \quad \frac{1}{2} m\left(\dot{u}^2 - 2b\cos(\theta-\alpha)\,\dot{u}\,\dot{\theta} + b^2\dot{\theta}^2\right) + \frac{1}{2} I_{\text{cm}}\dot{\theta}^2 \\ &= \quad \frac{1}{2} m(\dot{s}^2 + b^2\sin^2\phi\,\dot{\phi}^2) + \frac{1}{2} I_{\text{cm}}\dot{\phi}^2, \end{aligned} \tag{44}$$

and its potential energy is (recalling eq. (13),

$$V_{\text{beer}} = -mg(u\sin\alpha + b\cos\theta) = -mg(s\sin\alpha + b\cos\alpha\cos\phi). \tag{45}$$

The equations of motion for coordinates u and θ are therefore,

$$\left(m + m_{\rm can} + \frac{I_{\rm can}}{a^2}\right) \ddot{u} - mb[\cos(\theta - \alpha)\, \ddot{\theta} - \sin(\theta - \alpha)\, \dot{\theta}^2] \;=\; (m + m_{\rm can}) g \sin\alpha, \qquad (46)$$

$$ma^2\, \ddot{\theta} - mb\cos(\theta - \alpha)\, \ddot{u} \;=\; -mgb \sin\theta, \qquad (47)$$

recalling that $mb^2 + I_{\rm cm} = ma^2$, and the equations of motion for coordinates s and ϕ are,

$$\left(m + m_{\rm can} + \frac{I_{\rm can}}{a^2}\right) \ddot{s} + b\left(m_{\rm can} + \frac{I_{\rm can}}{a^2}\right)\left(\cos\phi\, \ddot{\phi} - \sin\phi\, \dot{\phi}^2\right)$$
$$= (m + m_{\rm can}) g \sin\alpha, \qquad (48)$$
$$(I_{\rm cm} + mb^2 \sin^2\phi)\, \ddot{\phi} + mb^2 \sin\phi \cos\phi\, \dot{\phi}^2 + \left(m_{\rm can} + \frac{I_{\rm can}}{a^2}\right)(b^2 \sin^2\phi\, \ddot{\phi} + b\cos\phi\, \ddot{s})$$
$$= -mgb \cos\alpha \sin\phi. \qquad (49)$$

These equations are complicated, but when $m_{\rm can}$ and $I_{\rm can}$ are small compared to m and $I_{\rm cm}$ of the "beer", they simplify to those found in sec. 2.3 above (keeping the equations there always in terms of b and $I_{\rm cm}$). That is, in this limit there exist small oscillations of the "beer" about an equilibrium, while the center of the can accelerates approximately as $g \sin\alpha$ down the incline, with small "hesitations" once a cycle.[4]

The case of a nearly empty can, in the approximation of no fluid friction, should be nearly the same as that of an empty can, with acceleration $(g/2)\sin\alpha$ down the incline. Such behavior was not reported in [1], where even smaller accelerations were observed, indicating that friction/viscosity is not negligible.

A Appendix: Motion for Beer Frozen to the Can

In this appendix we suppose that the "beer" is frozen in a half cylinder that is rigidly attached to the can (which still rolls without slipping).[5]

Referring to Fig. 3, the condition of rolling without slipping is that,

$$u = a\phi, \qquad \text{and so} \qquad s = a\phi - b\sin\phi, \qquad (50)$$

in the convention that $s = u = 0$ when $\phi = 0$. As the rolling constraint (50) is simpler in terms of coordinate u than s, we use the former.

Taking ϕ to be the single independent coordinate, the Lagrangian (17) becomes,

$$\mathcal{L}(\phi) = \frac{m}{2}\left(\frac{3a^2}{2} - 2ab\cos\phi\right) \dot{\phi}^2 + mg[a\phi \sin\alpha + b\cos(\alpha + \phi)], \qquad (51)$$

and the equation of motion for ϕ is,

$$\left(\frac{3a^2}{2} - 2ab\cos\phi\right) \ddot{\phi} + ab\sin\phi\, \dot{\phi}^2 = g[a\sin\alpha - b\sin(\alpha + \phi)]. \qquad (52)$$

[4]Ideally, the amplitude of the oscillations of the "beer" with respect to the can could be zero, in which case the acceleration of the center of the can would be constant, with value $g \sin\alpha\, (m + m_{\rm can})/(m + m_{\rm can} + I_{\rm can}/a^2) \approx g \sin\alpha\, (m + m_{\rm can})/(m + 2m_{\rm can})$, which lies between $g \sin\alpha$ and $(1/2) g \sin\alpha$.

