PH 205 LECTURE 7

IMPULSES

WE END OUR SURVEY OF GENERAL METHODS WITH SOME REMARKS ABOUT PROBLEMS CONTAINING IMPULSES. SEE ALSO B&O SEC 5-10, 11.

IN MANY PROBLEMS RAPID CHANGES IN THE MOTION ARE ASSOCIATED WITH VERY LARGE FORCES WHICH LAST ONLY A SHORT TIME. IT IS USUALLY IMPOSSIBLE TO GIVE A DETAILED DESCRIPTION OF $\bar{F}(t)$ DURING THIS SHORT TIME. WE CAN HOWEVER GAIN CONSIDERABLE INSIGHT BY INTRODUCING THE CONCEPT OF AN <u>IMPULSE</u>, WHICH IS THE TIME INTEGRAL OF THE LARGE FORCE OF SHORT DURATION.

$$\text{IMPULSE} \equiv \int_0^{\Delta t} \bar{F}\, dt \qquad \text{(HAS DIMENSIONS OF MOMENTUM)}$$

FURTHERMORE, WE REGARD THE DURATION, Δt, OF THE IMPULSE AS SO SMALL THAT THE <u>POSITIONS</u> OF PARTICLES IN THE SYSTEM DON'T HAVE TIME TO CHANGE. HOWEVER THE <u>VELOCITIES</u> MAY CHANGE. DURING THE TIME Δt WE ALSO REGARD ALL NON-IMPULSIVE FORCES AS TOO WEAK TO CHANGE EITHER POSITIONS OR VELOCITIES.

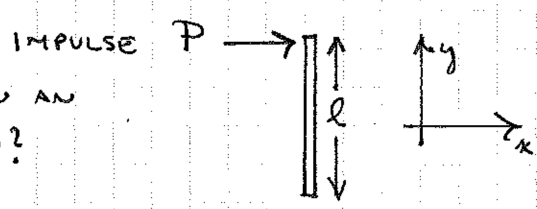
THE ELEMENTARY METHOD IS TO NOTE THAT IMPULSES CAUSE DISCRETE CHANGES IN MOMENTUM AND ANGULAR MOMENTUM:

$$\Delta \bar{P} = \int \bar{F}\, dt = M\, \Delta \bar{v}_{CM}$$

$$\Delta \bar{L} = \Delta(\bar{r} \times \bar{p}) = \bar{r} \times \Delta \bar{P} \qquad \text{SINCE } \Delta \bar{r} = 0 \text{ DURING AN IMPULSE}$$

<u>EXAMPLE 1</u> PROB 20 p198 B&O

A PENCIL ON A FLAT TABLE IS GIVEN AN IMPULSE AS SHOWN. WHAT IS THE MOTION?



MOTION OF THE C.M. $\Delta v_x = \frac{P_x}{M} \Rightarrow v_x = \frac{P}{M}$ $\quad \Delta v_y = 0 \Rightarrow v_y = 0$

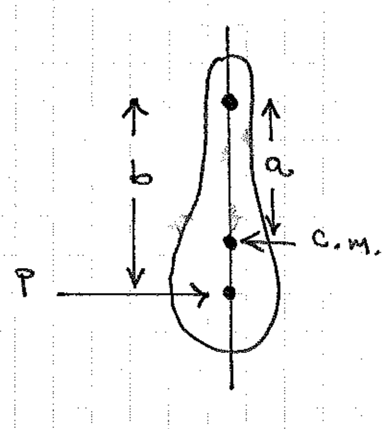
MOTION ABOUT THE C.M. $\Delta L = \frac{\ell}{2} P = I\omega$ WHERE $I = \frac{1}{12} M\ell^2$ ABOUT THE C.M.

SO $\omega = \frac{6P}{M\ell} = \frac{3v}{\ell/2} \Rightarrow$ ROTATION FASTER THAN 'ROLLING WITHOUT SLIPPING'

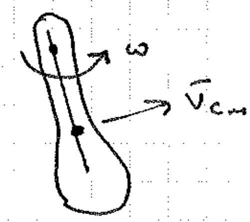
<u>EXAMPLE 2</u> <u>CENTER OF PERCUSSION</u>

AN OBJECT IS HANGING FROM A PIVOT TO FORM A PENDULUM. WHERE SHOULD YOU APPLY A HORIZONTAL IMPULSE SO THE OBJECT WILL START SWINGING WITHOUT ANY SHOCK TRANSMITTED TO THE PIVOT?

