Ph 205 Lecture 12

Collisions and the Scattering Cross Section

Suppose particles 1 and 2 collide resulting in particles 3 and 4 in the final state.

Classically we would expect that the initial and final particles could be identified with one another. i.e. $M_1 = M_3$ and $M_2 = M_4$. But in atomic and nuclear physics this need not be the case. For example, the nuclear reaction

$$P + Li_7 \to He_4 + He_4$$

As in disintegrations, conservation of momentum and energy hold:

$$\bar{P}_1 + \bar{P}_2 = \bar{P}_3 + \bar{P}_4$$

$$E_1 + \tfrac{1}{2} M_1 v_1^2 + E_2 + \tfrac{1}{2} M_2 v_2^2 = E_3 + \tfrac{1}{2} M_3 v_3^2 + E_4 + \tfrac{1}{2} M_4 v_4^2$$

We must know the change in internal energy, $Q = E_1 + E_2 - E_3 - E_4$, to proceed.

Note that, unlike the case of disintegrations, collisions with $Q < 0$ are possible — if the initial kinetic energies are larger than the 'threshold' energy $|Q|$.

Again a useful method of analysis is to consider the collision in the C.M. frame. The C.M. has velocity V_0 given by

$$\bar{V}_0 = \frac{M_1 \bar{V}_1 + M_2 \bar{V}_2}{M_1 + M_2}$$

In the C.M. frame $\quad \bar{V}_1^* = \bar{V}_1 - \bar{V}_0 = \dfrac{M_2}{M_1 + M_2} (\bar{V}_1 - \bar{V}_2)$

$$\bar{V}_2^* = \bar{V}_2 - \bar{V}_0 = -\frac{M_1}{M_1 + M_2} (\bar{V}_1 - \bar{V}_2)$$

So $\bar{P}_1^* = M_1 V_1^* = -P_2^*$ as expected.

Likewise $\bar{P}_3^* = -\bar{P}_4^*$ and $|\bar{P}_3^*|^2 = \dfrac{2 M_3 M_4}{M_3 + M_4} \left(Q + \tfrac{1}{2} M_1 V_1^2 + \tfrac{1}{2} M_2 V_2^2 \right)$

The direction of $\bar{P}_3^*$ cannot be determined from energy and momentum conservation alone.

FINALLY, WE TRANSFER BACK TO THE ORIGINAL FRAME:

$$\bar{v}_3 = \frac{\bar{P}_3^*}{m_3} + \bar{v}_0 \qquad \bar{v}_4 = \frac{\bar{P}_4^*}{m_4} + \bar{v}_0$$

ETC.

ELASTIC COLLISIONS

THIS IS THE CASE WHEN $m_3 = m_1$, $m_4 = m_2$ AND $Q = 0$ — THE MOST SENSIBLE CASE FROM A CLASSICAL VIEWPOINT.

IN THE C.M. FRAME, THE MAGNITUDES OF THE VELOCITIES ARE UNCHANGED BY THE COLLISION $|\bar{v}_3^*| = |\bar{v}_1^*|$, $|\bar{v}_4^*| = |\bar{v}_2^*|$, ALTHOUGH THE DIRECTIONS MAY CHANGE. (PROVE THIS TO YOURSELF — IT'S NOT OBVIOUS WHEN $m_1 \neq m_2$).

THEN FOR A HEAD-ON COLLISION $v_3^* = -v_1^*$ ETC.

$$v_3 = v_0 + v_3^* = v_0 - v_1^* = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} - \frac{m_2(v_1 - v_2)}{m_1 + m_2}$$

$$= \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \frac{2m_2}{m_1 + m_2} v_2$$

$$v_4 = v_0 + v_4^* = v_0 - v_2^* = \frac{2 m_1 v_1}{m_1 + m_2} - \frac{(m_1 - m_2)}{m_1 + m_2} v_2$$

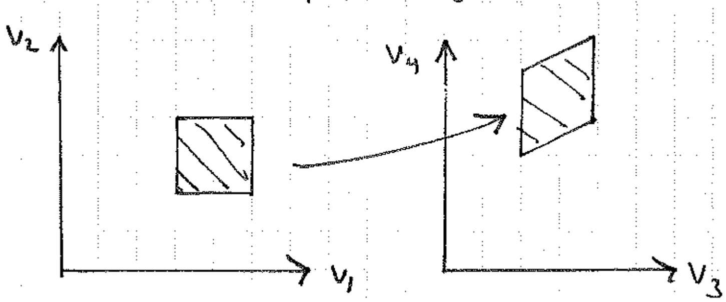
EXPERTS IN PARTIAL DERIVATES MAY NOTE AN AMUSING RESULT: IF WE CONSIDER A SET OF COLLISIONS BETWEEN PARTICLES IN THE RANGE OF VELOCITIES Δv_1 ABOUT v_1 WITH PARTICLES WHOSE VELOCITIES LIE IN RANGE Δv_2 ABOUT v_2, THEN THE RANGES OF FINAL VELOCITIES OBEY

