1. Wave Amplification in a Magnetic Medium

One way to prepare an optically active medium is to turn on a strong DC magnetic field at right angles to a static magnetic field that has initially aligned the dipoles of a magnetic medium. Then the dipoles will precess about the direction of the strong magnetic field, before eventually relaxing into alignment with that field. During those intervals while the dipoles \( \mathbf{m} \) are antialigned with the initial static field, they are in a state of high energy \( U = -\mathbf{m} \cdot \mathbf{B} \). When in this state, the medium can give up energy to a probe electromagnetic wave (with magnetic field along the direction of the strong DC field), thereby amplifying it.

Deduce the equations of motion for the magnetization \( \mathbf{M} = N\mathbf{m} \) of a medium that consists of \( N \) permanent dipoles \( \mathbf{m} \) (with angular momentum \( \mathbf{L} = \Gamma \mathbf{m} \)) per unit volume when the medium is immersed in a magnetic field \( \mathbf{B} \). Consider the specific example of a static magnetic field \( B_{0x}\hat{x} + B_{0y}\hat{y} \) where \( B_{0x} \ll B_{0y} \), and an oscillatory field \( B_y e^{-i\omega t}\hat{y} \).

You may suppose that \( \mathbf{M} \ll B_x \) and \( \mathbf{M} \ll B_y \).

A measure of the ability of the medium to amplify a probe wave is the frequency-dependent index of refraction \( n(\omega) = \sqrt{\mu} \), where \( \mu \) is the magnetic susceptibility related by \( B = \mu \mathbf{H} = \mathbf{H} + 4\pi \mathbf{M} \) (in Gaussian units, and in a medium of dielectric constant \( \epsilon = 1 \)). In the present example, the wave field has magnetic field along the \( y \) axis, so that you can write

\[
B_y(\omega) = \mu H_y(\omega) = H_y \left( 1 + 4\pi \frac{M_y}{H_y} \right),
\]

(1)

Since we assume that \( M \ll B_y \), we also have \( M_y \ll H_y \), and the index of refraction is given by

\[
n(\omega) = \sqrt{\mu} \approx 1 + 2\pi \frac{M_y}{H_y}.
\]

(2)

If the medium is to exchange energy with a wave, there must be additional processes occurring. For index of refraction to include absorption (or amplification), it suffices to suppose that there is a kind of damping mechanism that aligns the magnetic dipoles with the static magnetic field. A phenomenological form for this is

\[
\frac{d\mathbf{m}}{dt} = \gamma (\hat{\mathbf{m}} \times \hat{\mathbf{B}}) \times \mathbf{m} \approx -\gamma \mathbf{m} (\hat{\mathbf{m}} - \hat{\mathbf{y}}),
\]

(3)

where \( \gamma \) is the damping factor, and the approximation notes that the static field is largely along the \( y \) axis. Include this damping in the equations of motion, solve for the oscillatory behavior of \( M_y \propto e^{-i\omega t} \) assuming the damping is slow so that \( \gamma \ll \Gamma B_x \), and then calculate the index \( n(\omega) \). Show that when \( M_x \) has precessed to be opposite to \( B_{0x} \), the index of refraction implies amplification of a traveling wave of \( H_y \) (and \( M_y \)).

2. Spin Waves

Magnetostatics can be defined as the regime in which the magnetic fields \( \mathbf{B} \) and \( \mathbf{H} \) have no time dependence, and “of course” the electric fields \( \mathbf{D} \) and \( \mathbf{E} \) have no time dependence either. In this case, the divergence of the fourth Maxwell equation,

\[
\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{free}} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t},
\]

(4)
(in Gaussian units) implies that
\[ \nabla \cdot \mathbf{J}_{\text{free}} = 0, \tag{5} \]
\textit{i.e.}, that the free currents flow in closed loops. Likewise, the time derivative of the fourth Maxwell equation implies that \( \mathbf{J}_{\text{free}} \) has no time dependence in magnetostatics.

Often, magnetostatics is taken to be the situation in which \( \nabla \cdot \mathbf{J}_{\text{free}} = 0 \) and \( \mathbf{D}, \mathbf{E} \) and \( \mathbf{J}_{\text{free}} \) have no time dependence, without explicit assumption that \( \mathbf{B} \) and \( \mathbf{H} \) also have no time dependence. Discuss the possibility of waves of \( \mathbf{B} \) and \( \mathbf{H} \), consistent with the latter definition of magnetostatics.

Consider two specific examples of “magnetostatic” waves in which \( \mathbf{J}_{\text{free}} = 0 \):

a) Ferromagnetic spin waves in a medium subject to zero external field, but which has a uniform static magnetization that is large compared to that of the wave. That is, \( \mathbf{M} = M_0 \hat{z} + m_0 e^{i(kr - \omega t)} \), where \( m_0 \ll M_0 \). Here, the quantum mechanical exchange interaction is the dominant self interaction of the wave, which leads to an effective magnetic field in the sample given by \( \mathbf{B}_{\text{eff}} = \alpha \nabla^2 \mathbf{m} \), where \( \alpha \) is a constant of the medium.

b) Waves in a ferrite cylinder in a uniform external magnetic field parallel to its axis, supposing the spatial variation of the wave is slight, so the exchange interaction may be ignored. Again, the time-dependent part of the magnetization is assumed small compared to the static part. Show that the waves consist of transverse, magnetostatic fields that rotate with a “resonant” angular velocity about the axis.

In practice, the spin waves are usually excited by an external rf field, which is to be neglected here.

3. Self-Induced Transparency

Show that a distortionless electromagnetic pulse (a soliton) can propagate with its magnetic field at right angles to a steady magnetic field that is applied to a magnetic medium, the direction of propagation being perpendicular to both of these fields.

Deduce the equations of motion for the magnetization \( \mathbf{M} = N \mathbf{m} \) of a medium that consists of \( N \) permanent dipoles \( \mathbf{m} \) (with angular momentum \( \mathbf{L} = \Gamma \mathbf{m} \)) per unit volume when the medium is immersed in a magnetic field \( \mathbf{B} \). Consider the specific example of a static magnetic field \( B_0 \hat{x} \) and a pulse \( B_y(z - vt) \hat{y} \).

The physical picture is that the magnetic field \( B_y \) of the pulse precesses the dipoles in the \( x-z \) plane by exactly 360° as the pulse passes, restoring the medium to its initial condition – in which the dipoles are lined up with the static field \( B_0 \hat{x} \).

The solution can be deduced considering the pulse area function defined by
\[ A(z,t) = \Gamma \int_{-\infty}^{t} B_y(z,t') \, dt'. \tag{6} \]

Show that if \( B_0 = 0 \) then a formal solution for the behavior of the medium is \( M_x = M_0 \cos A, M_z = M_0 \sin A \), supposing that \( M_x(t = -\infty) = M_0, \) and \( M_z(t = -\infty) = 0 \) (even though \( B_x \) has been temporarily set to 0).
Generalize this solution to the case of nonzero $B_0$ (nonzero $\omega_0 \equiv \Gamma B_0$) by supposing that $M_z = F(\omega_0)M_0\sin A$ where $F(0) = 1$ and $M_z = M_0[F(\omega_0)\cos A - 1] + 1$, where the latter form preserves the condition that $M_z = M_0$ when $A = 0$. Show that the equations of motion for the magnetization imply that area function obeys the Mathieu equation

$$\ddot{A} = \frac{1}{\tau^2}\sin A,$$

where the function $F$ is given by

$$F(\omega_0) = \frac{1}{1 + \omega_0^2\tau^2},$$

in terms of a constant $\tau$ that will prove to be a measure of the pulse width.

Solve eq. (5) by multiplying by $\dot{A}$, etc., to show that a pulse solution with velocity $v$ is

$$A = 4 \tan^{-1}[e^{(t-z/v)}/\tau],$$

and the corresponding magnetic field pulse is

$$B_y = \frac{2}{\Gamma\tau}\text{sech} \left( \frac{t - z/v}{\tau} \right).$$

Also, give expressions for the components of the magnetization $M(t)$.

4. **Self focusing of laser beams**

When an isotropic dielectric (for example, a gas) is placed in a strong electric field, the induced polarization is a nonlinear function,

$$\mathbf{P} = \chi_1\mathbf{E} + \chi_3\mathbf{E}^2 + \ldots \quad \text{(Guassian units)}$$

[In MKSA units one writes $\mathbf{P} = \epsilon_0\chi_1\mathbf{E} + \ldots$, with the unfortunate result that $\chi_{\text{MKSA}} = 4\pi\chi_{\text{Gaussian}}$.]

a) Give an order of magnitude estimate of the susceptibilities $\chi_1$ and $\chi_2$ for a nitrogen gas at STP, supposing the nonlinear polarization is comparable to the linear term in an electric field just strong enough to ionize the gas. Fact: liquid nitrogen has index of refraction 1.2 and density 0.8 g/cm$^3$.

If the applied electric field is due to a laser beam, the nonlinear polarizability results in an index of refraction that is larger where the electric field is stronger. The radial gradient of the index of refraction is similar to that in a fiberoptic cable, so the laser beam can become trapped in a kind of “light pipe” of its own making.

b) Suppose a laser beam of wavelength $\lambda$ is focused at angle $\theta \approx \lambda/w$ into a slab of nonlinear dielectric over a radius of diameter $w$. What is the minimum power of the laser beam so that it becomes trapped in a channel of constant radius?
c) Deduce the pulse shape $E_x(y)$ of a 2-dimensional beam (which obeys $\nabla \cdot \mathbf{E} = 0$),

$$\mathbf{E} = E_x(y) \hat{x} e^{i(kz - \omega t)}, \quad (12)$$

that propagates with an invariant transverse profile in the nonlinear medium, assuming that the pulsewidth in $y$ is large compared to a wavelength so that $E_x$ is determined by time-averaged quantities.