(WHERE IS THE 'SWEET SPOT' OF A TENNIS RACKET OR A BASEBALL BAT?)



NO SHOCK WILL BE TRANSMITTED TO THE PIVOT IF $v_{PIVOT} = 0$ DUE TO THE EFFECT OF THE IMPULSE.



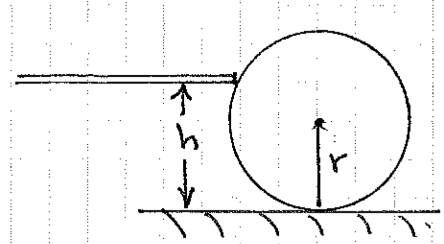
NOW $v_{PIVOT} = v_{cm} - a\omega$

BUT $v_{cm} = \dfrac{P}{M}$, WHILE $\Delta L_{\text{ABOUT PIVOT}} = bP = I_{PIVOT}\,\omega$

SO WE NEED $b = \dfrac{I}{Ma}$ $\equiv$ CENTER OF PERCUSSION

COMPARING TO P35 OF THE NOTES, OR PROB 7, SET 2, WE SEE THAT THIS IS ALSO THE CENTER OF OSCILLATION!

EXAMPLE 3 WHERE SHOULD YOU HIT A BILLIARD BALL WITH A HORIZONTAL CUE SO THAT IT ROLLS WITHOUT SLIPPING?



'CLEARLY' THE CUE SHOULD POINT TOWARDS THE CENTER OF PERCUSSION RELATIVE TO THE POINT OF CONTACT OF THE BALL WITH THE TABLE.

$$h = \frac{I}{Mr}, \quad I = I_{cm} + Mr^2 = \frac{7}{5}Mr^2 \Rightarrow h = \frac{7}{5}r.$$

LAGRANGE'S METHOD FOR IMPULSE PROBLEMS

IN VERY COMPLEX PROBLEMS, LAGRANGE'S METHOD OFFERS SOME ADVANTAGE.

IT IS USEFUL TO CONSIDER LAGRANGE'S EQUATIONS IN THE FORM DISPLAYING THE GENERALISED FORCES EXPLICITLY:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \qquad \text{(HOLONOMOUS CONSTRAINTS)}$$

DURING THE IMPULSE THE Q_j NOT DUE TO THE IMPULSE VANISH IN EFFECT. SO IF WE INTEGRATE THESE EQUATIONS DURING THE TIME $\Delta t = t_f - t_i$ OF THE IMPULSE:

$$\left.\frac{\partial T}{\partial \dot{q}_j}\right|_f - \left.\frac{\partial T}{\partial \dot{q}_j}\right|_i - \int_{t_i}^{t_f} \frac{\partial T}{\partial q_j}\,dt = \int Q_j\,dt \equiv I_j$$

WHERE I_j IS THE GENERALISED IMPULSE

NOW $\dfrac{\partial T}{\partial q_j}$ IS NOT ZERO, BUT IT IS FINITE, SO $\displaystyle\lim_{t_i \to t_f} \int_{t_i}^{t_f} \frac{\partial T}{\partial q_j}\,dt = 0$

AND WE CAN IGNORE THIS TERM — JUST LIKE WE IGNORE CHANGES IN POSITION DURING THE IMPULSE.

THUS LAGRANGE'S EQUATION FOR IMPULSES ARE

$$\Delta \frac{\partial T}{\partial \dot{q}_j} = \Delta P_j = I_j$$

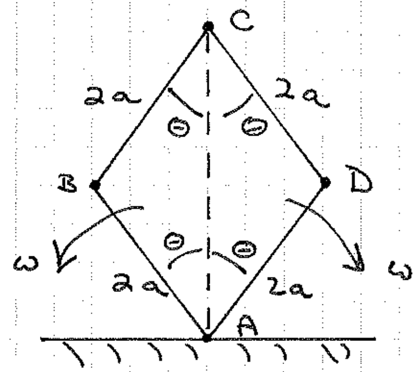
OR Δ(GENERALISED MOMENTUM) = GENERALISED IMPULSE

OF COURSE, YOU MUST USE $Q_j = \sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}$ TO CALCULATE