$$\Delta v_3 \, \Delta v_4 = \Delta v_1 \, \Delta v_2$$

THIS FOLLOWS FROM THE JACOBIAN RELATION:

$$\Delta v_3 \, \Delta v_4 = \frac{\partial(v_3, v_4)}{\partial(v_1, v_2)} \Delta v_1 \Delta v_2 = \begin{vmatrix} \left(\frac{m_1 - m_2}{m_1 + m_2}\right) & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{-(m_1 - m_2)}{m_1 + m_2} \end{vmatrix} \Delta v_1 \Delta v_2$$

$$= -\Delta v_1 \Delta v_2$$



In velocity space, the collision transformation is <u>area preserving</u>

This is an example of a more general theorem of classical statistical mechanics — Liouville's theorem — to be discussed briefly later in the course.

We now consider an amusing question. If we wish to slow an object down by <u>elastic</u> scattering off obstacles, should the obstacles be heavy or light?

Let particle 2 be the obstacle, initially at rest ($v_2 = 0$)

Then $$v_3 = \frac{m_1 - m_2}{m_1 + m_2} v_1 \qquad \text{(head-on collision)}$$

Hence it is best to collide the object with an obstacle of equal mass! This is familiar from games with colliding pennies...

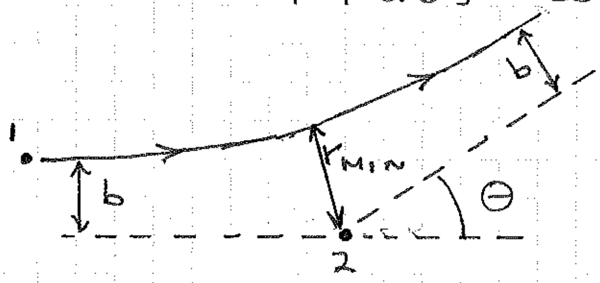
This result applies to the task of slowing down neutrons in a nuclear reactor. It is better to use a light substance such as carbon, rather than a heavy element like lead.

Impact Parameter

We have noted that the conservation laws alone are not sufficient to determine the scattering <u>angle</u> in a collision.

But if the force is known as a function of position during the collision, we can relate the scattering angle to the <u>impact parameter</u>.

We restrict ourselves to the case that particle 2 is very heavy and remains essentially at rest — scattering off a fixed target.



THE IMPACT PARAMETER, b, IS THE DISTANCE BETWEEN THE ORIGINAL LINE OF MOTION OF PARTICLE 1 AND A PARALLEL LINE THRU THE TARGET.

CONSERVATION OF ANGULAR MOMENTUM TELLS US THAT THE FINAL TRAJECTORY OF PARTICLE 1 IS ALSO PARALLEL TO A LINE THRU PARTICLE 2, AGAIN A DISTANCE b AWAY.

IF THE FORCE CAN BE EXPRESSED BY A POTENTIAL, $\bar{F} = -\partial V/\partial \bar{r}$, THEN CONSERVATION OF ENERGY CAN BE COMBINED WITH CONSERVATION OF ANGULAR MOMENTUM TO PROVIDE A RELATION BETWEEN b AND Θ, THE SCATTERING ANGLE.

THE METHOD IS VERY SIMILAR TO OUR SOLUTION FOR $\Theta(r)$ IN THE CENTRAL FORCE PROBLEM. REFER TO SEC 18 OF L & L FOR DETAILS.

$$\Theta = \pi - 2\phi \quad \text{WHERE} \quad \phi = \int_{r_{MIN}}^{\infty} \frac{b/r^2 \, dr}{\sqrt{1 - b^2/r^2 - \frac{2V(r)}{m v_0^2}}}$$

WHERE v_0 = INITIAL VELOCITY

NOTE THAT WHEN $r = r_{MIN}$ THE DENOMINATOR VANISHES.

ALSO, WHEN $r = r_{MIN}$, $\bar{v}$ IS $\perp$ TO $\bar{r}$ SO

ANGULAR MOMENTUM $L = m v_0 b = m v_{MIN} r_{MIN}$

AND $E = \frac{1}{2} m v_0^2 = \frac{1}{2} m v_0^2 + V(r_{MIN})$

SUPPOSING $V(\infty) = 0$. HENCE r_{MIN} CAN BE FOUND BE ELEMENTARY CONSIDERATIONS.

PERHAPS THE MOST IMPORTANT THING TO REMEMBER IS THAT IN CLASSICAL SCATTERING, $\Theta = \Theta(b)$.