5. Laser tweezers

It is well known that a charged particle cannot be held at rest by purely electrostatic fields (Earnshaw’s theorem).\footnote{On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether, Trans. Camb. Phil. Soc. 7, 97-112 (1839), particularly secs. 11-15; W.R. Smythe, Static and Dynamic Electricity, 3rd ed. (McGraw-Hill, New York, 1968), sec. 1.11.} Give a simple classical explanation of how a neutral atom of polarizability $\alpha$ can be “trapped” at the focus of a laser beam.

a). First, ignore magnetic interactions, and deduce that there is a (time-averaged) trapping force dependent on the electric field of the laser.

b). Atoms have some probability of absorbing photons from the laser beam, thereby being kicked along the direction of the beam. This processes can be modelled classically by supposing that the polarizability of the atom has an imaginary part: $\alpha = \alpha' + i\alpha''$. Deduce the (time-averaged) force on an atom along the direction of propagation of a linearly polarized plane electromagnetic wave in terms of $\alpha''$, the imaginary (absorptive) part of the polarizability.

c). For an idealized atom with a single natural frequency $\omega_0$, deduce the ratio $\alpha'/\alpha''$ at the frequency $\omega$ for which the real part, $\alpha'$, of the polarizability is a maximum. For this, you may use a classical model of an atom as an electron on a spring of frequency $\omega_0$, subject to a damping force $-\gamma m \dot{x}$, where $\gamma \ll \omega_0$ is the reciprocal of the lifetime of the ‘excited state’.

d). In practice, the trapping force a) must be larger than the longitudinal force b). This requires the laser beam to be tightly focused. Deduce the $f_\#$ of the lens needed for trapping under the conditions of part c).

6. Slow light

Consider a classical model of matter in which spectral lines are associated with oscillators. In particular, consider a gas with two closely spaced spectral lines, $\omega_{1,2} =$
\( \omega_0 \pm \Delta/2 \), where \( \Delta \ll \omega_0 \). Each line has the same damping constant (and spectral width) \( \gamma \), where \( \gamma \ll \Delta \).

Classically, one might expect these oscillators to correspond to the “V” configuration of atomic levels sketched in Fig. 1a), in which both higher levels can decay to the ground state by emission of photons of frequencies \( \omega_1 \) and \( \omega_2 \). However, quantum mechanics also permits the “Λ” configuration sketched in Fig. 1b) in which the highest level can decay to the ground state via emission of a photon of frequency \( \omega_1 \) as well as to the intermediate level via emission at frequency \( \omega_2 \).

\[
\begin{align*}
\text{a)} & \quad \omega_1 \quad \omega_2 \\
\text{b)} & \quad \omega_1 \quad \omega_2
\end{align*}
\]

Figure 1: Two possible level diagrams for a three-level atomic system: a) the “V” configuration; b) the “Λ” configuration.

The present problem is based on a “Λ” configuration in which a laser of frequency \( \omega_2 \) pumps oscillator 2. But since the lower level of this oscillator is not the ground state, the pumping does not result in an inverted population. The physics of this system can be fairly well described by a classical model in which the damping constant of the second oscillator is taken to be negative: \( \gamma_2 = -\gamma \). The strengths of both oscillators are positive.

Deduce an expression for the group velocity of a pulse of light centered on frequency \( \omega_0 \) in this medium. Show also that frequencies very near \( \omega_0 \) propagate without attenuation.

In a recent experiment\(^2\), the group velocity of light was reduced to 38 mph (17 m/s) by this technique in a sodium vapor of density \( N = 8 \times 10^{13} \text{ atoms/cm}^3 \) using a pair of lines for which \( \Delta/2\pi \approx 2 \times 10^7/\text{s} \).

7. Negative group velocity

Consider a variant on the physical situation of “slow light” (Prob. 6) in which two closely spaced spectral lines are now both optically pumped to show that the group velocity can be negative at the central frequency, which leads to apparent superluminal behavior.

a) In more detail, consider a classical model of matter in which spectral lines are associated with oscillators. In particular, consider a gas with two closely spaced spectral lines of angular frequencies \( \omega_{1,2} = \omega_0 \pm \Delta/2 \), where \( \Delta \ll \omega_0 \). Each line has the same damping constant (and spectral width) \( \gamma \).

Ordinarily, the gas would exhibit strong absorption of light in the vicinity of the spectral lines. But suppose that lasers of frequencies \( \omega_1 \) and \( \omega_2 \) pump the both oscillators into inverted populations. This can be described classically by assigning negative oscillator strengths to these oscillators.\(^3\)

Deduce an expression for the group velocity \( v_g(\omega_0) \) of a pulse of light centered on frequency \( \omega_0 \) in terms of the (univalent) plasma frequency \( \omega_p \) of the medium, given by

\[
\omega_p^2 = \frac{4\pi Ne^2}{m},
\]

where \( N \) is the number density of atoms, and \( e \) and \( m \) are the charge and mass of an electron. Give a condition on the line separation \( \Delta \) compared to the line width \( \gamma \) such that the group velocity \( v_g(\omega_0) \) is negative.

In a recent experiment by Wang et al.,\(^4\) a group velocity of \( v_g = -c/310 \), where \( c \) is the speed of light in vacuum, was demonstrated in cesium vapor using a pair of spectral lines with separation \( \Delta/2\pi \approx 2 \text{ MHz} \) and linewidth \( \gamma/2\pi \approx 0.8 \text{ MHz} \).

b) Propagation of a Monochromatic Plane Wave

Consider a wave with electric field \( E_0e^{i\omega(z/c - t)} \) that is incident from \( z < 0 \) on a medium that extends from \( z = 0 \) to \( a \). Ignore reflection at the boundaries, as is reasonable if the index of refraction \( n(\omega) \) is near unity. Particularly simple results can be obtained when you make the (unphysical) assumption that the \( \omega n(\omega) \) varies linearly with frequency about a central frequency \( \omega_0 \). Deduce a transformation that has a frequency-dependent part and a frequency-independent part between the phase of the wave for \( z < 0 \) to that of the wave inside the medium, and to that of the wave in the region \( a < z \).

c) Fourier Analysis

Apply the transformations between an incident monochromatic wave and the wave in and beyond the medium to the Fourier analysis of an incident pulse of form \( f(z/c - t) \).

d) Propagation of a Sharp Wave Front

In the approximation that \( \omega n \) varies linearly with \( \omega \), deduce the waveforms in the regions \( 0 < z < a \) and \( a < z \) for an incident pulse \( \delta(z/c - t) \), where \( \delta \) is the Dirac delta function. Show that the pulse emerges out of the gain region at \( z = a \) at time \( t = a/v_g \), which appears to be earlier than when it enters this region if the group velocity is negative. Show also that inside the negative group velocity medium a pulse propagates backwards from \( z = a \) at time \( t = a/v_g < 0 \) to \( z = 0 \) at \( t = 0 \), at which time it appears to annihilate the incident pulse.

e) Propagation of a Gaussian Pulse

\(^3\)This is in contrast to the “Λ” configuration of the three-level atomic system required for “slow light” (Fig. 1) where the pump laser does not produce an inverted population, in which case an adequate classical description is simply to reverse the sign of the damping constant for the pumped oscillator.

As a more physical example, deduce the waveforms in the regions $0 < z < a$ and $a < z$ for a Gaussian incident pulse $E_0 e^{-((z/c - t)/2\tau)^2/2\tau^2} e^{i\omega_0(z/c - t)}$. Carry the frequency expansion of $\omega n(\omega)$ to second order to obtain conditions of validity of the analysis such as maximum pulselength $\tau$, maximum length $a$ of the gain region, and maximum time of advance of the emerging pulse. Consider the time required to generate a pulse of risetime $\tau$ when assessing whether the time advance in a negative group velocity medium can lead to superluminal signal propagation.

8. An Electrostatic Wave

All electrostatic fields $E$ (i.e., ones with no time dependence) can be derived from a scalar potential $V$ ($E = -\nabla V$) and hence obey $\nabla \times E = 0$. The latter condition is sometimes considered to be a requirement for electrostatic fields. Show, however, that there can exist time-dependent electric fields for which $\nabla \times E = 0$, which have been given the name “electrostatic waves”.

In particular, show that a plane wave with electric field $E$ parallel to the wave vector $k$ (a longitudinal wave) can exist in a medium with no time-dependent magnetic field if the electric displacement $D$ is zero. This cannot occur in an ordinary dielectric medium, but can happen in a plasma. (Time-independent electric and magnetic fields could, of course, be superimposed on the wave field.) Compare the potentials for the “electrostatic wave” in the Coulomb and Lorentz gauges. Discuss energy density and flow for such a wave.

Deduce the frequency $\omega$ of the longitudinal wave in a hot, collisionless plasma that propagates transversely to a uniform external magnetic field $B_0$ in terms of the (electron) cyclotron frequency,

$$\omega_B = \frac{eB_0}{mc},$$

(in Gaussian units), the (electron) plasma frequency,

$$\omega_P^2 = \frac{4\pi Ne^2}{m},$$

and the electron temperature $T$, where $e > 0$ and $m$ are the charge and mass of the electron, $c$ is the speed of light, and $N$ is the electron number density.

For a simplified analysis, you may assume that the positive ions are at rest, that all electrons have the same transverse velocity $v_\perp = \sqrt{2KT/m}$, where $K$ is Boltzmann’s constant, $T$ is the temperature, and that the densities of the ions and unperturbed electrons are uniform. Then the discussion may proceed from an (approximate) analysis of the motion of an individual electron to the resulting polarization density and dielectric constant, etc.

Such waves are called electron Bernstein waves, following their prediction via an analysis based on the Boltzmann transport equation.\(^5\) Bernstein waves were first produced

in laboratory plasmas in 1964,\textsuperscript{6} following possible detection in the ionosphere in 1963. They are now being applied in plasma diagnostics where it is desired to propagate waves below the plasma frequency.\textsuperscript{7}

9.

10.

11.

12.


Solutions

1. The merits of an oscillatory magnetic field transverse to a static magnet field in the study of individual magnetic moments were emphasized by Rabi.\(^8\) Bloch\(^9\) extended this approach to magnetic media, but it was perhaps Dicke\(^10\) who realized that the optically active medium thereby created could lead to “super-radiance”, i.e., to laser beams.