THE GENERALISED IMPULSE $I_j = \sum_i \bar{I}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}$ WHERE $\bar{I}_i = \int \bar{F}_i \, dt$

<u>EXAMPLE</u> A RHOMBUS MADE OF RIGID RODS OF MASS M, LENGTH 2a FALLS WITH ITS DIAGONAL VERTICAL AND ANGLES θ CONSTANT. IT HITS THE FLOOR WITH VELOCITY v AND DEFORMS. POINTS A, B, C & D ARE FRICTIONLESS PIVOTS.

WHAT IS THE ANGULAR VELOCITY, ω, OF A ROD JUST AFTER STRIKING THE FLOOR.



NOTE THAT EVEN WHEN $\omega \neq 0$ THE RHOMBUS HAS NO ANGULAR MOMENTUM. THIS PROBLEM WILL NOT BE EASY BY ELEMENTARY METHODS.

<u>SOLUTION 1</u> FOR MOTION IN A VERTICAL PLANE IN WHICH THE DIAGONAL ALWAYS REMAINS VERTICAL, THIS SYSTEM HAS ONLY 2 DEGREES OF FREEDOM. WE NEED ONLY 2 GENERALISED COORDINATES WHICH WE CHOOSE AS y OF THE C.M., AND ANGLE θ OF ANY ROD TO THE VERTICAL.

IF WE ADOPT THE ELEMENTARY APPROACH WE WILL FIND 5 SEPARATE INDEPENDENT IMPULSE COMPONENTS, WHICH REQUIRE 5 EQUATIONS (SOLUTION 3). LAGRANGE'S METHOD WILL YIELD ONLY 2 EQUATIONS OF MOTION – A POSSIBLE SIMPLIFICATION!

THE ONLY <u>EXTERNAL</u> IMPULSE IS THE VERTICAL IMPULSE APPLIED AT POINT A. CALL THIS $\bar{P} = P\hat{y}$

WE MUST CALCULATE THE GENERALISED IMPULSES I_y AND I_θ

IN $I_j = \sum_i \bar{P}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}$ THE ONLY NON-ZERO TERM IS $P \frac{\partial y_A}{\partial q_j}$

WHERE y_A = POSITION OF POINT A = $y - 2a\cos\theta$

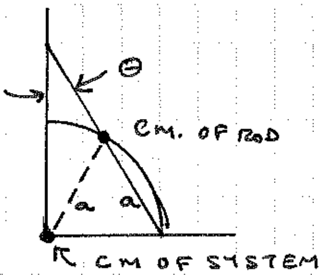
HENCE $I_y = P$ AND $I_\theta = 2aP\sin\theta$

FOR LAGRANGE'S METHOD, WE ALSO NEED THE KINETIC ENERGY.

$$T = T_{\text{OF C.M. MOTION}} + T_{\text{OF MOTION ABOUT THE C.M.}}$$

$$= \frac{1}{2}(4m)\dot{y}^2 + 4\cdot\left\{\begin{array}{l} T_{\text{OF C.M. MOTION OF A ROD RELATIVE TO THE OVERALL C.M.}} \\ + T_{\text{OF ROTATION ABOUT THE C.M. OF A ROD}}\end{array}\right\}$$

NOW $T_{\text{ROD ROTATION}} = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2}\,\frac{1}{12}\, m(2a)^2\,\dot{\theta}^2 = \frac{ma^2}{6}\dot{\theta}^2$



AS THE RHOMBUS DEFORMS THE CENTER OF A ROD IS CONSTRAINED TO MOVE IN A CIRCLE OF RADIUS a ABOUT THE C.M. OF THE WHOLE SYSTEM.