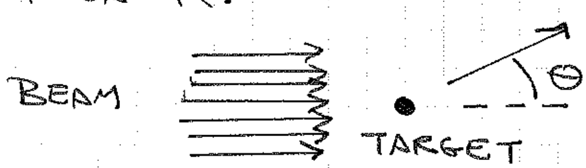
SCATTERING CROSS SECTION

SCATTERING HAS NOT PLAYED A VERY FUNDAMENTAL ROLE IN THE DEVELOPMENT OF CLASSICAL MECHANICS. NEWTON'S LAW OF GRAVITATION CAME FROM A STUDY OF BOUND ORBITS OF PLANETS, RATHER THAN FROM THE 'SCATTERING' OF COMETS OFF THE SUN.

BUT WHEN ONE TURNS TO THE STUDY OF THE MICROSCOPIC FEATURES OF THE UNIVERSE, IT OFTEN TURNS OUT THAT 'BOUND STATE' SYSTEMS ARE DIFFICULT TO OBSERVE DIRECTLY. ATOMS CONTAIN NUCLEI AND ELECTRONS; NUCLEI CONTAIN PROTON AND NEUTRONS; PROTONS AND NEUTRONS CONTAIN QUARKS AND GLUONS ....
IT IS TEMPTING TO THINK OF THESE SYSTEMS AS A KIND OF MINIATURE SOLAR SYSTEM — BUT THIS VIEW HAS NOT PROVED COMPLETELY EFFECTIVE.

INSTEAD, MUCH OF OUR KNOWLEDGE OF THE MICRO WORLD COMES FROM SCATTERING EXPERIMENTS. HOWEVER, EVEN FROM CLASSICAL MECHANICS WE CAN APPRECIATE THAT THE RELATION BETWEEN THE SCATTERING ANGLE AND THE FORCE LAW IS SOMEWHAT COMPLICATED.

AN ADDITIONAL DIFFICULTLY IS THAT IT IS VERY HARD TO ARRANGE A SCATTER WITH A KNOWN IMPACT PARAMETER WHEN VERY TINY DISTANCES ARE INVOLVED. RATHER IT IS MORE PRACTICAL TO SHOOT A BEAM OF PARTICLES AT THE TARGET, SO THAT A RANGE OF IMPACT PARAMETERS IS OBTAINED. TYPICALLY THE BEAM COULD CONTAIN N PARTICLES UNIFORMLY DISTRIBUTED OVER SOME AREA A.



WE ARE LED TO A STATISTICAL DESCRIPTION. THE # OF PARTICLES SCATTERED INTO RANGE OF ANGLES $d\Theta$ ABOUT Θ IS $dN = N\,F(\Theta)\,d\Theta$.

OF COURSE IF THE BEAM OCCUPIES A SMALLER AREA, MORE PARTICLES ARE LIKELY TO HIT THE TARGET. SO IT IS CONVENTIONAL TO DISPLAY THE DEPENDENCE ON BEAM AREA A DIRECTLY:

$$dN = \frac{N}{A} f(\theta)\, d\theta$$

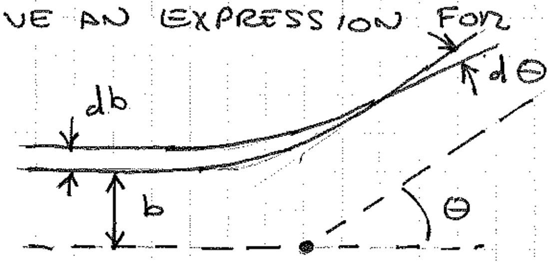
NOW $f(\theta)$ HAS THE DIMENSIONS OF AN AREA

PEOPLE USUALLY WRITE $f(\theta) = \frac{d\sigma}{d\theta} \equiv$ DIFFERENTIAL CROSS SECTION

AND $\sigma = \int \frac{d\sigma}{d\theta}\, d\theta$ = TOTAL CROSS SECTION - AN AREA!

EXAMPLE: FOR THE SCATTERING OF BB'S OFF A BILLIARD BALL WE EXPECT σ SHOULD BE JUST THE CROSS-SECTIONAL AREA OF THE BALL - πR^2.

IF WE KNOW THE RELATION $\theta(b)$ OF SCATTERING ANGLE TO THE IMPACT PARAMETER WE CAN GIVE AN EXPRESSION FOR THE DIFFERENTIAL CROSS SECTION.



PARTICLES SCATTERED INTO RANGE $d\theta$ ABOUT θ MUST HAVE HAD IMPACT PARAMETERS IN RANGE

$$db = \frac{db}{d\theta}\, d\theta \quad \text{ABOUT } b(\theta)$$

THE AREA OF THE BEAM CONTAINING THESE PARTICLES IS JUST

$$2\pi\, b\, db$$

HENCE THE NUMBER OF PARTICLES SCATTERED INTO $d\theta$ IS

$$dN = N \cdot \frac{2\pi\, b\, db}{A} = \frac{N}{A}\, 2\pi\, b\, \frac{db}{d\theta}\, d\theta$$

THUS

$$\underline{\frac{d\sigma}{d\theta}} = 2\pi\, b\, \frac{db}{d\theta} = \underline{\pi \frac{d(b^2)}{d\theta}}$$