When a magnetic dipole \(\mathbf{m}\) is subject to a magnetic field \(\mathbf{B}\) it experiences a torque \(\mathbf{m} \times \mathbf{B}\) that precesses the angular momentum \(\mathbf{L} = \mathbf{m}/\Gamma\), where \(\Gamma = m/L\) is the gyromagnetic ratio of the dipole. If the magnetic dipoles are electrons, then \(\Gamma = e/2m_ec \approx 10^7\) Hz/gauss, where \(e\) and \(m_e\) are the charge and mass of the electron, and \(c\) is the speed of light. Thus,

\[
\mathbf{m} \times \mathbf{B} = \frac{d\mathbf{L}}{dt} = \frac{1}{\Gamma} \frac{d\mathbf{m}}{dt}.
\]

The precession frequency is \(\Gamma B \approx 10^7 B\) for \(B\) in gauss. We will consider magnetic fields \(B_y(t)\) of optical frequencies, \(\approx 10^{15}\) Hz, so the precession will be very slow compared to the wave frequency.

The equation of motion of a single moment, including the damping (3) of the moment to alignment with the static magnetic field that is predominantly along the \(y\) axis, is

\[
\frac{d\mathbf{m}}{dt} = \Gamma \mathbf{m} \times \mathbf{B} - \gamma \mathbf{m} (\mathbf{m} - \mathbf{\hat{y}}).
\]

The equation of motion for the magnetization \(\mathbf{M} = N\mathbf{m}\) is therefore

\[
\frac{d\mathbf{M}}{dt} + \gamma \mathbf{M} = \Gamma \mathbf{M} \times \mathbf{B} + \gamma \mathbf{M} \mathbf{\hat{y}}.
\]

For a magnetic field \(B_0\mathbf{\hat{x}} + (B_{0y} + B_y(t))\mathbf{\hat{y}} = (H_x + 4\pi M_x)\mathbf{\hat{x}} + (H_y + 4\pi M_y)\mathbf{\hat{y}},\) the components of eq. (18) are

\[
\frac{dM_x}{dt} + \gamma M_x = -\Gamma M_y B_y, \quad \frac{dM_y}{dt} + \gamma M_y = \Gamma M_x B_x + \gamma M, \quad \frac{dM_z}{dt} + \gamma M_z = \Gamma (M_x B_y - M_y B_x).
\]

The desired physical picture is that the magnetization \(\mathbf{M}\) precesses around the \(y\) axis (subject to the “slow” damping \(\gamma\), with the oscillatory magnetization \(M_y\) being only a small perturbation about this dominant motion. From eqs. (19) and (21) we see that this is a good approximation so long as \(M_y B_x \ll M_x B_y\). We choose \(B_{0y}\) to be small compared to \(B_{0y}\), and prepare the medium in an initial state with \(M_y \ll M_x\).

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latter might be accomplished, for example, by starting with $B_{0y} = 0$ so the dipoles line up with $B_{0x}$, and then turning on the field $B_y$ quickly; if the damping time is long compared to the precession period, then there is a useful interval during which the desired behavior obtains.

We are principally interested in the behavior of $M_y$ for use in calculating the index of refraction, so we take the derivative of eq. (20), noting that $M$ is constant since the medium is comprised of permanent dipoles, and insert eq. (21) to find

$$\frac{d^2 M_y}{dt^2} + \gamma \frac{dM_y}{dt} = \Gamma \frac{dM_z}{dt} B_x = \Gamma B_x [\Gamma (M_x B_y - M_y B_x) - \gamma M_z]$$

$$= \Gamma^2 B_x (M_x H_y - M_y H_x) - \gamma \left( \frac{dM_y}{dt} + \gamma M_y - \gamma M \right).$$ (22)

Assuming that $M \ll B_x$, then $H_x \approx B_x$ and we may approximate $\Gamma^2 B_x H_x \equiv \omega_0^2$ as being constant ($\omega_0 \approx \Gamma B_x$). Then,

$$\frac{d^2 M_y}{dt^2} + 2\gamma \frac{dM_y}{dt} + (\gamma^2 + \omega_0^2) M_y = \Gamma^2 B_x H_x \frac{M_x}{H_x} H_y + \gamma^2 M = \omega_0^2 \frac{M_x}{H_x} H_y + \gamma^2 M.$$ (23)

The term $\gamma^2 M$ leads to a constant component $M_y = \gamma^2 M / (\gamma^2 + \omega_0^2)$, which we can ignore since we assume that the damping constant $\gamma$ is small compared to the frequency $\omega_0 \approx \Gamma B_x$. Our main interest is the behavior of the system when a wave is present, $H_y = H_{0y} e^{-i\omega t}$ and $M_y = M_{0y} e^{-i\omega t}$, at frequency $\omega \gg \omega_0$, in which case we can regard $M_x$ as effectively constant over a few cycles of the high frequency wave. Inserting this hypothesis in eq. (23), we find that the high-frequency part of $M_y$ obeys

$$M_y = \frac{M_x}{H_x} \frac{\omega_0^2 H_y}{\omega_0^2 - \omega^2 + \gamma^2 - 2i\gamma \omega}.$$ (24)

Recall that we need $M_y H_x \ll M_x H_y$ for the dominant behavior of the magnetization to be precession about the $y$ axis. From eq. (24) we see that this would not hold for frequency $\omega$ close to $\omega_0$ (since we assume that $\gamma \ll \omega_0$). But we consider $\omega$ of optical frequencies, so $\omega \gg \omega_0$ for any reasonable value of $B_x$, as noted previously.

The index of refraction for a wave propagating in the $z$ direction with magnetic field along the $y$ axis is therefore

$$n(\omega) = \sqrt{\mu} \approx 1 + 2\pi \frac{M_y}{H_y} = 1 + 2\pi \frac{M_x}{H_x} \frac{\omega_0^2 (\omega_0^2 - \omega^2 + \gamma^2 + 2i\gamma \omega)}{(\omega_0^2 - \omega^2 + \gamma^2)^2 + 4\gamma^2 \omega^2}.$$ (25)

In particular, during the part of the precession cycle when the magnetization $M_x$ is antialigned with $B_x \approx H_x$, $\text{Im}(n) < 0$, and a propagating wave $H_{0y} e^{i\omega(nz - ct)}$ is amplified during its passage through the medium.

It appears difficult to realize the desired precession of $\mathbf{M}$ about the $y$ axis as suggested, since $B_y$ would have to reach full strength in less than the damping time $1/\gamma$, and no actual laser has (I believe) been built utilizing a magnetic medium. The interest of this problem is in providing a classical viewpoint of how wave amplification is possible in principle by preparing a medium in an optically active state.
2. In both definitions of magnetostatics the electric field \( \mathbf{E} \) has no time dependence, \( \partial \mathbf{E} / \partial t = 0 \), so the magnetic field \( \mathbf{B} \) obeys \( \partial^2 \mathbf{B} / \partial t^2 = 0 \), as follows on taking the time derivative of Faraday’s law,

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \tag{26}
\]

(in Gaussian units). In principle, this is consistent with a magnetic field that varies linearly with time, \( \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r})t \). However, this leads to arbitrarily large magnetic fields at early and late times, and is excluded on physical grounds. Hence, any magnetic field \( \mathbf{B} \) that coexists with only static electric fields is also static.

There remains the possibility of a “magnetostatic wave” in a magnetic medium that involves the magnetic field \( \mathbf{H}_{\text{wave}} \) and magnetization density \( \mathbf{M}_{\text{wave}} \) which are related by

\[
0 = \mathbf{B}_{\text{wave}} = \mathbf{H}_{\text{wave}} + 4\pi \mathbf{M}_{\text{wave}}. \tag{27}
\]

If there are no free currents in the medium, and any electric field is static, then the fourth Maxwell equation is simply

\[
\nabla \times \mathbf{H} = 0, \tag{28}
\]

which defines a subset of magnetostatic phenomena.

a) **Ferromagnetic Spin Waves**

Consider a ferromagnetic material that consists of a single macroscopic domain with magnetization density \( \mathbf{M} = M_0 \hat{z} + \mathbf{m}(\mathbf{r}, t) \), where \( M_0 \) is constant and \( m \ll M_0 \). We suppose there are no external electromagnetic fields. Associated with the magnetization \( \mathbf{M} \) are magnetic fields \( \mathbf{B} \) and \( \mathbf{H} \) whose values depend on the geometry of the sample. We suppose that the weak time-dependent magnetic fields due to \( \mathbf{m} \) lead to even weaker time-dependent electric fields, such that the situation is essentially magnetostatic. The consistency of this assumption will be confirmed at the end of the analysis.

The ferromagnetism is due to electron spins, whose dominant interaction is the quantum mechanical exchange interaction, in the absence of external fields. For a weak perturbation \( \mathbf{m} \) of the magnetization, the exchange interaction preserves the magnitude of the magnetization, so its time evolution has the form of a precession,

\[
\frac{d\mathbf{M}}{dt} = \mathbf{\Omega} \times \mathbf{M}. \tag{29}
\]

As this is the same form as the precession of a magnetic moment in an external magnetic field (Prob. 1), the precession vector \( \mathbf{\Omega} \) is often written as a gyromagnetic factor \( \Gamma = e/2m_e c \approx 10^7 \text{ Hz/gauss} \) times an effective magnetic field \( \mathbf{B}_{\text{eff}} \) (or \( \mathbf{H}_{\text{eff}} \)). Here, \( e > 0 \) and \( m_e \) are the charge and mass of the electron, and \( c \) is the speed of light. For a weak perturbation in an isotropic medium,\(^{11}\)

\[
\mathbf{B}_{\text{eff}} = \alpha \nabla^2 \mathbf{m}, \tag{30}
\]

where $\alpha$ is a constant of the medium.