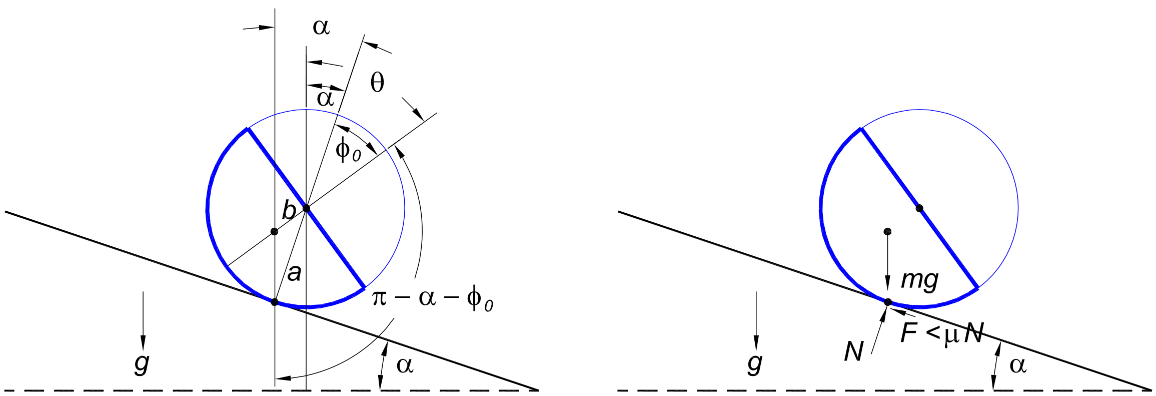
[5]A variant of this case has been discussed in [8], where the "beer" is instead considered to be a solid cylinder of radius $r < a$ that is somehow rigidly attached to the can.

A.1 Static Equilibrium

A stable equilibrium (with $\dot{\phi} = 0 = \ddot{\phi}$) exists at angle ϕ_0 related by,

$$\sin(\alpha + \phi_0) = \frac{a}{b}\sin\alpha = \frac{3\pi}{4}\sin\alpha, \tag{53}$$

using eq. (52), and recalling eq. (7). Here, the center of mass of the half cylinder is directly above the point of contact with the incline,[6] as illustrated in the left figure below, in which the law of sines tells us that $b/\sin\alpha = a/|\sin(\pi - \alpha - \phi_0)| = a/\sin(\alpha + \phi_0)$.



This equilibrium exists only for $\sin\alpha \le 4/3\pi$, *i.e.*, for $\alpha \le 25°$, and also only if the coefficient μ of static friction is larger than $\tan\alpha$.[7]

A.2 Small Oscillations about Equilibrium

To identify the angular frequency ω of small oscillations about the equilibrium at angle ϕ_0. we consider motion of the form,

$$\phi = \phi_0 + \epsilon\cos\omega t, \qquad \cos\phi \approx \cos\phi_0 - \epsilon\cos\omega t\sin\phi_0,$$
$$\sin(\alpha + \phi) \approx \sin(\alpha + \phi_0) - \epsilon\cos\omega t\cos(\alpha + \phi_0), \tag{54}$$

for small ϵ. Using this in eq. (52), the terms in $\epsilon\cos\omega t$ tell us that,

$$\omega^2 = \frac{gb\cos(\alpha + \phi_0)}{3a^2/2 - ab\cos\phi_0}. \tag{55}$$

If the "frozen beer can" is launched with $\phi(0) = 0 = \dot{\phi}(0)$, then it does not roll, but executes oscillations with frequency approximately given by eq. (55).[8]

---

[6]This contrasts with the case considered in sec. 2 where the half cylinder slides without friction inside the can, and a stable equilibrium exists only for $\alpha = 0$.

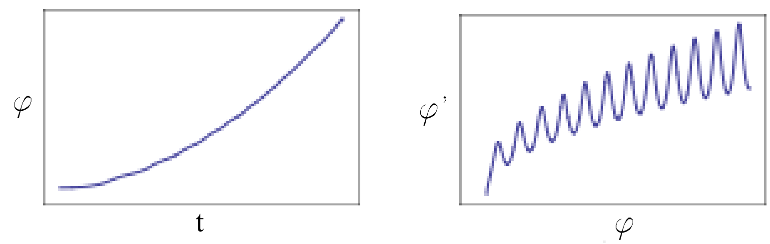
[7]This follows from the right figure above, where the horizontal components of the forces obey $N\sin\alpha = F\cos\alpha \le \mu N\cos\alpha$.

[8]This can be verified using Wolfram Alpha with input $(1.5(3)^2 - 2(3)(1)\cos(x))x'' + 3(0.5)\sin(x)(x')^2 = 980(3\sin(\mathrm{pi}/18) - 1\sin(\mathrm{pi}/18 + x)),\ x(0) = 0,\ x'(0) = 0$, which corresponds to $a = 3$ cm, $b = 1$ cm, $\alpha = 10°$, $\phi(0) = 0$ and $\dot{\phi}(0) = 0$.

