The Force on an Antenna Array

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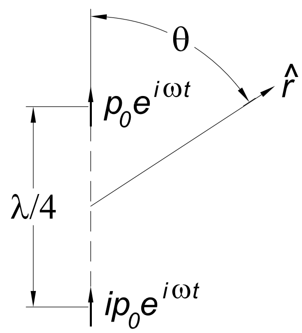
1 Problem

Antennas radiate (vector) momentum as well as energy, but if the radiation pattern is symmetric, the total radiated momentum is zero. Consider a simple antenna array that consists of two short linear dipole antennas whose centers are $\lambda/4$ apart and whose conductors point along the line of centers of the antennas. If the two antennas are excited with a 90° phase difference, deduce the asymmetric radiation pattern and the time-average rate of momentum $d\langle \mathbf{P}\rangle/dt$, that is radiated. The antenna experiences a time-average reaction force $\langle \mathbf{F}\rangle = -d\langle \mathbf{P}\rangle/dt$.



2 Solution

This problem is based on prob. 6, p. 400 of [1].[1] For another application of the concept of the radiation reaction to antennas, see [2].

The momentum density $\mathbf{p}$ and energy density u of electromagnetic radiation that moves in direction $\hat{\mathbf{x}}$ with speed of light c are related by $u = cp$. This suggests that the reaction force will be related to the radiated power dU/dt as

$$\langle F\rangle = \frac{K}{c}\frac{dU}{dt}, \tag{1}$$

where K is a dimensionless constant (possibly zero) dependent on details of the radiated momentum distribution.

The time-average pattern of radiated energy, $d\langle U\rangle/d\Omega\, dt$, of an antenna with electric dipole moment $\mathbf{p}$ can be deduced from the radiation fields in the far zone via the time-average Poynting vector,

$$\langle \mathbf{S}_{\rm rad}\rangle = \frac{c}{8\pi}Re(\mathbf{E}_{\rm rad}\times\mathbf{B}^\star_{\rm rad}), \tag{2}$$

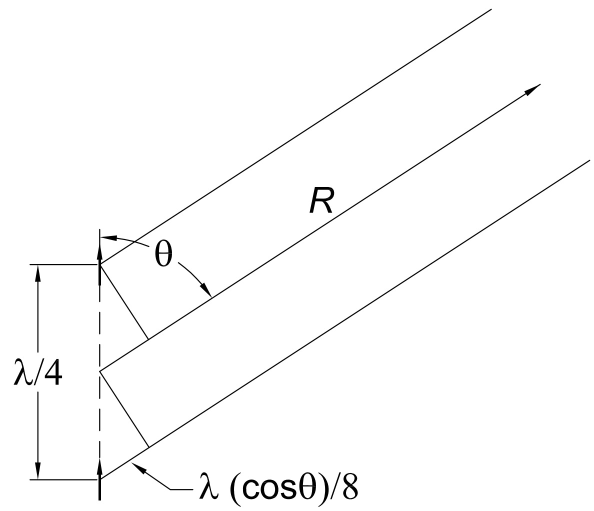
[1]This paper is dedicated to the memory of W.K.H. “Pief” Panofsky, who passed away on Sept. 24, 2007 at age 88.

in Gaussian units, according to

$$\frac{d\langle U\rangle}{d\Omega\, dt} = r^2 \langle \mathbf{S}_{\rm rad}\rangle \cdot \hat{\mathbf{r}} = \frac{|[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}|^2}{8\pi c^3}\,, \tag{3}$$

where $[\ddot{\mathbf{p}}] = \ddot{\mathbf{p}}(t' = t - r/c)$ is the second time derivative of the dipole moment evaluated at the retarded time, $t' = t - r/c$ for an observer at distance r from the dipole.

In the present example the dipole moment is the sum of two terms, $p_0 e^{i\omega t}\,\hat{\mathbf{z}}$ due to a small antenna at $z = \lambda/8$, and $ip_0 e^{i\omega t}\,\hat{\mathbf{z}}$ due to a second antenna at $z = -\lambda/8$ that is driven 90° out of phase with respect to the first. If the distance from the center of the array to the observer is R, then the distance to the upper antenna is $R - \lambda(\cos\theta)/8$ while the distance to the lower antenna is $R + \lambda(\cos\theta)/8$, as sketched in the figure below.