IF WE HAD CHOSEN TO DESCRIBE THE INTERVAL IN SCATTERING ANGLE BY $d\cos\theta$, WE WOULD HAVE

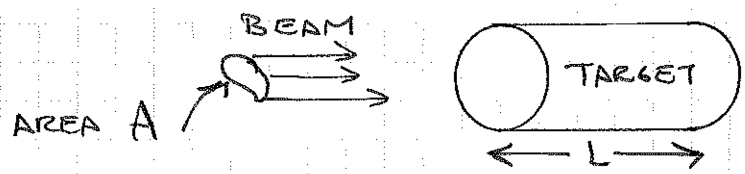
$$\underline{\frac{d\sigma}{d\cos\theta}} = \frac{1}{\sin\theta}\frac{d\sigma}{d\theta} = \frac{\pi}{\sin\theta}\frac{db^2}{d\theta} = \underline{\frac{\pi\, db^2}{d\cos\theta}}$$

IT IS ALSO COMMON TO MEASURE THE CROSS SECTION PER UNIT OF SOLID ANGLE. IT IS EASY TO CONVERT TO THIS, NOTING THAT THE TOTAL SOLID ANGLE IN A BAND $d\theta$ WIDE IS

$$d\Omega = 2\pi \sin\theta \, d\theta = 2\pi \, d\cos\theta$$

$$\text{SO} \quad \frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta} = \frac{1}{2}\frac{db^2}{d\cos\theta} \quad \text{ETC.}$$

IN REAL EXPERIMENTS ONE SELDOM USES A TARGET CONSISTING OF A SINGLE PARTICLE — WHEN THE TARGET IS A PROTON. INSTEAD A MULTI-PARTICLE TARGET IS USED.



NOW HOW MANY SCATTERS OCCUR? WE MUST MULTIPLY OUR PREVIOUS RESULTS FOR dN BY THE NUMBER OF TARGET PARTICLES SWEPT OVER BY THE BEAM. OF COURSE IF THE TARGET IS MUCH LARGER THAN THE BEAM (TRANSVERSELY) THEN NOT ALL TARGET PARTICLES SHOULD BE COUNTED.

THE EFFECTIVE # OF TARGET PARTICLES $= \rho L A$

WHERE ρ = # OF TARGET PARTICLES PER UNIT VOLUME

A = AREA OF THE BEAM

$$\text{THEN} \quad dN = \frac{N}{A}\, d\sigma \cdot \rho L A = N\rho L \, d\sigma$$

NOTE HOW THE AREA OF THE BEAM DROPS OUT! THIS IS CONVENIENT SINCE THE AREA IS OFTEN NOT WELL KNOWN.

THE EXPERIMENT THEN DETERMINES $d\sigma = \dfrac{dN}{N\rho L}$

AND WE TRY TO EXPLAIN $d\sigma$ IN TERMS OF A FORCE LAW WHICH RELATES $\theta = \theta(b)$.

RUTHERFORD SCATTERING

THE MOST FAMOUS APPLICATION OF THE CLASSICAL SCATTERING ANALYSIS IS FOR THE SCATTERING OF CHARGED PARTICLES OFF ATOMS.

HERE WE BELIEVE WE KNOW THE FORCE LAW $F = \frac{kq_1q_2}{r^2}$ = COULOMB'S LAW. BUT WHAT IS THE DISTRIBUTION OF CHARGE IN AN ATOM. IF THE + AND − IS UNIFORMLY DISTRIBUTED THE ATOM IS COMPLETELY NEUTRAL AND NO SCATTERING SHOULD OCCUR. BUT IF THE ATOM IS SOMETHING LIKE A SOLAR SYSTEM WITH ALL THE + CHARGE IN THE CENTER, THERE SHOULD BE A STRONG SCATTERING OFF THIS 'NUCLEUS'.

SINCE THE FORCE LAW IS $\sim 1/r^2$ WE CAN USE OUR SOLUTION TO THE KEPLER PROBLEM TO FIND $\Theta(b)$, AND HENCE THE SCATTERING CROSS SECTION.

FOLLOWING THE FORMAL METHOD OF L&L SEC 18, WE READILY ARRIVE AT

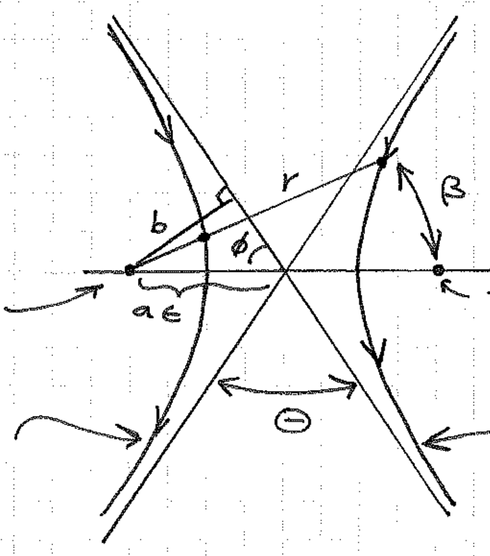
$$\sin \Theta/2 = \frac{\alpha / M v_0^2}{\sqrt{\frac{\alpha^2}{M^2 v_0^4} + b^2}} \quad \left(\text{REPLACING } kq_1q_2 \text{ BY } \alpha\right)$$