A.3 Rolling Motion with Large ϕ

A solid cylinder that rolls without slipping would have acceleration $(2/3)g \sin\alpha$ down the incline, corresponding to $\phi = g \sin\alpha\, t^2/3a$.[9]

A numerical integration of eq. (52) for the "frozen beer can" using Wolfram Alpha is show below for $a = 3$ cm, $b = 1$ cm, $\alpha = 10°$, $\phi(0) = \pi$ and $\dot{\phi}(0) = 0$.



Angle ϕ varies almost quadratically in time, with a small oscillation about this, as found in sec. A.2.

A.4 Hopping

An interesting phenomenon not reported for cans rolling with liquid inside, but which exists for a can/hoop with an asymmetric mass distribution, is "hopping", in which the can leaves the surface briefly.

The literature thereon appears somewhat inconsistent [9]-[20], mainly due to varying assumptions as to the character of the system. Hopping is easier to obtain if the can/hoop is somewhat elastic (as for a loaded hula hoop), and only occurs when the can/hoop is rolling/slipping and the mass of the hoop (without load) is nonzero. A rigid can that is somehow constrained to roll without slipping cannot hop.

This problem seems to have been posed on various British exams in the early 20$^{\text{th}}$ century, as delightfully recalled by Littlewood [9].

> **A weight is attached to a point of a rough weightless hoop, which then rolls in a vertical plane, starting near the position of unstable equilibrium. What happens, and is it intuitive ?**
>
> **The hoop lifts off the ground when the radius vector to the weight becomes horizontal. I don't find the lift directly intuitive ; one can, however, 'see' that the motion is equivalent to the weight's sliding smoothly under gravity on the cycloid it describes, and it is intuitive that it will sooner or later leave *that*. (But the 'seeing' involves the observation that W is instantaneously rotating about P (Fig. 5).)**
>
> **Mr. H. A. Webb sets the question annually to his engineering pupils, but I don't find it in books.**
>
> **In actual practice the hoop skids first.**

[9]This corresponds to eq. (52) with $b = 0$.

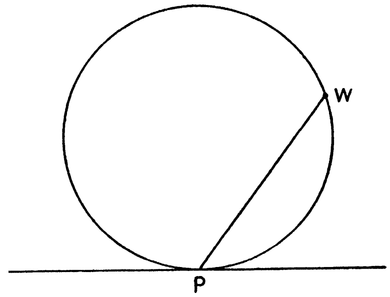


Fig. 5

The author likes the solution given by Pritchett [12], which included the figures below.[10]

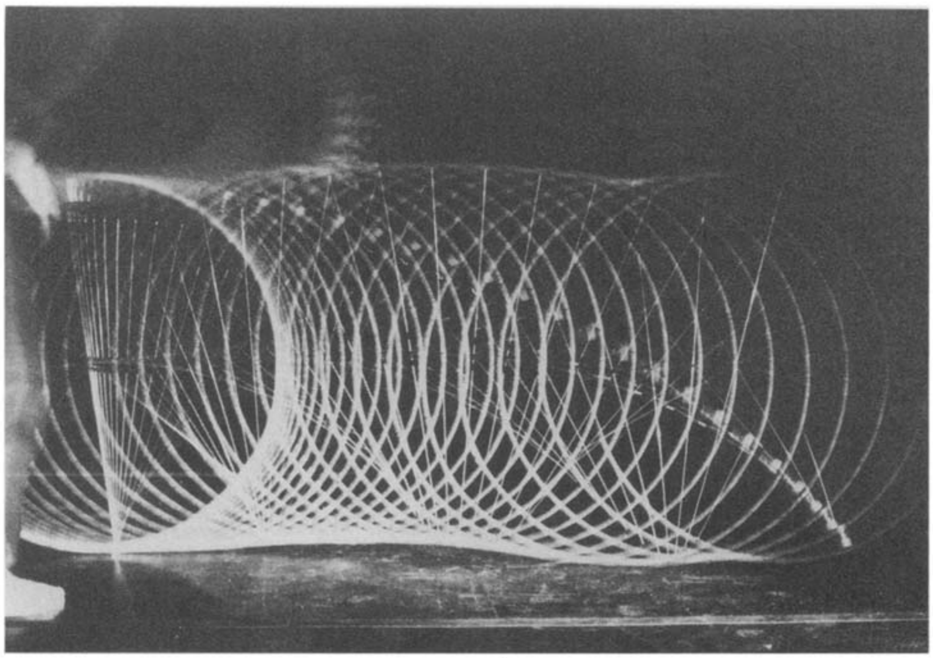


Figure 2. A stroboscopic photo by Dan Schwalbe and Stan Wagon shows a small hop of the hoop, which is a plastic hula hoop and four brass rods.