Then, the equation of motion of the magnetization $\mathbf{m}$ is

$$\frac{d\mathbf{m}}{dt} = \alpha \Gamma \nabla^2 \mathbf{m} \times \mathbf{M}. \tag{31}$$

For a plane-wave perturbation, whose phase factor is $e^{i(kr-\omega t)}$, the equation of motion (31) becomes

$$i\omega \mathbf{m} = \alpha \Gamma k^2 \mathbf{m} \times \mathbf{M}_0 \hat{z}. \tag{32}$$

This is satisfied by a circularly polarized wave,

$$\mathbf{m} = m(\hat{x} + i\hat{y})e^{i(kr-\omega t)}, \tag{33}$$

that obeys the quadratic dispersion relation\(^{12}\)

$$\omega = \alpha \Gamma M_0 k^2, \tag{34}$$

which implies that $\omega \ll c k$ in physical materials, where $c$ is the speed of light.

Hence, the electric fields are much smaller than the magnetic fields associated with the time-dependent magnetization $\mathbf{m}$, so that $\nabla \times \mathbf{H} = 0$ to a good approximation, and we may use the term “magnetostatic” to describe the waves. These waves of magnetization are, however, better termed “spin waves”, whose quanta are called “magnons”.

b) **Rotating Magnetostatic Modes in a Ferrite Cylinder**

In the magnetostatic approximation the fields $\mathbf{B}$ and $\mathbf{H}$ obey

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\mathbf{H} + 4\pi \mathbf{M}) = 0, \quad \nabla \times \mathbf{H} = 0, \tag{35}$$

where the field $\mathbf{B}$ but not $\mathbf{H}$ and $\mathbf{M}$ must be static (or at least so slowly varying in time that the resulting electric field is small compared to $\mathbf{B}$). We first consider a ferrite of arbitrary shape of characteristic length $a$ in a uniform external magnetic field $\mathbf{B}_{\text{ext}} = \mathbf{H}_{\text{ext}} = H_0 \hat{z}$. We suppose that this field is strong enough to induce a uniform magnetization $M_0 \hat{z}$ throughout the sample.

For waves with weak spatial dependence as we shall assume, the exchange interaction is negligible, since it varies as the second spatial derivative of $\mathbf{M}$. Then, the spins interact primarily with the local magnetic field $\mathbf{B}$ according to

$$\frac{d\mathbf{M}}{dt} = \Gamma \mathbf{B} \times \mathbf{M} = \Gamma \mathbf{H} \times \mathbf{M}. \tag{36}$$

We consider a perturbation $\mathbf{m}$ to the magnetization that has frequency $\omega$ and wavelength large compared to the size of the the sample. Then the total magnetization can be written

$$\mathbf{M} = M_0 \hat{z} + \mathbf{m} e^{-i\omega t}, \tag{37}$$

where \( m \ll M_0 \). Similarly, we write the magnetic field inside the sample as
\[
\mathbf{B} = B_z \hat{z} + be^{-i\omega t}, \quad \mathbf{H} = H_z \hat{z} + he^{-i\omega t},
\] (38)
where \( B_z = H_z + 4\pi M_0 \) and \( H_z = H_0 - 4\pi N_z M_0 \) are the sum of the external field and that due to the uniform magnetization \( M_0 \hat{z} \), and so are also uniform for spheroidal (and cylindrical) samples whose axis is the \( z \) axis. The “demagnetization” factor \( N_z \) varies between 1 for a disk and 0 for a cylinder. The perturbation \( m \) exists only inside the sample, but the corresponding perturbations \( b \) and \( h \) exist outside the sample as well.

Inserting eqs. (37) and (38) in the equation of motion (36), we keep only the first-order terms to find
\[
-i\omega \mathbf{m} = \Gamma \hat{z} \times (M_0 \mathbf{h} - H_z \mathbf{m}),
\] (41)
whose components are
\[
m_x = \frac{i\Gamma}{\omega} (M_0 h_y - H_z m_y),
\]
\[
m_y = -\frac{i\Gamma}{\omega} (M_0 h_x - H_z m_x),
\]
\[
m_z = 0.
\] (42)

We solve for \( \mathbf{m} \) in terms of \( \mathbf{h} \) as
\[
m_x = \alpha h_x - i\beta h_y,
\]
\[
m_y = i\beta h_x + \alpha h_y,
\]
\[
\alpha = \frac{\Gamma^2 H_z M_0}{\Gamma^2 H_z^2 - \omega^2}, \quad \beta = \frac{\Gamma M_0 \omega}{\Gamma^2 H_z^2 - \omega^2}.
\] (45)

For example, a disk with \( c = 0 \) has \( \zeta = 0 \) also, and \( \mathbf{H} = -4\pi \mathbf{M}, \mathbf{B} = 0 \). For a sphere, \( c = 1 \) is a sphere, \( 1 < c < \infty \) is a prolate spheroid, and \( c = \infty \) is a cylinder. For an oblate spheroid of aspect ratio \( c \), the “radial” coordinate is \( \zeta = c/\sqrt{1-c^2} \), and the magnetic field due to the uniform magnetization is
\[
\mathbf{H} = -4\pi \mathbf{M} \{1 - \zeta \left[(1 + \zeta^2) \cot^{-1} \zeta - \zeta\right]\}.
\] (39)

For a prolate spheroid of aspect ratio \( c \), the “radial” coordinate is \( \eta = c/\sqrt{c^2 - 1} \), and
\[
\mathbf{H} = -4\pi \mathbf{M} \{1 - \eta \left[(1 - \eta^2) \coth^{-1} \eta + \eta\right]\}.
\] (40)

For a cylinder with \( c \to \infty \), we have \( \eta = 1 \), \( \coth^{-1} \eta = 0 \) and \( \mathbf{H} = 0, \mathbf{B} = 4\pi \mathbf{M} \). The fields for a sphere can also be obtained from the limit \( c \to 1 \), \( \eta \to \infty \) and \( \coth^{-1} \eta \to 1/\eta + 1/3\eta^3 \). The expressions in braces in eqs. (39) and (40) correspond to the demagnetization factor \( N_z \) introduced in the main text.
For later use, we note that in cylindrical coordinates, \((r, \theta, z)\), eq. (44) becomes

\[
\begin{align*}
m_r &= \alpha h_r - i\beta h_\theta, \\
m_\theta &= i\beta h_r + \alpha h_\theta.
\end{align*}
\] (46)

As we are working in the magnetostatic limit (35), we also have

\[
\nabla \cdot \mathbf{b} = \nabla \cdot (\mathbf{h} + 4\pi \mathbf{m}) = 0, \quad \nabla \times \mathbf{h} = 0.
\] (47)

Hence, the perturbation \(\mathbf{h}\) can be derived from a scalar potential,

\[
\mathbf{h} = -\nabla \phi,
\] (48)

and so,

\[
\nabla^2 \phi = 4\pi \nabla \cdot \mathbf{m}.
\] (49)

Outside the sample the potential obeys Laplace’s equation,

\[
\nabla^2 \phi = 0 \quad \text{(outside)},
\] (50)

while inside the sample we find, using eq. (44),

\[
(1 + 4\pi\alpha) \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{(inside)}.
\] (51)

The case of an oblate or prolate spheroid with axis along the external field has been solved with great virtuosity by Walker,\(^{14}\) following the realization that higher-order modes deserved discussion.\(^{15}\) Here, we content ourselves with the much simpler case of a long cylinder whose axis is along the external field, for which the lowest-order spatial mode was first discussed by Kittel.\(^{16}\) We consider only the case of waves with no spatial dependence along the axis of the cylinder.

With these restrictions, both eqs. (50) and (51) reduce to Laplace’s equation in two dimensions. We can now work in a cylindrical coordinate system \((r, \theta, z)\), where appropriate 2-D solutions to Laplace’s equation have the form

\[
\begin{align*}
\phi(r < a, \theta) &= \sum \frac{r^n}{a^n}(A_n e^{in\theta} + B_n e^{-in\theta}), \\
\phi(r > a, \theta) &= \sum \frac{a^n}{r^n}(A_n e^{in\theta} + B_n e^{-in\theta}),
\end{align*}
\] (52, 53)

which is finite at \(r = 0\) and \(\infty\), has period \(2\pi\) in \(\theta\), and is continuous at the boundary \(r = a\).

---


The boundary conditions at \( r = a \) in the magnetostatic limit (47) are that \( b_r \) and \( h_\theta \) are continuous. The latter condition is already satisfied, since

\[
h_\theta = -(1/r)\partial \phi/\partial \theta.
\]

We note that

\[
b_r = h_r + 4\pi m_r = (1 + 4\pi \alpha)h_r - 4\pi i\beta h_\theta,
\]

recalling eq. (46). Using eqs. (52) and (53) we find that continuity of \( b_r \) at \( r = a \) requires

\[
\sum_{n} \frac{1}{a} \left[ (1 + 2\pi \alpha + 2\pi \beta)A_n e^{in\theta} + (1 + 2\pi \alpha - 2\pi \beta)B_n e^{-in\theta} \right] = 0.
\]

(55)

Nontrivial solutions are possible only if \( 2\pi (\alpha \pm \beta) = -1 \), in either of which case there is an infinite set of modes that are degenerate in frequency. Using eq. (45), we find the “resonance” frequency to be

\[
\omega = \pm \Gamma(H_0 + 2\pi M_0),
\]

(56)

noting that for a cylinder the demagnetization factor is \( N_z = 0 \), so that \( H_z = H_0 \), as is readily deduced by elementary arguments. Since we consider frequency \( \omega \) to be positive, we see that the two solutions (56) correspond to two signs of \( H_0 \), and are essentially identical.

For spheroidal samples, the modes are enumerated with two integer indices, and are not all degenerate in frequency, as discussed by Walker.

We close our discussion by showing that the electric field of the wave is much smaller than the magnetic field. The scalar potential for mode \( n \) is

\[
\phi_n(r < a) = \frac{r^n}{a^n} e^{i(n\theta - \omega t)}, \quad \phi_n(r > a) = \frac{a^n}{r^n} e^{i(n\theta - \omega t)}.
\]

(57)

We see that for \( n > 0 \) the potential rotates with angular velocity \( \Omega_n = \omega/n \) about the \( z \) axis. The potential is maximal at \( r = a \), so consistency with special relativity requires that

\[
v(r = a) = \frac{a\omega}{n} \ll c,
\]

(58)

which appears to have been (barely) satisfied in typical experiments. We also see that for high mode number the spatial variation of the wave becomes rapid, and the neglect of the exchange interaction is no longer justified.