AND HENCE $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4 M^2 v_0^4 \sin^4 \Theta/2} = \frac{\alpha^2}{16 E^2 \sin^4 \Theta/2}$$

THE CROSS SECTION IS VERY BIG AT SMALL ANGLES, BUT EVEN AT $\Theta = 180°$ THE CROSS SECTION IS FINITE $\left(= \alpha^2/16E^2\right)$

THIS NON-ZERO CROSS SECTION AT LARGE ANGLES IS VERY UNLIKE THE RESULT EXPECTED FOR A UNIFORM ATOM. ITS OBSERVATION LED TO OUR PRESENT VIEW OF THE ATOM.

<u>RUTHERFORD SCATTERING VIA ORBIT EQUATIONS</u>. IT MAY BE INTERESTING TO OBTAIN THE RELATION $\Theta(b)$ DIRECTLY FROM THE EQUATIONS OF THE <u>HYPERBOLIC</u> ORBITS.

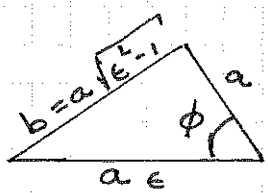


$$\frac{1}{r} = \frac{\epsilon\cos\beta + 1}{a(\epsilon^2-1)} \qquad\qquad \frac{1}{r} = \frac{\epsilon\cos\beta - 1}{a(\epsilon^2-1)}$$

FOR HYPERBOLAE, $\epsilon > 1$

THE IMPACT PARAMETER IS $b = a\sqrt{\epsilon^2-1}$

AND $a = \left|\frac{\alpha}{2E}\right|$ $(E>0)$ (α CHANGES SIGN FOR THE TWO ORBITS SHOWN)



FROM THE TRIANGLE, $\cos\phi = \frac{1}{\epsilon}$

BUT $\pi = \Theta + 2\phi \Rightarrow \phi = \pi/2 - \Theta/2$

SO $\sin\frac{\Theta}{2} = \frac{1}{\epsilon}$

ALSO $\epsilon^2 = \frac{b^2}{a^2} + 1 = \frac{1}{\sin^2\Theta/2}$

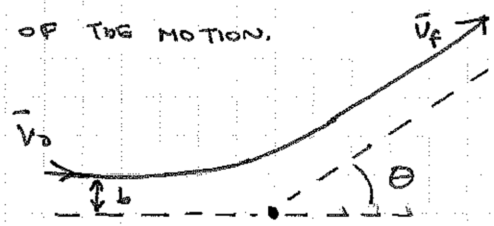
THUS $b^2 = a^2\frac{\cos^2\Theta/2}{\sin^2\Theta/2} = \frac{\alpha^2}{4E^2}\cot^2\Theta/2$

AGAIN $\frac{d\sigma}{d\Omega} = \frac{1}{2}\frac{db^2}{d\cos\Theta} = \frac{\alpha^2}{16E^2\sin^4\Theta/2}$

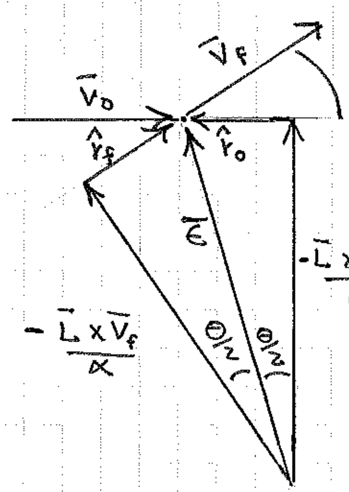
IN THE TRANSITION TO QUANTUM MECHANICS MANY OF THE CLASSICAL INSIGHTS INTO PROBLEMS OF BOUNDED ORBITAL MOTION LOST THEIR EFFECTIVENESS. MIRACULOUSLY, HOWEVER, THE RUTHERFORD SCATTERING RESULT SURVIVES ALMOST UNCHANGED. IT MAY BE USEFUL THEREFORE TO GIVE DERIVATIONS FROM ADDITIONAL POINTS OF VIEW.

RUTHERFORD SCATTERING VIA THE ECCENTRICITY VECTOR

ON p 109 WE SAW THAT $\bar{\epsilon} = \hat{r} + \frac{\bar{L}\times\bar{v}}{(-\alpha)}$ IS A CONSTANT VECTOR IN MOTION UNDER A FORCE $\bar{F} = +\frac{\alpha\hat{r}}{r^2}$. THIS ALLOWS AN INTERESTING GRAPHICAL CONSTRUCTION. NOTE THAT $|\bar{L}| = m V_0 b$ AND THAT $\bar{L}$ IS $\perp$ TO THE PLANE OF THE MOTION.