SO $T_{\text{ROD C.M. REL TO SYSTEM C.M.}} = \frac{1}{2} m a^2 \dot{\theta}^2$

$$T = 2m\dot{y}^2 + 4\left(\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{6}ma^2\dot{\theta}^2\right) = 2m\left(\dot{y}^2 + \tfrac{4}{3}a^2\dot{\theta}^2\right)$$

LAGRANGE'S METHOD NOW TELLS US THAT

$$\Delta\frac{\partial T}{\partial \dot{y}} = 4m(\dot{y}_f - \dot{y}_i) = I_y = P$$

$$\Delta\frac{\partial T}{\partial \dot{\theta}} = \frac{16}{3}ma^2(\dot{\theta}_f - \dot{\theta}_i) = I_\theta = 2aP\sin\theta$$

BUT $\dot{y}_i = -v$, $\dot{\theta}_i = 0$, $\dot{\theta}_f = \omega$ AND $\dot{y}_f = \frac{d}{dt}(2a\cos\theta) = -2a\sin\theta\,\omega$

WHICH LEADS TO $\omega = \dfrac{3v\sin\theta}{2a(1+3\sin^2\theta)}$

<u>SOLUTION 2</u> A MORE CLEVER CHOICE OF GENERALISED COORDINATES MIGHT HAVE BEEN y_A AND θ

THEN $I_{y_A} = P$ AND $I_\theta = 0$

SO $\left.\dfrac{\partial T}{\partial \dot{\theta}}\right|_f = \left.\dfrac{\partial T}{\partial \dot{\theta}}\right|_i$

AND THE SIZE OF THE IMPULSE P WILL NEVER APPEAR IN OUR SOLUTION.

BUT NOTE THAT T IS MORE COMPLICATED.

WE SAW THAT $T = 2m\left(\dot{y}_{cm}^2 + \tfrac{4}{3}a^2\dot{\theta}^2\right)$

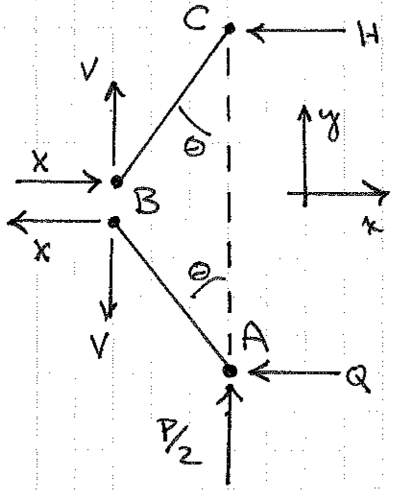
NOW $y_{cm} = y_A + 2a\cos\theta \Rightarrow T = 2m\left(\dot{y}_A^2 - 4a\sin\theta\,\dot{y}_A\dot{\theta} + \left(4a^2\sin^2\theta + \tfrac{4}{3}a^2\right)\dot{\theta}^2\right)$

$$\frac{\partial T}{\partial \dot\theta} = 4m\left(-2a \sin\theta\, \dot y_A + \tfrac{4}{3} a^2 (1+3\sin^2\theta)\dot\theta\right)$$

$\dot y_A|_i = -v \qquad \dot\theta_i = 0 \qquad \dot y_A|_f = 0 \qquad \dot\theta_f = \omega$

$$\omega = \frac{3v\sin\Theta}{2a(1+3\sin^2\theta)} \qquad \text{AS BEFORE}$$

SOLUTION 3 WE OUGHT TO BE ABLE TO SOLVE THIS PROBLEM BY ELEMENTARY MEANS. WE MUST BE VERY CAREFUL TO IDENTIFY ALL THE INTERNAL IMPULSES.



AT POINT C THERE CAN BE ONLY A HORIZONTAL IMPULSE, BY SYMMETRY.

THE IMPULSE AT D ARE THE SAME AS B.

IN COMPONENT FORM THERE ARE 5 INDEPENDENT IMPULSES ARBITRARILY CALLED P, Q, X, V & H.