The magnetic field \( \mathbf{h} = -\nabla \phi \) of mode \( n \) has components

\[
h_r(r < a) = -n \frac{r^n}{a^n} e^{i(n\theta - \omega t)} = -\frac{n}{r} \phi_n, \quad h_r(r > a) = n \frac{a^n}{r^{n+1}} e^{i(n\theta - \omega t)} = \frac{n}{r} \phi_n,
\]

(59)

\[
h_\theta(r < a) = ih_r(r < a), \quad h_\theta(r > a) = -ih_r(r > a).
\]

(60)

The monopole mode, \( n = 0 \), does not exist. The lowest mode is \( n = 1 \), which corresponds to a uniform, transverse field \( \mathbf{h} \) that rotates about the \( z \) axis with angular velocity \( \omega \).
From eq. (46) we find the magnetization to be
\[
m = -\frac{h}{2\pi}
\] (61)
for all modes (for \(r < a\) only, of course), so the magnetization of mode \(n\) also rotates with angular velocity \(\omega/n\).

The magnetic field \(b = h + 4\pi m\) is then,
\[
b(r < a) = -h(r < a), \quad b(r > a) = h(r > a).
\] (62)

Using either the \(r\) or \(\theta\) component of Faraday’s law, we find that the associated electric field \(e\) has only a \(z\) component,
\[
e = \frac{\omega}{c} \hat{\phi} \hat{z},
\] (63)
both inside and outside the cylinder (consistent with continuity of the tangential component of the electric field at a boundary, and with \(\nabla \cdot e = 0\)). The ratio of the electric to the magnetic field of mode \(n\) at \(r = a\) is
\[
\left| \frac{e_z}{b_r} \right| = \frac{a\omega}{nc},
\] (64)
which is small so long as condition (58) is satisfied. Hence, the condition that \(a\omega/c \ll 1\) is doubly necessary for the validity of this analysis.

Because the magnetization \(m\) is moving (rotating), there is an associated electric polarization \(p\) according to special relativity,\(^{17}\)
\[
p = \gamma \frac{v}{c} \times m.
\] (65)

For mode \(n\) we have \(v = \omega r \hat{\theta}/n \ll c\), so \(\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1\), and
\[
p = -\frac{\omega r m_r}{nc} \hat{z} = \frac{\omega r h_r}{2\pi nc} \hat{z} = -\frac{\omega}{2\pi c} \phi \hat{z} = -\frac{e}{2\pi}.
\] (66)

The electric displacement is related by \(d = e + 4\pi p\), which has the value
\[
d(r < a) = -e = -\frac{\omega}{c} \phi \hat{z}, \quad d(r > a) = e = \frac{\omega}{c} \phi \hat{z},
\] (67)

The fourth Maxwell equation now implies that
\[
\nabla \times h = -\nabla \times \nabla \phi = \frac{1}{c} \frac{\partial d}{\partial t} = \pm i \frac{\omega^2}{c^2} \phi \hat{z}
\] (68)

Thus, there is a small violation of the magnetostatic conditions (47), but this is second order in the small quantity \(a\omega/c\) (noting that \(\nabla \phi \approx \phi/a\)).

3. The phenomenon of self-induced transparency for electromagnetic pulses in a magnetic medium was first analyzed and demonstrated in the laboratory by McCall and Hahn.\textsuperscript{18} See also the review by Lamb.\textsuperscript{19}

As in the related problem of wave amplification in a magnetic medium (Prob. 1), we note that when a magnetic dipole $\mathbf{m}$ is subject to a magnetic field $\mathbf{B}$ it experiences a torque $\mathbf{m} \times \mathbf{B}$ that precesses the angular momentum $\mathbf{L} = \mathbf{m}/\Gamma$, where $\Gamma = m/L$ is the gyromagnetic ratio of the dipole. If the magnetic dipoles are electrons, then $\Gamma = e/2m_e c \approx 10^7$ Hz/ gauss, where $e$ and $m_e$ are the charge and mass of the electron, and $c$ is the speed of light. Thus,

$$\mathbf{m} \times \mathbf{B} = \frac{d\mathbf{L}}{dt} = \frac{1}{\Gamma} \frac{d\mathbf{m}}{dt}. \tag{69}$$

The equation of motion of a single moment is

$$\frac{d\mathbf{m}}{dt} = \Gamma \mathbf{m} \times \mathbf{B}, \tag{70}$$

so the equation of motion for the magnetization $\mathbf{M} = N\mathbf{m}$ is therefore

$$\frac{d\mathbf{M}}{dt} = \Gamma \mathbf{M} \times \mathbf{B}. \tag{71}$$

For a magnetic field $B_x \hat{x} + B_y(t) \hat{y}$, the components of eq. (71) are

$$\frac{dM_x}{dt} = -\Gamma B_y M_z, \tag{72}$$

$$\frac{dM_y}{dt} = \Gamma B_x M_z \equiv \omega_0 M_z, \tag{73}$$

$$\frac{dM_z}{dt} = \Gamma (B_y M_x - B_x M_y) = \Gamma B_y M_x - \omega_0 M_y, \tag{74}$$

where $\omega_0 \equiv \Gamma B_x$.

The hint is to consider the pulse area function,

$$A(z, t) = \Gamma \int_{-\infty}^{t} B_y(z, t') \, dt', \tag{75}$$

whose time derivatives are

$$\dot{A} = \Gamma B_y, \quad \ddot{A} = \Gamma \ddot{B}_y. \tag{76}$$

First we consider the case that $\omega_0 = 0$, for which the equations of motion (72)-(74) reduce to

$$\dot{M}_x = -\Gamma B_y M_z, \quad \dot{M}_y = \text{const.}, \quad \dot{M}_z = \Gamma B_y M_x. \tag{77}$$


We readily find two solutions for $M_x$ and $M_z$:

$$M_x = M_0 \sin A, \quad M_z = -M_0 \cos A,$$

and

$$M_x = M_0 \cos A, \quad M_z = M_0 \sin A.$$

We seek a pulse solution for $B_y$, so $A(z, t = -\infty) = 0$ for any finite $z$, with the initial (and final) condition that the magnetization is aligned along the $x$ axis, i.e., $M_x(t = -\infty) = M_0$ and $M_z(t = -\infty) = 0$. Clearly, the solution (79) is of the desired character.

Turning to the case of nonzero $B_0$, i.e., nonzero $\omega_0$, we extrapolate solution (79) by supposing that

$$M_z = F(\omega_0) M_0 \sin A,$$

where $F(0) = 1$. A related trial solution for $M_x$ is

$$M_x = M_0 [F(\omega_0)(\cos A - 1) + 1],$$

which preserves the initial condition that $M_x(t = -\infty) = M_0$.

To verify these trial solutions, we differentiate eq. (74) and combine with eqs. (72)-(73) and (76):

$$\ddot{M}_z = \Gamma \dot{B}_y M_x + \Gamma B_y \dot{M}_x - \omega_0 \dot{M}_y = \ddot{A} M_x - \Gamma^2 B_y^2 M_z - \omega_0^2 M_z$$

$$= M_0 \ddot{A} [F(\cos A - 1) + 1] - M_0 \dot{A}^2 F \sin A - M_0 \omega_0^2 F \sin A.$$

But also,

$$\ddot{M}_z = \frac{d^2}{dt^2} (F M_0 \sin A) = \frac{d}{dt} (F M_0 \dot{A} \cos A) = F M_0 \ddot{A} \cos A - F M_0 \dot{A}^2 \sin A.$$  

Combing eqs. (82) and (83) we find

$$\ddot{A} = \frac{\omega_0^2 F}{1 - F} \sin A = \frac{1}{\tau^2} \sin A,$$

where the constant $\tau$ is defined by

$$\frac{1}{\tau^2} = \frac{\omega_0^2 F}{1 - F}, \quad \text{and so} \quad F(\omega_0) = \frac{1}{1 + \omega_0^2 \tau^2},$$

which obeys $F(0) = 1$ as required.

It turns out that we do not need the most general solution of the Mathieu equation (84). It suffices to use the particular solution found on multiplying eq. (84) by $\dot{A}$:

$$\dot{A} \ddot{A} = \frac{1}{\tau^2} \dot{A} \sin A,$$
which integrates to
\[ \frac{\dot{A}^2}{2} = K - \frac{\cos A}{\tau^2}. \]  
(87)

The pulse area \( A \) and its time derivative vanish at \( t = -\infty \) for any finite \( z \), so the constant of integration is \( K = 1/\tau^2 \). Thus,
\[ \frac{\dot{A}^2}{2} = \frac{1 - \cos A}{\tau^2} = \frac{2}{\tau^2} \sin^2 \frac{A}{2}. \]  
(88)

Taking the square root, we have
\[ \frac{\dot{A}}{2} = \frac{1}{\tau} \sin \frac{A}{2}, \]  
(89)

or
\[ \frac{d(A/2)}{\sin(A/2)} = \frac{dt}{\tau}, \]  
(90)

which integrates to
\[ \ln[\tan(A/4)] = \frac{t}{\tau} + K. \]  
(91)

A clever trick is to evaluate the integration constant \( K \) at the time \( t_0(z) \) such that half the pulse has arrived at position \( z \): \( A(t_0) = A_{\text{max}}/2 \). If we define \( A_{\text{max}} \) to be \( 2\pi \), a compact solution is obtained. Thus,
\[ A(t_0) = \pi, \quad \tan \frac{A(t_0)}{4} = 1, \]  
(92)

\[ \ln[\tan(A(t_0)/4)] = 0 = \frac{t_0}{\tau} + K, \]  
(93)

so \( K = -t_0/\tau \). The solution (93) is now
\[ \ln[\tan(A/4)] = \frac{t - t_0}{\tau}, \]  
(94)

so
\[ \tan \frac{A}{4} = e^{(t-t_0)/\tau}, \]  
(95)

and
\[ A = 4 \tan^{-1}[e^{(t-t_0)/\tau}]. \]  
(96)

Since we desire a solution that describes a traveling pulse with velocity \( v \), we identify \( t_0 \) at point \( z \) as \( z/v \), and write
\[ A = 4 \tan^{-1} f, \quad \text{with} \quad f = e^{(t-z/v)/\tau}. \]  
(97)

We see that \( \tau \) is the characteristic width of the pulse (although this is even clearer once we have deduced eq. (108).)
To evaluate the components of the magnetization $M$, we need explicit forms for $\sin A$ and $\cos A$, which will also permit a confirmation that the solution (97) satisfies the Mathieu equation (84).