REPULSIVE FORCE, $\alpha > 0$ $\Longrightarrow$



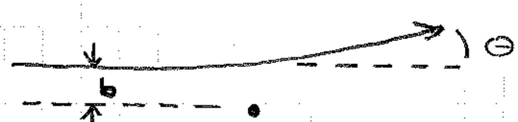
$$\bar{\epsilon} = \hat{r}_0 - \frac{\bar{L}\times\bar{v}_0}{\alpha} = \hat{r}_f - \frac{\bar{L}\times\bar{v}_f}{\alpha}$$

SO AT ONCE $\tan\Theta/2 = \frac{\alpha}{m v_0^2 b} = \frac{\alpha}{2Eb}$

OR $b^2 = \frac{\alpha^2}{4E^2}\cot^2\Theta/2$ AND $\frac{d\sigma}{d\Omega} = \frac{1}{2}\frac{d b^2}{d\cos\Theta} = \frac{\alpha^2}{16E^2\sin^4\Theta/2}$

QUICK APPROXIMATION TO RUTHERFORD SCATTERING AT SMALL ANGLES

WE CAN GIVE AN APPROXIMATE DERIVATION WHICH DOES NOT USE ANY OF OUR FANCY TRICKS. WE CONSIDER THE EFFECT OF THE



COULOMB FORCE ON A FAST MOVING PARTICLE IS TO GIVE IT AN IMPULSE PERPENDICULAR TO THE INITIAL MOTION.

THEN $\Theta \sim \frac{\Delta P}{P}$ WHERE $\Delta P = F\Delta t$

WE ESTIMATE $F \sim F_{MAX} \sim \frac{\alpha}{b^2}$; $\Delta t \approx \frac{2b}{v_0} \sim$ TIME SPENT WITH $r \sim b$

THUS $\Delta P \sim \frac{2\alpha}{b v_0} \Rightarrow \Theta \sim \frac{2\alpha}{b p v_0} = \frac{\alpha}{E b}$ OR $b^2 \sim \frac{\alpha^2}{E^2\Theta^2}$

NOW $\underline{\frac{d\sigma}{d\Omega}} = \frac{1}{2}\frac{d b^2}{d\cos\Theta} = \frac{1}{2\sin\Theta}\frac{d b^2}{d\Theta} \sim \frac{1}{2\Theta}\frac{d b^2}{d\Theta} = \underline{\frac{\alpha^2}{E^2\Theta^4}}$

A FINAL, IMPORTANT CONSIDERATION IN RUTHERFORD SCATTERING CAN BE UNDERSTOOD WITHOUT ANY OF THE FOREGOING ANALYSIS.

HOW BIG IS THE NUCLEUS?

IF THE BEAM PARTICLE PENETRATED THE NUCLEUS, THE SCATTERING WOULD BE LESS THAN THAT PREDICTED BY THE ABOVE RESULTS — SINCE BY GAUSS' LAW THE BEAM PARTICLE FEELS THE EFFECT ONLY OF THAT CHARGE AT A RADIUS SMALLER THAN THE DISTANCE OF THE BEAM PARTICLE FROM THE CHARGE CENTER.

THUS IF NO DEVIATIONS FROM RUTHERFORD'S FORMULA ARE OBSERVED, THEN $r_{NUCLEUS} < r_{MIN}$ DURING SCATTER.

NOW r_{MIN} OCCURS WHEN ALL THE INITIAL KINETIC ENERGY IS CONVERTED TO POTENTIAL ENERGY $\Rightarrow \Theta_{SCAT} = 180°$

$$\text{HENCE} \quad E = V_{MIN} = \frac{\alpha}{r_{MIN}} \quad (\text{IF } r_{MIN} > r_{NUCLEUS})$$

$$\text{AND} \quad \underline{r_{MIN} = \frac{\alpha}{E}}$$

THIS SHOWS THAT HIGH ENERGY BEAMS ARE NEEDED TO PROBE SMALL DISTANCES IN THE ATOM. THIS HAS LED TO THE GROWTH OF 'HIGH-ENERGY PHYSICS' - THE STUDY OF MATTER ON THE VERY SMALL SCALE.