The retarded derivative of the total dipole moment is therefore,

$$\begin{aligned}[\ddot{\mathbf{p}}] &= -\omega^2 p_0 e^{i\omega(t-R/c)}\left(e^{i\omega\lambda(\cos\theta)/8c} + ie^{-i\omega\lambda(\cos\theta)/8c}\right)\hat{\mathbf{z}} \\ &= -\omega^2 p_0 e^{i\omega(t-R/c)} e^{i\pi(\cos\theta)/4}\left(1 + ie^{-i\pi(\cos\theta)/2}\right)\hat{\mathbf{z}}.\end{aligned} \tag{4}$$

Then,

$$[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}} = -\omega^2 p_0 e^{i\omega(t-R/c)} e^{i\pi(\cos\theta)/4}\left(1 + ie^{-i\pi(\cos\theta)/2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{5}$$

$$|[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}|^2 = 2\omega^4 p_0^2 \sin^2\theta \left(1 + \sin\frac{\pi}{2}\cos\theta\right), \tag{6}$$

and

$$\frac{d\langle U\rangle}{d\Omega\, dt} = \frac{|[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}|^2}{8\pi c^3} = \frac{\omega^4 p_0^2 \sin^2\theta}{4\pi c^3}\left(1 + \sin\frac{\pi}{2}\cos\theta\right). \tag{7}$$

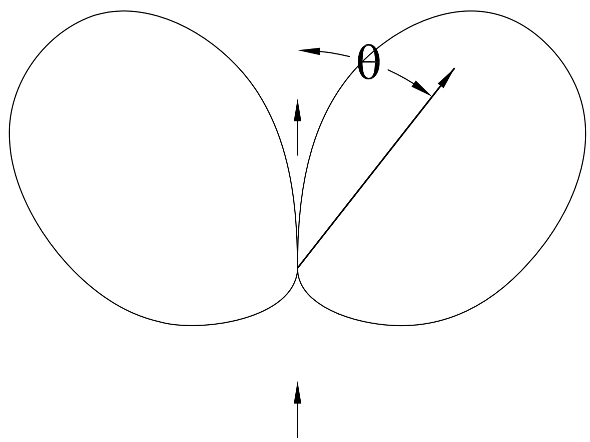
This angular distribution is sketched in the figure on the next page. More energy is radiated into the forward hemisphere ($z > 0$) than the backward.

The total radiated energy is obtained by integration of eq. (7) over solid angle,

$$\frac{d\langle U\rangle}{dt} = \int \frac{d\langle U\rangle}{d\Omega\, dt}\, d\Omega = \frac{2\omega^4 p_0^2}{3c^3}\,. \tag{8}$$

Associated with the radial flow of energy $\langle U \rangle$ in the far zone is a radial flow of momentum,[2] $\langle \mathbf{P} \rangle = \langle U \rangle\, \hat{\mathbf{r}}/c$. Hence, the angular distribution of time-average momentum radiated by the antenna follows from eq. (7) as

$$\frac{d\langle \mathbf{P} \rangle}{d\Omega\, dt} = \frac{\omega^4 p_0^2 \sin^2\theta}{4\pi c^4} \left(1 + \sin\frac{\pi}{2}\cos\theta\right) \hat{\mathbf{r}}. \tag{9}$$



On integrating this over solid angle to find the total momentum radiated, only the z component is nonzero,

$$\begin{aligned} \frac{d\langle \mathbf{P} \rangle_z}{dt} &= 2\pi \frac{\omega^4 p_0^2}{4\pi c^4} \int_{-1}^{1} \sin^2\theta \left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right] \cos\theta\ d\cos\theta \\ &= \frac{4\omega^4 p_0^2}{\pi^2 c^4} \int_0^{\pi/2} x \left(1 - \frac{4x^2}{\pi^2}\right) \sin x\ dx \qquad \text{where } x = \frac{\pi}{2}\cos\theta \\ &= \frac{8\omega^4 p_0^2}{\pi^2 c^4} \left(\frac{12}{\pi^2} - 1\right) = \frac{12}{\pi^2 c} \frac{d\langle U \rangle}{dt} \left(\frac{12}{\pi^2} - 1\right) \approx \frac{0.26}{c} \frac{d\langle U \rangle}{dt}. \end{aligned} \tag{10}$$

The radiation reaction force on the antenna is $F_z = -dP_z/dt$. For a broadcast antenna radiating 10^5 Watts, the reaction force would be only $\approx 10^{-4}$ N.

It is useful to re-express the dipole moment in terms of the current distribution in the antennas. To a good first approximation the latter can be written for each of the two antennas as the triangular form

$$I_j(z_j, t) = I_0 \left(1 - \frac{2\,|z_j|}{L}\right) e^{i(\omega t + \phi_j)}, \tag{11}$$

where $j = 1, 2$. The current vanishes at the tips of the antenna at $z_j = \pm |L|/2$ and has a peak value of I_0 at the feedpoint at $z_j = 0$.[3] The associated charge distributions $\rho(z_j, t)$ are

[2] This is the classical version of the quantum relation for photons that $U = \hbar\omega$ and $\mathbf{P} = \hbar\,\mathbf{k} = \hbar\omega\,\hat{\mathbf{k}}/c = U\,\hat{\mathbf{k}}/c$.