Since $\tan(A/4) = f$, we have

$$
\cos^2 \frac{A}{4} = \frac{1}{1+f^2}, \quad \text{and} \quad \sin^2 \frac{A}{4} = \frac{f^2}{1+f^2}.
$$

(98)

Then,

$$
\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} = 4 \sin \frac{A}{4} \cos \frac{A}{4} \left( \cos^2 \frac{A}{4} - \sin^2 \frac{A}{4} \right)
= 4 \sqrt{\frac{f^2}{1+f^2}} \cdot \frac{1}{1+f^2} \left( \frac{1-f^2}{1+f^2} \right) = 4 \frac{f(1-f^2)}{(1+f^2)^2} = 4 \frac{1-f}{(1+f)^2}
= -2 \tanh \left( \frac{t-z/v}{\tau} \right) \text{sech} \left( \frac{t-z/v}{\tau} \right).
$$

(99)

Likewise,

$$
\cos A = 1 - 2 \sin^2 \frac{A}{2} = 1 - 8 \sin^2 \frac{A}{4} \cos^2 \frac{A}{4} = 1 - \frac{8f^2}{(1+f^2)^2} = 1 - \frac{8}{(1+f)^2}
= 1 - 2 \text{sech}^2 \left( \frac{t-z/v}{\tau} \right).
$$

(100)

Also,

$$
\dot{A} = \frac{4}{1+f^2} \dot{f} = \frac{4}{\tau} \frac{f}{1+f^2} = \frac{4}{\tau f} = \frac{2}{\tau} \text{sech} \left( \frac{t-z/v}{\tau} \right),
$$

(101)

noting that $\dot{f} = f/\tau$. Hence,

$$
\ddot{A} = \frac{4}{\tau} \left( \frac{\ddot{f}}{1+f^2} - \frac{2f^2 \dot{f}}{(1+f^2)^2} \right) = \frac{4}{\tau^2} \frac{f(1-f^2)}{(1+f^2)^2} = \frac{\sin A}{\tau^2},
$$

(102)

in agreement with eq. (84).

The components of the time-dependent magnetization are obtained as follows.

$$
\frac{M_x}{M_0} = 1 + F(\cos A - 1) = 1 - 2F \text{sech}^2 \left( \frac{t-z/v}{\tau} \right).
$$

(103)

We see that $M_x(t = +\infty) = 1$, and $M_x(0) \approx -1$ for $\omega_0 \tau \gg 1 \ (F \approx 1)$.

$$
\frac{M_z}{M_0} = F \sin A = -2F \tanh \left( \frac{t-z/v}{\tau} \right) \text{sech} \left( \frac{t-z/v}{\tau} \right).
$$

(104)
In the limit that \( \omega_0\tau \ll 1 \) (\( F \approx 1 \)), we have
\[
\frac{M_x^2 + M_z^2}{M_0^2} \approx 1 - 4 \sech^2 \left( \frac{t - z/v}{\tau} \right) + 4 \sech^4 \left( \frac{t - z/v}{\tau} \right) + 4 \tanh^2 \left( \frac{t - z/v}{\tau} \right) \sech^2 \left( \frac{t - z/v}{\tau} \right) = 1.
\]
(105)

The behavior of the magnetization in the \( x-z \) plane is essentially a single revolution about the \( y \) axis, beginning and ending with \( M_x = M_0 \), \( M_z = 0 \), and with \( M_x = -M_0 \) at the position of the peak of the traveling pulse.

To find \( M_y \) we use eq. (74) for \( \dot{M}_z \) together with eqs. (76), (80)-(81), (85) and (101):
\[
\omega_0 M_y = \Gamma B_y M_x - \dot{M}_z = \Gamma B_y M_0 [1 + F(\cos A - 1)] - F\dot{A} \cos A = \Gamma B_y M_0 (1 - F)
\]
\[
= \Gamma B_y \omega_0^2 \tau^2 F M_0 = \omega_0^2 \tau^2 F M_0 \dot{A} = 2 \omega_0^2 \tau F M_0 \sech \left( \frac{t - z/v}{\tau} \right).
\]
(106)

Thus,
\[
M_y = 2 \omega_0 \tau^2 F M_0 \sech \left( \frac{t - z/v}{\tau} \right),
\]
(107)

and
\[
B_y = \frac{M_y}{\Gamma \omega_0 \tau^2 F M_0} = \frac{2}{\Gamma \tau} \sech \left( \frac{t - z/v}{\tau} \right) = \frac{2 B_0}{\omega_0 \tau} \sech \left( \frac{t - z/v}{\tau} \right).
\]
(108)

This hyperbolic secant pulse propagates without distortion or attenuation, with the leading “edge” of the pulse putting energy into the medium by flipping the magnetic dipoles, and the trailing edge of the pulse extracting the same energy by flipping the dipoles back to their original position.

Since \( M_y \) is proportional to \( B_y \), we see that the equations of motion (72)-(74) are nonlinear. The pulse height of the soliton wave (108) cannot be chosen arbitrarily, as in linear wave propagation, but must be inversely proportional to the pulsewidth \( \tau \). In the interesting limit that \( \omega \tau \ll 1 \), i.e., where the pulse width is short compared to the Larmor precession period, the peak field strength of the pulse is large compared to the static field \( B_0 \) although the wave magnetization \( M_y \) is small compared to \( M_0 \).

From Faraday’s law, we deduce that the electric field of the pulse is in the \( x \) direction, with
\[
E_x = \frac{v}{c} B_y.
\]
(109)

Taking the curl of the fourth Maxwell equation (assuming the medium to have dielectric constant \( \epsilon = 1 \)), we find
\[
\nabla^2 \mathbf{H} - \nabla (\nabla \cdot \mathbf{H}) = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}.
\]
(110)

Since
\[
H_y = B_y - 4\pi M_y = (1 - 4\pi \Gamma \omega_0 \tau^2 F M_0) B_y = \left( 1 - 4\pi \omega_0^2 \tau^2 F \frac{M_0}{B_0} \right) B_y,
\]
(111)
the y component of eq. (110) tells us that
\[
v = c \sqrt{1 - 4\pi\omega_0^2\tau^2 \frac{F_1 M_0}{B_0}}. \tag{112}
\]

The ratio of $M_0$ to $B_0$ in a magnetic medium can be as large as the effective permeability, i.e., of order $10^3$. In practice, not only is $\omega_0 \tau \ll 1$, but also $\omega_0^2 \tau^2 M_0 / B_0 \ll 1$, so the soliton velocity $v$ is approximately $c$.

4. This problem is based on a paper by R.Y. Chiao, E. Garmire and C.N. Townes, Phys. Rev. Lett. 13, 479 (1964). We work in Gaussian units.

a) We recall that the electric displacement is related by
\[
D = E + 4\pi P = \epsilon E, \tag{113}
\]
so the dielectric “constant” $\epsilon$ follows from eq. (113) as
\[
\epsilon = 1 + 4\pi\chi_1 + 4\pi\chi_3 E^2 + \ldots \equiv \epsilon_1 + \epsilon_3 E^2 + \ldots. \tag{114}
\]
The index of refraction $n$ is, of course, related by
\[
n = \sqrt{\epsilon} \approx 1 + 2\pi\chi_1 + 2\pi\chi_3 E^2 + \ldots. \tag{115}
\]

For a typical gas at STP, $n - 1 \approx \text{few} \times 10^{-4}$, so we estimate the linear susceptibility as
\[
\chi_1 \approx \frac{n - 1}{2\pi} \approx 10^{-4}. \tag{116}
\]

For a somewhat more precise estimate we use the given facts about nitrogen. The index of liquid nitrogen is related to its susceptibility $\chi_L$ via $n_L = 1.2 \approx \sqrt{1 + 4\pi\chi_L}$, so $\chi_L \approx 0.44/4\pi = 0.035$. The susceptibility is proportional to the density, so the susceptibility of nitrogen gas, whose density is $\rho_{\text{gas}} = 28 \text{ g}/22.4 \text{ l} = 1.25 \times 10^{-3} \text{ g/cm}^3$, is given by $\chi_L = \chi_L \rho_{\text{gas}} / \rho_L = 0.035 \times 1.25 \times 10^{-3} / 0.8 = 5.5 \times 10^{-5}$. The index of the gas is $n \approx \sqrt{1 + 4\pi\chi_1} \approx 1 + 2\pi\chi_1 \approx 1.00035$. The actual value for the index of nitrogen gas at STP is $n = 1.00030$.