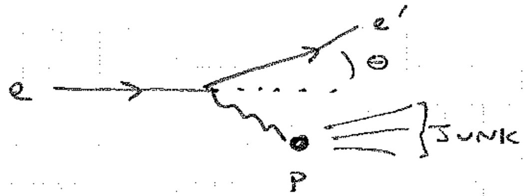
RUTHERFORD'S COLLEAGUES EVENTUALLY FOUND SMALL DEVIATIONS FROM THE RUTHERFORD SCATTERING LAW WHEN THE ENERGY OF THE BEAM PARTICLE WAS GREATER THAN A FEW MeV. ($1 \text{ MeV} \sim 1.6 \times 10^{-13}$ JOULE). IN MKS UNITS, $\alpha \approx 3 \times 10^{-28} \cdot Z_1 Z_2$

WHERE Z_1 = CHARGE OF THE NUCLEUS IN UNITS OF ELECTRON CHARGE.
Z_2 = " " " BEAM " " " " "

THUS $r_{NUCLEUS} \approx \underline{10^{-15} \text{ METERS}}$

Substructure of the Proton

In recent experiments one scatters electrons off protons, causing the proton to break up. Is this



evidence for substructure of the proton?

Not exactly! The final 'junk' always weighs MORE than the proton itself. This is an example of $E = mc^2$ in practice: some of the energy of the electron has been converted into new particles — but these particles cannot be said to have existed inside the proton originally.

However, a sophisticated analysis of the data does provide evidence for substructure! The claim is that the observed scattering is equivalent to elastic scattering off a constituent of the proton — called a PARTON by Feynman.

We sketch some aspects of an analysis that could lead to the above conclusion. This requires use of special relativity, as the experiments involve $v \lesssim c$

$$a = (a_0, \vec{a}) \equiv 4 \text{ vector}$$

Examples: $(ct, \vec{x}) = 4$ distance

$(\gamma, \gamma\vec{\beta}) = 4$ velocity, where $\vec{\beta} = \frac{\vec{v}}{c}$, and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$(E, \vec{p}c) = 4$ momentum of a particle

The $(\text{length})^2$ of a 4 vector is $a^2 = a_0^2 - \vec{a}^2$.

This is the same in all reference frames related by

LORENTZ TRANSFORMATIONS

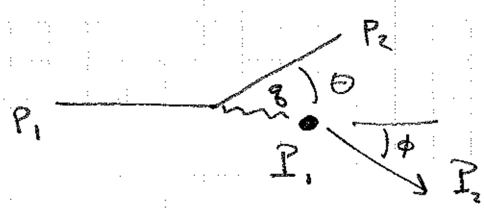
IF $\bar{v}$ = VELOCITY OF THE TRANSFORMATION

THEN IN THE TRANSFORMED FRAME $a = (a_0^*, \bar{a}^*)$

WHERE
$$a_0^* = \gamma(a_0 - \bar{a}\cdot\bar{\beta})$$
$$\bar{a}_{\parallel}^* = \gamma((\bar{a}\cdot\hat{\beta})\hat{\beta} - \bar{\beta}a_0)$$
$$\bar{a}_{\perp}^* = \bar{a}_{\perp} = \bar{a} - (\bar{a}\cdot\hat{\beta})\hat{\beta}$$

IN PARTICULAR, THE $(\text{LENGTH})^2$ OF THE 4-MOMENTUM IS $E^2 - p^2c^2$. IN A FRAME IN WHICH THE PARTICLE IS AT REST, ITS ENERGY IS JUST mc^2, WHILE ITS MOMENTUM IS ZERO. HENCE $mc^2 = \sqrt{E^2 - p^2c^2}$ HOLDS IN ANY FRAME.
WE THEN SAY THAT m IS THE INVARIANT MASS OF THE $(E, \bar{p})$ 4-VECTOR.

BACK TO SCATTERING. FIRST WE CONSIDER ELASTIC SCATTERING



IN THIS REACTION ENERGY AND MOMENTUM ARE CONSERVED:

$$P_1 + \mathcal{P}_1 = P_2 + \mathcal{P}_2$$

THE PICTURE SUGGESTS THE MODEL OF HOW ELECTROMAGNETIC SCATTERING PROCEEDS AT HIGH ENERGIES: $P_1 \rightarrow P_2$ + PHOTON q (EMISSION)

$q + \mathcal{P}_1 \rightarrow \mathcal{P}_2$ (ABSORPTION)

THE PHOTON HAS 4-MOMENTUM $q = P_1 - P_2 = \mathcal{P}_2 - \mathcal{P}_1$

THE INVARIANT MASS OF THE PHOTON IS $q^2 = (P_1 - P_2)^2 = P_1^2 + P_2^2 - 2(P_1\cdot P_2)$
[WHERE $(P_1\cdot P_2) \equiv P_{10}P_{20} - \bar{P}_1\cdot\bar{P}_2$]

IN THE LAB FRAME WE COULD WRITE (SETTING $c = 1$ FROM NOW ON!)

