Physics 105: Notes on complex variables

16 November 2001

We don’t really know what aspects of complex variables you learned about in high school, so the goal here is to start more or less from scratch. Feedback will help us to help you, so let us know what you do and don’t understand. Also, if something is not immediately clear you should work through examples ... as usual.

The introduction to square roots in school often makes the point that the square root of a negative number is not defined, since after all when we square a number we always get something positive. Then at some point you are told about imaginary numbers, where the basic object is \( i \neq \sqrt{-1} \). It is not clear, perhaps, whether this is some sort of joke (calling them “imaginary” probably doesn’t help!). Here we are asking you to take these things very seriously.

Remember that when you first learned about negative numbers (a long time ago ... ) there was some mystery about what you do when you add, multiply, etc.. In the end the answer is that the rules are the same, and you have to apply them in a consistent way. This is true also for complex or imaginary numbers.

We begin by recalling that with \( x \) and \( y \) real numbers, we can form the complex number \( z = x + iy \). The object \( i \) is the square root of negative one, \( i = \sqrt{-1} \). Then, if we have two of these numbers,

\[
\begin{align*}
z_1 &= x_1 + iy_1 \\
z_2 &= x_2 + iy_2
\end{align*}
\tag{1}
\]

we can go through all the usual operations of arithmetic:

\[
\begin{align*}
z_1 + z_2 &\equiv (x_1 + iy_1) + (x_2 + iy_2) \\
&= (x_1 + x_2) + i(y_1 + y_2);\tag{3}
\\z_1 - z_2 &\equiv (x_1 + iy_1) - (x_2 + iy_2) \\
&= (x_1 - x_2) + i(y_1 - y_2);\tag{5}
\\(z_1) \times (z_2) &\equiv (x_1 + iy_1) \times (x_2 + iy_2) \\
&= x_1x_2 + x_1(iy_2) + iy_1x_2 + iy_1(iy_2) \\
&= x_1x_2 + i(x_1y_2 + x_2y_1) + (i^2)y_1y_2 \\
&= x_1x_2 + i(x_1y_2 + x_2y_1) - y_1y_2 \tag{9}
\\&= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1);\tag{11}
\end{align*}
\]
where in the second to last step we use the fact that $i^2 = -1$. Note that this list leaves out division, which we’ll get back to in a moment.

One very useful operation that is new for complex numbers is called “taking the complex conjugate,” or “complex conjugation.” For every complex number $z = x + iy$, the complex conjugate is defined to be $z^* = x - iy$. Note that in elementary physics we usually use $\bar{z}$ to denote the complex conjugate of $z$; in the math department and in some more sophisticated physics problems it is conventional to write the complex conjugate of $z$ as $\overline{z}$, but of course this is just notation. The crucial fact is that

$$z \times z^* \equiv (x + iy) \times (x - iy)$$

$$= x^2 + x(-iy) + iyx + (i)(-i)y^2$$

$$= x^2 + i(-xy + yx) - (i^2)y^2$$

$$= x^2 + y^2. \quad (15)$$

Often we write $zz^* = |z|^2$, just the way we write the length of a vector in terms of its dot product with itself, $\vec{x} \cdot \vec{x} = |\vec{x}|^2$. This is an important thing by itself, as we will see, but also it makes division a lot easier, which we do now.

There is a trick, which is to clear the complex numbers from the denominator any time we divide:

$$\frac{z_1}{z_2} \equiv \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$

$$= \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} \quad \frac{z_1 z_2^*}{z_2 z_2^*} \quad \frac{z_1 z_2^*}{|z_2|^2} \quad \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2} \quad \frac{z_1 z_2^*}{|z_2|^2}$$

$$= \frac{z_1 z_2^*}{|z_2|^2} \quad \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2} \quad \frac{|z_2|^2}{z_2 z_2^*} \quad \frac{z_1 z_2^*}{|z_2|^2} \quad \frac{z_1 z_2^*}{|z_2|^2}$$

Test yourself. You should be able to add, subtract, multiply and divide these pairs of complex numbers: (a) $z_1 = 3 + 4i, z_2 = 4 + 3i$. (b) $z_1 = 3 + 4i, z_2 = 4 - 3i$. (c) $z_1 = 7 - 9i, z_2 = 27 + 12i$. And you should be able to make up your own examples!

It is useful to think about a complex number as being a vector in a two dimensional space, where the x axis in the real part and the y axis is the imaginary part (as hinted when we write $z = x + iy$). Then the length of the vector is

$$|z| \equiv \sqrt{|z|^2} \quad \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2} \quad \frac{|z_2|^2}{z_2 z_2^*} \quad \frac{z_1 z_2^*}{|z_2|^2} \quad \frac{z_1 z_2^*}{|z_2|^2} \quad \frac{z_1 z_2^*}{|z_2|^2}$$

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$$|z| \equiv \sqrt{|z|^2} \quad (21)$$
\[
\sqrt{zz^*}, \quad (22)
\]

and the angle that this makes with the x axis is given by

\[
\theta = \tan^{-1}\left( \frac{y}{x} \right). \quad (23)
\]

In this notation,

\[
z = x + iy \quad (24)
\]

\[
= \sqrt{x^2 + y^2} \left( \frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right) \quad (25)
\]

\[
= |z|(\cos \theta + i \sin \theta). \quad (26)
\]

Now there is a very pretty thing, which is that if we multiply two complex numbers, the magnitudes get multiplied and the angles just add:

\[
z_1 \times z_2 \equiv |z_1|(\cos \theta_1 + i \sin \theta_1) \times |z_1|(\cos \theta_1 + i \sin \theta_1) \quad (27)
\]

\[
= (|z_1||z_2|)[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \sin \theta_1 \cos \theta_1)] \quad (28)
\]

\[
= (|z_1||z_2|)[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \quad (29)
\]

where in the last step we use the trigonometric identities

\[
\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \quad (30)
\]

\[
\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \sin \theta_1 \cos \theta_1. \quad (31)
\]

**Test yourself.** In terms of these angles, what is the condition for multiplying two complex numbers and getting a real answer? What happens to the angles when we divide rather than multiply?

In class we showed one more thing, namely that the combination of sin and cos which appear in Eq. (26) can be written as the exponential of an imaginary number:

\[
e^{i\theta} = \cos \theta + i \sin \theta. \quad (32)
\]

This is an exponential that obeys all the rules, in particular

\[
e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} \quad (33)
\]

\[
e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 - \theta_2)}. \quad (34)
\]
Notice that when we divide we end up taking the difference of angles, which is something you should have shown above by explicit manipulation of the sin and cos terms plus application of the appropriate trig identities. Indeed, this is a different path to proving that Eq. (32) must be true. That is, we can take known trig identities and use them to prove that if the object $e^{i\theta}$ obeys all the usual rules for arithmetic with exponentials then the real part must be the cosine and imaginary part must be the sign. An alternative view is to take Eq. (32) for granted (after all, we showed it in class!) and then use this prove the trig identities. This way you’ll never need to remember them!

**Test yourself.** Derive the sum and difference angle identities by multiplying and dividing the complex exponentials. For more fun, use the same sort of trick to derive and expression for $\cos(3\theta)$ in terms of $\sin \theta$ and $\cos \theta$. 