We estimate that the nonlinear term in the polarizability is comparable to the linear term when the external field is barely sufficient to ionize a nitrogen atom. For this the field strength would need to be about $6 \text{ V/angstrom}$ (since air is transparent for light up to about $6 \text{ eV}$, and the size of a nitrogen atom is about $1 \text{ angstrom}$). Recalling that $1 \text{ V} = 300 \text{ statvolt}$, we need field $E_{\text{ionize}} \approx (6/300) \times 10^8 = 2 \times 10^6 \text{ statvolt/cm}$. Thus, we estimate
\[
\chi_3 \approx \frac{\chi_1}{E_{\text{ionize}}^2} \approx \frac{5 \times 10^{-5}}{(2 \times 10^6)^2} \approx 1.25 \times 10^{-17}. \tag{117}
\]
b) We consider the fate of a ray that is initially at angle $\theta \approx \lambda/w$ to the axis of the laser beam, shown as a dashed line in the figure. The index of refraction is varying with position along this ray, reaching a value $n_{\text{max}}$ on the axis of the laser beam. In a simplified view, this ray would like to continue across the axis of the beam back into the region of low index of refraction, where $n \approx 1$. However, a ray is subject to total internal reflection when going from high index to low index if

$$n_{\text{max}} \sin \theta > 1,$$

(118)

according to Snell’s law. From the figure, we see that

$$\sin \theta_i \approx \cos \theta.$$

(119)

Taking $n_{\text{max}}$ from eq. (114), the ray will be totally internally reflected, and hence trapped in a tube, if

$$(1 + 2\pi \chi_1 + 2\pi \chi_3 E^2) \left(1 - \frac{\theta^2}{2}\right) \approx 1.$$

(120)

This requires a laser electric field strength of

$$E^2 \approx \frac{\theta^2}{4\pi \chi_3} \approx \frac{(\lambda/w)^2}{4\pi \chi_3}.$$

(121)

The corresponding laser power $P$ is given in terms of the Poynting vector $S$ as

$$P = S \cdot \text{Area} \approx \frac{c}{4\pi} E^2 w^2 \approx \frac{c(\lambda/w)^2 w^2}{(4\pi)^2 \chi_3} = \frac{c}{4\pi} \left(\frac{\lambda}{4\pi}\right)^2 \approx 3 \times 10^{10} \cdot (5 \times 10^{-6})^2 \approx 10^{17} \, \text{erg/s} = 10^{10} \, \text{W} = 10 \, \text{GW}. \quad (122)$$

Such powers can be achieved in table-top lasers, where a pulse of is compressed to 100 ps and amplified to 1 J. We note that for the focused laser beam to be just below the ionization threshold, the focus angle $\theta$ obeys

$$\theta \approx \sqrt{4\pi \chi_3 E^2} \approx \sqrt{4\pi \chi_1} \approx \sqrt{7 \times 10^{-4}} \approx \frac{1}{40}. \quad (123)$$

Since the focus angle is roughly the ratio $D/f$ of the diameter $D$ of the focusing lens to its focal length $f$, we see that a “long” lens of $f/D \approx 40$ must be used. The spot size $w$, and hence the approximate diameter of the self-trapped filament of light, is given by $w \approx \lambda/\theta \approx 40\lambda \approx 20 \, \mu\text{m}$.
c) The wave equation in the nonlinear medium is
\[ \nabla^2 E = \frac{n^2}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{\epsilon_1 + \epsilon_3 E^2}{c^2} \frac{\partial^2 E}{\partial t^2}, \]  
(124)
where \( \epsilon_1 = 1 + 4\pi\chi_1 \) and \( \epsilon_3 = 4\pi\chi_3 \). Inserting the pulse (12) into the wave equation, we find
\[ E''_x - k_2^2 E_x = -\frac{\omega^2}{c^2} (\epsilon_1 + \epsilon_3 E^2) E_x, \]  
(125)
where \( E'_x = dE_x/dy \).

We suppose that the slowly varying waveform \( E_x \) responds only to the time average of \( E^2 \) that appears in eq. (125); of course \( \langle E^2 \rangle = E^2_x/2 \).

We define
\[ k \equiv \frac{\omega}{c}, \quad k_1 \equiv \frac{\omega}{c/n_1} = \sqrt{\epsilon_1 \omega}, \quad \text{and} \quad \Gamma^2 \equiv k_2^2 - k_1^2. \]  
(126)
Then, eq. (125) becomes
\[ E''_x = \Gamma^2 E_x - \frac{\epsilon_3 k^2}{2} E^3_x \equiv \Gamma^2 \left( E_x - \frac{2}{a^2} E^3_x \right), \]  
(127)
where the constant \( a \) is defined by
\[ a = \frac{2\Gamma}{\sqrt{\epsilon_3 k}}. \]  
(128)
Perhaps surprisingly, this nonlinear differential equation is easily integrated after multiplying by \( E_x' \),
\[ E'_x E''_x = \frac{1}{2} \frac{dE'^2_x}{dx} = \Gamma^2 \left( E_x E'_x - \frac{2}{a^2} E^3_x E'_x \right) = \Gamma^2 \left( \frac{1}{2} \frac{dE^2_x}{dx} - \frac{1}{a^2} \frac{dE^4_x}{dx} \right). \]  
(129)
Hence,
\[ E'^2_x = \Gamma^2 \left( E_x^2 - \frac{1}{a^2} E^4_x \right) + C. \]  
(130)
For a pulse, we desire that \( E_x(\pm\infty) = 0 = E'_x(\pm\infty) \), so the integration constant \( C \) is zero. Then,
\[ E'_x = \frac{dE_x}{dy} = \frac{\Gamma}{a} E_x \sqrt{a^2 - E^2_x}, \]  
(131)
which integrates to
\[ \frac{\Gamma}{a} y + \frac{D}{a} = \int \frac{dE_x}{E_x \sqrt{a^2 - E^2_x}} = -\frac{1}{a} \text{sech}^{-1} \frac{E_x}{a}, \]  
(132)
or
\[ E_x = a \text{sech}(\Gamma y + D). \]  
(133)
For a pulseshape that is symmetric about $y = 0$, we set the integration constant $D$ to zero, and have

$$E_x = \frac{2\Gamma}{\sqrt{\epsilon_3 k}} \text{sech} \Gamma y. \quad (134)$$

This pulse exhibits the hyperbolic secant shape found in many distortionless solutions (solitons) to nonlinear wave equations. Note also that the pulse amplitude is larger for narrower pulses.

5. The concept of “laser tweezers” was proposed by Ashkin in 1977, and first realized by Ashkin et al. in 1985. Aspects of this problem have been discussed from a semiclassical view by Shimizu and Sasada. Here, a completely classical approach displays the essential results quickly.

Undergraduate laboratory experiments with laser tweezers have been described by Smith et al.

a) The important hint is that the atom is polarizable, so it takes on an induced dipole moment $\mathbf{p} = \alpha \mathbf{E}$ where $\mathbf{E}$ is the electric field strength, and $\alpha$ is the atomic polarizability. We suppose this simple relation holds for a wave field as well as for a static field.

We now want the force $\mathbf{F}$ on the dipole. This is obtained from the form $(\mathbf{p} \cdot \nabla) \mathbf{E}$ which holds even when $\nabla \times \mathbf{E} \neq 0$,

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E} = \alpha (\mathbf{E} \cdot \nabla) \mathbf{E} = \frac{\alpha}{2} \nabla E^2. \quad (135)$$

So the polarizable atom can be “trapped” at a point where $E^2$ takes on a local maximum, since the force (135) is restoring for departures in any direction from that point.

There is no such place in a charge-free region of an electrostatic field, as demonstrated in part e). Thus, Earnshaw’s theorem can be extended to include the case of a polarizable atom.

However, $E^2$ can have a local maximum in a nonelectrostatic field, such as at the focus of a laser beam.

For oscillatory fields, it is appropriate to restrict the discussion to the time-averaged behavior of the atom. The time-average of the trapping force (135) is written

$$\langle \mathbf{F} \rangle = \frac{\alpha}{2} \nabla \langle E^2 \rangle. \quad (136)$$

---

b) The induced, oscillating dipole is equivalent to an oscillating charge with velocity along the direction of \( \mathbf{E} \). Then the \( \mathbf{v} \times \mathbf{B} \) force is in the direction of \( \mathbf{E} \times \mathbf{B} \), i.e., along the direction of propagation of the wave. This is consistent with the force being due to absorption of photons from the wave.

Since it is stated that the polarizability has an imaginary part, we describe the plane wave using complex notation. It suffices to consider the dipole as being at the origin. The wave is polarized in, say, the \( x \) direction. Then

\[
\mathbf{E} = E_0 \hat{x} e^{i \omega t}, \quad \mathbf{B} = E_0 \hat{y} e^{i \omega t}, \quad \text{and} \quad \mathbf{p} = \alpha E_0 \hat{x} e^{i \omega t} = e \mathbf{x}(t). \tag{137}
\]

The oscillating dipole is equivalent to a charge \( e \) at distance \( x(t) \) from the nucleus. The velocity of the charge is, of course, \( \dot{x} \). Then, the Lorentz force on the moving charge is (in Gaussian units)

\[
\mathbf{F} = e \frac{\dot{x}}{c} \times \mathbf{B} = \frac{\dot{p}}{c} \times \mathbf{B} \tag{138}
\]

More precisely, since we are using complex notation, the time-average force is

\[
\langle \mathbf{F} \rangle = \frac{1}{2} \text{Re} \left( \frac{\dot{p}}{c} \times \mathbf{B}^* \right) = \frac{1}{2} \text{Re} \left( i \frac{\omega}{c} \alpha E_0^2 \right) \hat{z} = -\frac{k}{2} \alpha'' E_0^2 \hat{z}, \tag{139}
\]

where \( k = 2\pi/\lambda \) is the wave number.

c) To model the polarizability, it suffices to consider the response of the model atom to the electric field. The equation of motion is then

\[
\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{e E_0}{m} e^{i \omega t}. \tag{140}
\]

The trial solution \( x = x_0 e^{i \omega t} \) leads to

\[
x_0 = \left( \frac{e E_0}{m} \right) \frac{\omega_0^2 - \omega^2 - i \gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}. \tag{141}
\]

Since the magnitude of the dipole moment is \( p_0 = e x_0 = \alpha E_0 \), the polarizability is

\[
\alpha' = \left( \frac{e^2}{m} \right) \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}, \quad \alpha'' = -\left( \frac{e^2}{m} \right) \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}. \tag{142}
\]

Since \( \alpha'' < 0 \), the force (139) is in the +z direction, as expected for photon absorption.