$P_1 = (E_1, 0, 0, P_1) \approx (E_1, 0, 0, E_1)$ IF $m \ll E_1$, $P_1^2 \approx 0$

$P_2 = (E_2, P_2\sin\theta, 0, P_2\cos\theta) \approx (E_2, E_2\sin\theta, 0, E_2\cos\theta)$, $P_2^2 \approx 0$

THEN $q^2 \approx -2(P_1 \cdot P_2) = -2E_1E_2(1-\cos\theta) = -4E_1E_2 \sin^2\theta/2$

BUT ALSO $P_2 = P_1 + q$ SO $P_2^2 = P_1^2 + q^2 + 2(P_1 \cdot q)$

NOW $P_1^2 = M^2 = P_2^2$, WHERE M IS THE INVARIANT (= REST) MASS OF THE TARGET PARTICLE. HENCE $q^2 = -2(P_1 \cdot q)$

NOW IN THE LAB, $P_1 = (M, 0, 0, 0)$

$$q = P_1 - P_2 = (E_1 - E_2, \bar{q}) \equiv (\nu, \bar{q})$$

SO $P_1 \cdot q = M\nu$

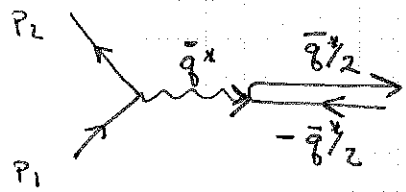
THUS $\underline{q^2 = -2M\nu}$ RELATES THE VARIABLES E_2 AND θ OF THE BEAM PARTICLE IN <u>ELASTIC</u> SCATTERING (IF $m \ll M$)

IN AN <u>INELASTIC</u> SCATTERING, WITH BREAKUP OF THE PROTON, THE VARIABLES E_2 AND θ OF THE SCATTERED ELECTRON ARE NOT SO RELATED, BUT CAN VARY SEPARATELY.

WE RETURN TO THE CONJECTURE THAT THE INELASTIC SCATTERING MIGHT BE AN ELASTIC SCATTERING OFF A PARTON WITHIN THE PROTON. THE INSIGHT OF BJORKEN & FEYNMAN IS THAT THE SCATTERING SHOULD BE A FUNCTION ONLY OF THE FRACTION OF THE PROTON'S MOMENTUM THAT IS CARRIED BY THE PARTON.

THIS IDEA IS <u>NOT</u> EASY TO UNDERSTAND IN THE LAB FRAME, WHERE THE PROTON INITIALLY HAS NO MOMENTUM!

RATHER, WE TRANSFORM TO A FRAME IN WHICH THE PARTON IS SCATTERED BY 180°, KEEPING THE MAGNITUDE OF ITS MOMENTUM CONSTANT (THE 'BREIT FRAME').



IN THIS FRAME THE PHOTON'S 4-MOMENTUM IS $q = (0, \bar{q}^*)$

THE PHOTON ENERGY q_0^* IS 0, AS NO ENERGY IS GIVEN TO THE PARTON. THEN $q^2 = -|\bar{q}^*|^2$ or $|\bar{q}^*| = \sqrt{-q^2}$

BACK IN THE LAB FRAME, WE NOW CHOOSE THE Z-AXIS ALONG THE PHOTON'S MOMENTUM, RATHER THAN THAT OF THE BEAM PARTICLE.

SO IN THE LAB, $q = (E_1 - E_2, 0, 0, q_z) = (\nu, 0, 0, q_z)$

THUS $q^2 = \nu^2 - q_z^2$ AND $q_z = \sqrt{\nu^2 - q^2}$

WE CAN NOW CONSTRUCT THE TRANSFORMATION FROM THE LAB FRAME TO THE BREIT FRAME. WE CONSIDER THE PHOTON ENERGY:

$$\nu^* = 0 = \gamma\left(\nu - \beta\sqrt{\nu^2 - q^2}\right) \quad \text{WHERE } \beta = \frac{v}{c} = \text{VELOCITY OF TRANS}$$

HENCE $\beta = \dfrac{\nu}{\sqrt{\nu^2 - q^2}}$ AND $\gamma = \dfrac{\sqrt{\nu^2 - q^2}}{\sqrt{-q^2}}$

IN THE LAB FRAME THE INITIAL PROTON HAS 4-MOMENTUM $P_1 = (M, \bar{0})$, SO IN THE BREIT FRAME ITS 3-MOMENTUM IS JUST $P^* = \gamma\beta M = \dfrac{\nu M}{\sqrt{-q^2}}$

INTRODUCING $x \equiv$ FRACTION OF PROTON'S MOMENTUM CARRIED BY THE PARTON

$$\underline{x} = \frac{|\bar{q}^*/2|}{P^*} = \frac{\sqrt{-q^2}/2}{\nu M/\sqrt{-q^2}} = \underline{-\frac{q^2}{2M\nu}}$$

FOR ELASTIC SCATTERING, $x = 1$ ONLY AS FOUND ABOVE.

FOR INELASTIC SCATTERING, IF THE CROSS SECTION IS OBSERVED TO BE A FUNCTION OF x, $0 < x < 1$, AND NOT OF BOTH ν AND q^2, THEN WE INFER THE VALIDITY OF THE PARTON HYPOTHESIS.

THE EXPERIMENTAL EVIDENCE FOR THIS EARNED THE 1990 NOBEL PRIZE.