WE CAN PROCEED IN SEVERAL WAYS, BUT WE CERTAINLY NEED SEVERAL EQUATIONS. WE WILL TRY TO AVOID THE USE OF IMPLULSES P & Q.

a) HORIZONTAL MOMENTUM CHANGE OF ROD BC

$$m(V_{x_f} - V_{x_i}) = X - H$$

$V_{x_i} = 0 \qquad x_f = -a\sin\theta \quad \text{so} \quad V_{x_f} = -a\cos\theta\,\dot\theta = -a\cos\theta\,\omega$

$$\text{so} \quad \underline{H - X = ma\omega\sin\theta} \qquad (1)$$

b) VERTICAL MOMENTUM CHANGE OF ROD BC

$$m(v_{y_f} - v_{y_i}) = V$$

$v_{y_i} = -v \qquad y_f = 3a\cos\theta \quad \text{so} \quad V_{y_f} = -3a\omega\sin\theta$

$$\text{so} \quad \underline{V = mv - 3ma\omega\sin\theta} \qquad (2)$$

c) ANGULAR MOMENTUM CHANGE ABOUT C.M. OF ROD BC

$$L_f - L_i = H + Xa\cos\theta - Va\sin\theta \qquad \text{(+ IN RIGHT-HAND SENSE)}$$

$L_i = 0 \quad , \quad L_f = -\frac{1}{12} m(2a)^2\omega \qquad \text{(NOTE SIGN)}$

$$\text{so} \quad \underline{-\tfrac{1}{3} ma^2\omega = (H+X)a\cos\theta - Va\sin\theta} \qquad (3)$$

d) WE NEED ONE MORE EQUATION AS WE HAVE 4 UNKNOWNS: ω, H, X, & V. WE CONSIDER THE ANGULAR MOMENTUM OF ROD AB ABOUT POINT A (WHICH IS FIXED DURING THE IMPULSE) TO AVOID INTRODUCING IMPULSES P & Q.

$$L_f - L_i = V 2a \sin\theta + X 2a \cos\theta$$

$$L_i = m v a \sin\theta \neq 0 ! \qquad L_f = \tfrac{1}{3} m (2a)^2 \omega$$

$$\text{so} \quad \tfrac{4}{3} m a^2 \omega = m v a \sin\theta + 2a V \sin\theta + 2a X \cos\theta \quad (4)$$

REWRITE (3) AS $\quad -\frac{1}{3} m a^2 \omega = (H - X) a \cos\theta - V a \sin\theta + 2a X \cos\theta \quad (3')$

SUBSTITUTE (1) AND (2) INTO (3')

$$m a^2 \omega \left(\tfrac{4}{3} - 2 \sin^2\theta \right) = - m a v \sin\theta + 2a X \cos\theta \quad (5)$$

SUBSTITUTE (2) INTO (4) AND SUBTRACT FROM (5)

$$m a^2 \omega \left(\frac{8}{3} + 8 \sin^2\theta \right) = 4 m v a \sin\theta$$

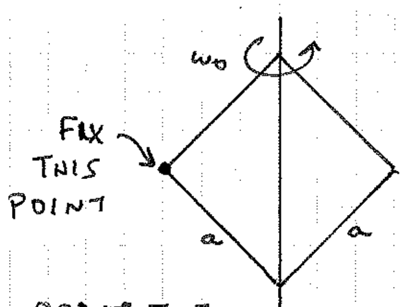
$$\omega = \frac{3 v \sin\theta}{2a (1 + 3 \sin^2\theta)}$$

SUDDEN FIXTURES

A SPECIAL CLASS OF IMPULSE PROBLEMS IS THAT IN WHICH SOME POINT OR LINE IN A MOVING BODY BECOMES FIXED BY AN IMPULSE. THE ANGULAR MOMENTUM ABOUT THAT POINT OR AXIS WILL BE THE SAME BEFORE AND AFTER THE IMPULSE.

EXAMPLE 1 PROB 9 SET 1 IS OF THIS TYPE, VIEWED IN THE MOVING FRAME.

EXAMPLE 2 A SQUARE PLATE IS ROTATING ABOUT A DIAGONAL WITH ANGULAR VELOCITY ω_0 WHEN SUDDENLY A CORNER NOT ON THE DIAGONAL BECOMES FIXED. WHAT IS THE NEW ω?



$$I_0 \omega_0 = I \omega \quad = \text{ANGULAR MOMENTUM ABOUT THE FIXED POINT}$$

$$I = I_0 + m a^2/2 \quad \text{BY THE PARALLEL AXIS THEOREM}$$

[RECALL THE RESULT ON P. 16]

$$I_0 = 4 \frac{m}{a^2} \int_0^{\frac{\sqrt{2}a}{2}} dx \int_0^{\frac{\sqrt{2}a}{2} - x} dy \, x^2 = \frac{m a^2}{12}$$

$$\text{so} \quad I = \frac{7}{12} m a^2 \qquad \text{AND} \qquad \omega = \omega_0 / 7.$$