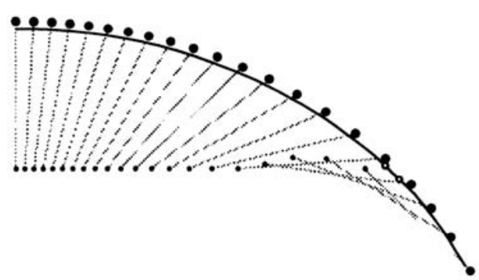
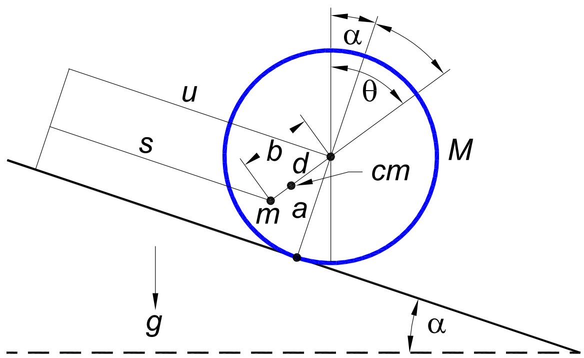


Figure 3. Simulated hop of a real hoop: results of a numerical simulation using $\lambda = 0.95$, $\epsilon = 0.01$, and $c = 0.1$ and coefficients of friction $\mu_s = 1.0$ and $\mu_k = 0.8$. The large dots indicate the positions of the attached object and the small dots indicate the corresponding positions of the center of the hoop. The trajectory of the center of mass of the system is indicated by the black line; the two open circles along the center of mass trajectory indicate the position of the center of mass at the onset of slipping and at the instant of loss of contact with the supporting surface.

[10]Pritchett [12] pointed out that the conclusions of Littlewood [9] are valid only if the mass of the unloaded hoop is nonzero, in contrast to Littlewood's idealization of a "weightless hoop".

B Appendix: Wheel with Pendulum

Another model of the rolling of a partially filled can of beer is a wheel of mass M and radius a that rolls without slipping on the incline, with a simple pendulum of mass m and length b suspended from its axle, as illustrated in the figure below. In this model, the can has mass M, which was neglected in the previous analysis, while the liquid is approximated as a point mass m, subject to no friction, at fixed distance b from the center of the wheel



This model relates to the considerations in sec. 3 above with $m_{\rm can} \to M$, $I_{\rm can}/a^2 \to M$ and $I_{\rm cm} \to 0$, Then, the Lagrangian for coordinates u and θ follows from eqs. (39)-(40) and (44)-(45) as,

$$\mathcal{L}(u,\theta) = M\dot{u}^2 + \frac{m}{2}[\dot{u}^2 + 2b\cos(\theta-\alpha)\,\dot{u}\,\dot{\theta}] + (M+m)gu\sin\alpha + mgb\cos\theta. \tag{56}$$

The equations of motion for these coordinates are,

$$(m+2M)\,\ddot{u} - mb[\cos(\theta-\alpha)\,\ddot{\theta} - \sin(\theta-\alpha)\,\dot{\theta}^2] = (m+M)g\sin\alpha, \tag{57}$$

$$mb^2\,\ddot{\theta} - mb\cos(\theta-\alpha)\,\ddot{u} = -mgb\sin\theta, \tag{58}$$

Similarly, for coordinates s and ϕ, the Lagrangian is,

$$\begin{aligned}\mathcal{L}(s,\phi) &= M\left(\dot{s}^2 + 2b\cos\phi\,\dot{s}\,\dot{\phi} + b^2\sin^2\phi\,\dot{\phi}^2\right) + \frac{1}{2}m(\dot{s}^2 + b^2\sin^2\phi\,\dot{\phi}^2) \\ &\quad + Mg(s + b\sin\phi)\sin\alpha + mg(s\sin\alpha + b\cos\alpha\cos\phi)\end{aligned} \tag{59}$$

and the equations of motion are,

$$(m+2M)\ddot{s} + 2Mb\left(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2\right) = (m+M)g\sin\alpha, \tag{60}$$

$$mb^2(\sin^2\phi\,\ddot{\phi} + \sin\phi\cos\phi\,\dot{\phi}^2) + 2M(b^2\sin^2\phi\,\ddot{\phi} + b\cos\phi\,\ddot{s}) = -mgb\cos\alpha\sin\phi. \tag{61}$$

These complicated equations can be analyzed simply only in the limits that $m \ll M$ or $m \gg M$, in which case the motion has the same qualitative character as that discussed in secs. 2 and 3 above. That is, the case of a wheel plus pendulum does not offer any insights not accessible in the example of a partially filled can of liquid.

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