[3] The current distribution (11) does not satisfy the metallic boundary condition that the electric field have zero tangential component at the surface of the conductors. However, eq. (11) suffices for a good understanding of the far-zone radiation pattern. What is lost in the approximation (11) is the relation between voltage and current at the antenna feedpoint, *i.e.*, the impedance of the antenna. For further discussion see [3].

related by current conservation, $\boldsymbol{\nabla}\cdot\mathbf{J} = -\dot\rho$, which for a 1-D distribution is simply

$$\dot\rho_j = i\omega\rho_j = -\frac{\partial I_j}{\partial z_j} = -I_0\left(\mp\frac{2}{L}\right)e^{i(\omega t+\phi_j)}, \tag{12}$$

where the upper sign is for $z_j > 0$ and the lower for $z_j < 0$, so that

$$\rho_j = \pm\frac{2I_0}{i\omega L}e^{i(\omega t+\phi_j)}. \tag{13}$$

The dipole moment is given by

$$p_j = \int_{-L/2}^{L/2}\rho_j z_j\ dz_j = \frac{I_0 L}{2i\omega}e^{i(\omega t+\phi_j)}. \tag{14}$$

Thus, the magnitude p_0 of the dipole moment of the antenna is related to the peak current I_0 by

$$p_0 = \frac{I_0 L}{2\omega}\,, \tag{15}$$

Using eq. (15) in eqs. (7)-(10) and noting that $\omega = 2\pi c/\lambda$, the radiated power can be written as

$$\frac{d\langle U\rangle}{dt} = \frac{I_0^2}{2}\frac{4\pi^2}{3c}\frac{L^2}{\lambda^2} = \frac{I_0^2 R_{\rm rad}}{2}\,, \tag{16}$$

where (using $1/c = 30\,\Omega$)

$$R_{\rm rad} = \frac{4\pi^2}{3c} = 395\,\Omega \tag{17}$$

is the radiation resistance of the antenna, and the reaction force on the antenna is

$$\langle F_z\rangle = -\frac{d\langle\mathbf{P}\rangle_z}{dt} = -\frac{8I_0^2L^2}{c^2\lambda^2}\left(\frac{12}{\pi^2}-1\right) = -\frac{12}{\pi^2 c}\frac{d\langle U\rangle}{dt}\left(\frac{12}{\pi^2}-1\right) \approx -\frac{0.26}{c}\frac{d\langle U\rangle}{dt}\,. \tag{18}$$

Recalling eq. (1), the magnitude of constant K is 0.26 for the present example.

The radiation reaction force (18) cannot, in general, be deduced as the sum over charges of the radiation reaction force of Lorentz [4], $\mathbf{F}_{\rm rad} = 2e^2\ddot{\mathbf{v}}/3c^3$. Lorentz' result is obtained by an integration by parts of the integral of the radiated power over a period. This procedure can be carried out if the power is a sum/integral of a square, as holds for an isolated radiating charge. But it cannot be carried out when the power is the square of a sum/integral as holds for (coherent) radiation by an extended charge/current distribution. Rather, the radiation reaction force on an extended current distribution must be deduced from the rate of radiation of momentum, as done here.[4,5]

[4]The radiation reaction force (18) of Lorentz is still useful for macroscopic current distributions whose extent is small compared to the wavelength of the emitted radiation. See [2].

[5]An antenna that emits met momentum can be considered as a kind of "electromagnetic rocket." The momentum transfer to the antenna can also be computed directly by integating the Lorentz force over the antenna. See, for example, [5].

References

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[2] K.T. McDonald, *The Radiation Reaction Force and the Radiation Resistance of Small Antennas* (Jan. 21, 2006), http://physics.princeton.edu/~mcdonald/examples/resistance.pdf

[3] K.T. McDonald, *Currents in a Center-Fed Linear Dipole Antenna*, (June 27, 2007),
http://physics.princeton.edu/~mcdonald/examples/transmitter.pdf

[4] H.A. Lorentz, *La Théorie Électromagnétique de Maxwell et son Application aux Corps Mouvants*, Arch. Neérl. **25**, 363-552 (1892), reprinted in *Collected Papers* (Martinus Nijhoff, The Hague, 1936), Vol. II, pp. 64-343,
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[5] K.T. McDonald, *Tuval's Electromagnetic Spaceship* (Nov. 30, 2015),
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