Ph 205 LECTURE 7

SUDDEN REMOVAL OF SUPPORT

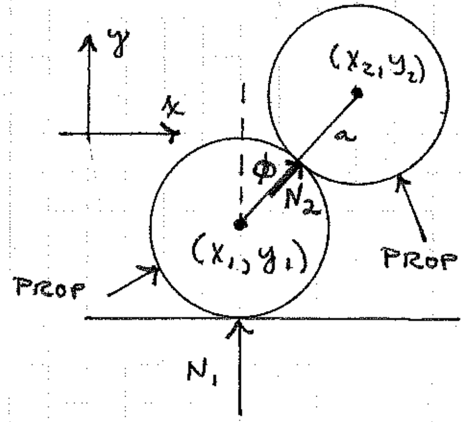
WE ALL KNOW THAT IF OUR SUPPORT IS REMOVED, OUR SYSTEM FEELS A SHOCK. A TYPICAL MECHANICS QUESTION IS HOW BIG ARE THE REMAINING FORCES JUST AFTER THE REMOVAL OF SUPPORT. ONCE THESE ARE KNOWN, THE SUBSEQUENT MOTION CAN BE STUDIED BY OUR USUAL METHODS.

EXAMPLE PROB. 6 SET 1

A METHOD OF ATTACK IS TO WRITE DOWN THE EQUATIONS OF MOTION JUST AFTER THE REMOVAL, INCLUDING EXPLICITLY THE UNKNOWN FORCES.

THEN WRITE DOWN THE CONSTRAINT EQUATION(S) AMONG THE COORDINATES q_j. WE CAN DIFFERENTIATE THESE TWICE AND EVALUATE THE RESULTING DIFFERENTIAL EQUATIONS AT THE MOMENT OF REMOVAL — WHEN THE VELOCITIES ARE STILL ZERO. NOW WE CAN ELIMINATE THE $\ddot{q}_j$ FROM THE EQUATIONS OF MOTION TO FIND THE FORCES.

EXAMPLE



TWO SPHERES OF EQUAL MASS AND RADIUS ARE ARRANGED AS SHOWN. WHAT ARE THE NORMAL FORCES N_1 AND N_2 JUST AFTER THE PROPS ARE REMOVED?
ASSUME NO FRICTION ANYWHERE.

FIRST THE EQUATIONS OF MOTION:
(NO LAGRANGIAN NEEDED!)

$$\left.\begin{aligned} m\ddot{x}_1 &= -N_2 \sin\phi \\ m\ddot{x}_2 &= N_2 \sin\phi \end{aligned}\right\} \ddot{x}_1 + \ddot{x}_2 = 0 \quad \text{- THERE IS NO NET } x \text{ FORCE}$$

$$m\ddot{y}_1 = 0 = N_1 - mg - N_2 \cos\phi$$

$$m\ddot{y}_2 = N_2 \cos\phi - mg$$

THE CONSTRAINTS ARE $x_2 = x_1 + 2a\sin\phi$ & $y_2 = y_1 + 2a\cos\phi$

DIFFERENTIATING, AND PUTTING $\dot{x}_1 = \dot{y}_1 = \dot{x}_2 = \dot{y}_2 = \ddot{y}_1 = 0$ AND $\dot{\phi} = 0$

$$\ddot{x}_2 = \ddot{x}_1 + 2a\cos\phi\,\ddot{\phi} \qquad \ddot{y}_2 = -2a\sin\phi\,\ddot{\phi}$$

SO $m(\ddot{x}_2 - \ddot{x}_1) = 2ma\cos\phi\,\ddot{\phi} = 2N_2\sin\phi \Rightarrow \ddot{\phi} = \frac{N_2}{ma}\tan\phi$

$$m\ddot{y}_2 = -2ma\sin\phi\,\ddot{\phi} = -2N_2\frac{\sin^2\phi}{\cos\phi} = N_2\cos\phi - mg$$

FINALLY $N_2 = \dfrac{mg\cos\phi}{\cos^2\phi + 2\sin^2\phi} = \dfrac{mg\cos\phi}{1+\sin^2\phi}$ & $N_1 = mg + N_2\cos\phi = \dfrac{2mg}{1+\sin^2\phi} = \dfrac{2N_2}{\cos\phi}$

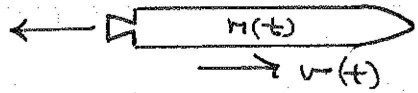
Ph 205 Lecture 7

Rockets (see B & O sec 3-1)

We'd like to consider this famous problem sometime in the course - so we'll squeeze it in here.

A rocket engine spews out mass at rate $\frac{dM}{dt} = -C$ at velocity w <u>relative</u> to the rocket. What is the rocket's velocity $v(t)$?

$v_{EXHAUST} = v - w$



$m(t) = M_0 - ct$

There are no external forces, so $\bar{P}_{TOTAL}$ = constant

But $\bar{P}_{TOTAL}$ is split between the rocket and the exhaust.

We can get the equation of motion by comparing P_{TOTAL} at two times t and $t+dt$ for small dt.

$$P(t) = m(t)\, v(t) + P_{EXHAUST}(t)$$

$$P(t+dt) = \underbrace{(m+dm)(v+dv)}_{\text{new momentum of rocket}} + P_{EXHAUST}(t) + \underbrace{(-dm)(v-w)}_{\text{momentum of exhaust created during time } dt}$$

[Tricky point: $dm < 0$ by our convention that $dm/dt = -c$]

$$= mv + P_{ex}(t) + m\,dv + w\,dm + \text{2nd order}$$

Conservation of momentum then tells us that

$$m\,dv + w\,dm = 0$$

$$dv = -w\frac{dm}{m} \qquad (w \text{ is constant})$$

$$\text{so} \quad v = -w \log\left(\frac{m}{M_0}\right) = -w \log\left(1 - \frac{ct}{M_0}\right) = w \log\left(\frac{M_0}{M}\right)$$

If the rocket can burn until $\frac{M_0}{M} > e = 2.718\ldots$, then $v > w$ and even the exhaust moves in the same direction as the rocket.

As $m \to 0$, $v \to \infty$, which violates the theory of relativity. See the homework set!

For an alternate derivation you might try considering the rocket and exhaust as seperate systems. Then use $\bar{F} = d\bar{P}/dt$ to find the equation of motion.

SO FAR WE HAVE IGNORED GRAVITY. FOR SHORT ROCKET FLIGHTS WE CAN ASSUME THE FORCE OF GRAVITY IS JUST $M(t)\,g$, SO FOR PURELY VERTICAL MOTION WITH $v = \dot{y}$ THE EQUATION OF MOTION IS $M\ddot{y} = -u\dot{M} - Mg$.

IF WE CONTINUE TO ASSUME THAT $M(t) = M_0 - Ct$ YOU CAN READILY INTEGRATE TO FIND y_{MAX}, ASSUMING $M_1 > 0$ IS THE FINAL MASS.

BUT SUPPOSE WE ASK A MORE GENERAL QUESTION: WHAT FORM OF $M(t)$ MAXIMIZES y_{MAX}?

THIS IS A PROBLEM FOR THE CALCULUS OF VARIATIONS. APPARENTLY, HOWEVER, THERE IS NO GENERAL SOLUTION! SEE EXAMPLE 1.6 AND SECTIONS 5.12 AND 10.2 OF THE BOOK BY G.M. EWING. IF WE ADD THE REASONABLE REQUIREMENT THAT $\dot{M}(t) > -C_0$, A CONSTANT, THEN THEN ANSWER IS SIMPLY TO BURN THE FUEL AS QUICKLY AS POSSIBLE AT RATE $\dot{M} = -C_0$ UNTIL IT RUNS OUT. WE SHOULD KEEP THE FINAL MASS $M_1 > 0$ TO AVOID INFINITIES. A PROOF OF THE SOLUTION FOR THE RESTRICTED PROBLEM STILL REQUIRES CONSIDERABLE EFFORT!