To find the frequency at which \( \alpha' \) is maximum, we take the derivative:

\[
0 = \frac{-2 \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} - \frac{(\omega_0^2 - \omega^2) [-4 \omega (\omega_0^2 - \omega^2) + 2 \omega \gamma^2]}{[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2]^2}. \tag{143}
\]

Thus, \( \gamma \omega_0 = \omega_0^2 - \omega^2 \) at the maximum, and \( \alpha' \) is maximal for \( \omega \) slightly less than \( \omega_0 \):

\[
\omega_{\text{max}} \approx \omega_0 - \gamma/2. \tag{144}
\]
Approximating $\gamma \omega_{\text{max}}$ by $\gamma \omega_0$, we have

$$\alpha'(\omega_{\text{max}}) = \frac{e^2}{m} \frac{1}{2\gamma \omega_0} = -\alpha''(\omega_{\text{max}}). \quad (145)$$

The desired ratio is $\alpha'/\alpha'' = -1$.

d) Comparing the results for a) and b), for trapping to work we must have

$$\frac{\alpha' \partial \langle E^2 \rangle}{2} > \frac{\pi}{\lambda} (-\alpha'') E_0^2 = \frac{2\pi}{\lambda} (-\alpha'') \langle E^2 \rangle. \quad (146)$$

From Prob. 7, Set 9, we know that the characteristic length for changes in the electric field along the axis near the focus is the Rayleigh range,

$$z_0 \approx \pi f_\#^2 \lambda, \quad (147)$$

where the $f_\#$ is the usual $f/D$ ratio of the focusing lens. That is, we can approximate

$$\frac{\partial \langle E^2 \rangle}{\partial z} \approx \frac{\langle E^2 \rangle}{z_0} \approx \frac{\langle E^2 \rangle}{\pi f_\#^2 \lambda}. \quad (148)$$

The maximum trapping force occurs about one Rayleigh range away from the focus. Using (148) in (146), we find the requirement

$$f_\# < \frac{1}{2\pi} \sqrt{\frac{\alpha'}{-\alpha''}}. \quad (149)$$

From part c), if we run the trap at the frequency $\omega = \omega_0 - \gamma/2$ where the trapping term is maximal, the detrapping term is large: $\alpha'' = -\alpha'$. In this case, we need

$$f_\# < \frac{1}{2\pi} = 0.16, \quad (150)$$

which is a very strong focus! For $\omega < \omega_0 - \gamma/2$, the ratio $\alpha'/(-\alpha'')$ grows rapidly, and a softer focus can be used. But the trapping is not so strong, so other detrapping effects become important.

e) We show here that an electrostatic field $\mathbf{E}$ (or magnetostatic field $\mathbf{B}$) cannot have a local maximum of $E^2 = |\mathbf{E}|^2$ at a charge-free point.

The demonstration makes use of the mean-value theorem,\textsuperscript{25} that the average value of the electrostatic field in a charge-free sphere is equal to the value of the field at the center of the sphere.

If $E^2$ has a local maximum at some point $P$ in a charge-free region, then there is a nonzero $r$ such that $E^2 < E^2(P)$ for all points (other than $P$) within a sphere of radius $r$ about $P$. Consequently, $E < E(P)$ in that sphere.

Let $\hat{z}$ point along $\mathbf{E}(P)$. Then the mean-value theorem can be written

$$\int E_z \, d\text{Vol} = \frac{4\pi r^3}{3} E(P), \quad (151)$$

for the sphere about $P$. In general, $E_z \leq E$, and by assumption $E < E(P)$ for all points other than $P$ within the sphere, so
\[
\int E_z d\text{Vol} \leq \int E d\text{Vol} < \int E(P) d\text{Vol} = \frac{4\pi r^3}{3} E(P),
\]
which contradicts eq. (151). Hence, $E^2$ cannot be locally maximal at $P$.

However, $E^2$ can take on a local minimum. This has been shown by explicit examples,\textsuperscript{26} which also provides an alternative demonstration that $E^2$ cannot have a local maximum. A brief discussion of this issue by Ketterle and Pritchard\textsuperscript{27} shows that $\nabla^2 E^2 \geq 0$, which does not exclude the possibility that at a maximum both the first and second derivatives of $E^2$ vanish. However, as noted by Ketterle and by Berry,\textsuperscript{28} the condition that $\nabla^2 E^2 \geq 0$ is sufficient to exclude the possibility of trapping.

6. In a medium of index of refraction $n(\omega)$, the dispersion relation can be written
\[
k = \frac{\omega n}{c},
\]
where $k$ is the wave number and $c$ is the speed of light. The group velocity is then given by
\[
v_g = \frac{d\omega}{dk} = \frac{1}{d\omega/dk} = \frac{c}{n + \omega \frac{dn}{d\omega}}.
\]
We next recall the classical oscillator model for the index of refraction. The index $n$ is the square root of the dielectric constant $\epsilon$, which is in turn related to the atomic polarizability $\alpha$ according to (in Gaussian units)
\[
D = \epsilon E = E + 4\pi P = E(1 + 4\pi N\alpha),
\]
where $D$ is the the electric displacement, $E$ is the electric field, $P$ is the polarization density, and $N$ is the number density of atoms. Then,
\[
n = \sqrt{\epsilon} \approx 1 + 2\pi N\alpha,
\]
for a dilute gas with index near 1.

The polarizability $\alpha$ is obtained from the dipole moment $p = ex = \alpha E$ induced by electric field $E$. In the case of a single spectral line of frequency $\omega_0$, we say that the electron of charge $e$ and mass $m$ is bound to the (fixed) nucleus by a spring of constant $K = m\omega_0^2$, and the motion is subject to damping $-m\gamma \dot{x}$ where the dot indicates

\textsuperscript{26}W.H. Wing, On Neutral Particle Trapping in Quasistatic Electromagnetic Fields, Prog. Quant. Electr. 8, 181-199 (1984).
differentiation with respect to time. The equation of motion in the presence of a wave of frequency \( \omega \) is

\[
\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{eE}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} e^{\omega t}.
\]

(157)

Hence,

\[
x = \frac{eE}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{eE \omega_0^2 - \omega^2 + i\gamma\omega}{m (\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2},
\]

(158)

and so the polarizability is

\[
\alpha = \frac{e^2 \omega_0^2 - \omega^2 + i\gamma\omega}{m (\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}.
\]

(159)

In the present problem, we have two spectral lines, \( \omega_{1,2} = \omega_0 \pm \Delta/2 \), both of unit oscillator strength, but line 2 is pumped so that \( \gamma_2 = -\gamma_1 = -\gamma \). In this case, the polarizability is given by

\[
\alpha = \frac{e^2 (\omega_0 - \Delta/2)^2 - \omega^2 + i\gamma\omega}{m ((\omega_0 - \Delta/2)^2 - \omega^2)^2 + \gamma^2 \omega^2} + \frac{e^2 (\omega_0 + \Delta/2)^2 - \omega^2 - i\gamma\omega}{m ((\omega_0 + \Delta/2)^2 - \omega^2)^2 + \gamma^2 \omega^2}
\]

\[
\approx \frac{e^2 \omega_0^2 - \Delta \omega_0 - \omega^2 + i\gamma\omega}{m (\omega_0^2 - \Delta \omega_0 - \omega^2)^2 + \gamma^2 \omega^2} + \frac{e^2 \omega_0^2 + \Delta \omega_0 - \omega^2 - i\gamma\omega}{m (\omega_0^2 + \Delta \omega_0 - \omega^2)^2 + \gamma^2 \omega^2},
\]

(160)

where the approximation is obtained by the neglect of terms in \( \Delta^2 \) compared to those in \( \Delta \omega_0 \). The index of refraction (156) corresponding to polarizability (160) is shown in Fig. 2.

Figure 2: The real and imaginary parts of the index of refraction corresponding to the polarizability (156) in a medium with one of a pair of spectral lines pumped so as to produce a large rate of change of the index with nearby frequency.

We next consider the issue of attenuation of a pulse of frequency \( \omega \). Since \( k = \omega n/c \approx \omega (1 + 2\pi N\alpha)/c \), the spatial dependence \( e^{ikz} \) of a pulse propagating in the \( z \) direction includes attenuation if the imaginary part of the index \( n \) is positive. However, the pumping described by \( \gamma_2 = -\gamma_1 \) leads to \( Im[\alpha(\omega_0)] = 0 \). Hence, there is no attenuation
of a probe pulse at frequency $\omega_0$. This phenomenon has been called electromagnetically induced transparency.\(^{29}\)

Since $\alpha(\omega_0) = 0$, we have $n(\omega_0) = 1$, and the phase velocity at $\omega_0$ is exactly $c$.

To obtain the group velocity at frequency $\omega_0$, we need the derivative

$$\left. \frac{dRe(n)}{d\omega} \right|_{\omega_0} = 2\pi N \left. \frac{dRe(\alpha)}{d\omega} \right|_{\omega_0} = \frac{4\pi Ne^2(\Delta^2 - \gamma^2)}{m\omega_0(\Delta^2 + \gamma^2)^2} \approx \frac{4\pi Ne^2}{m\omega_0\Delta^2}, \quad (161)$$

where the approximation holds when $\gamma \ll \Delta$ as is the case here. For large density $N$, the group velocity (154) is given by

$$v_g \approx \frac{\Delta^2}{4\pi N r_0 c}, \quad (162)$$

where $r_0 = e^2/mc^2 \approx 3 \times 10^{-13}$ cm is the classical electron radius. The group velocity is lower in a denser medium.

In the experiment of Hau et al., the medium was sodium vapor, cooled to less than 100 nK to increase the density. An additional increase in density by a factor of 16 was obtained when the vapor formed a Bose condensate. Plugging in the experimental parameters, $N = 8 \times 10^{13}/\text{cm}^3$ and $\Delta \approx 1.2 \times 10^8/\text{s}$, we find $v_g \approx 1700 \text{ cm/s}$ as observed in the lab.

7. The concept of group velocity appears to have been first enunciated by Hamilton in 1839 in lectures of which only abstracts were published.\(^{30}\) The first recorded observation of the group velocity of a (water) wave is due to Russell in 1844.\(^{31}\) However, widespread awareness of the group velocity dates from 1876 when Stokes used it as the topic of a Smith’s Prize examination paper.\(^{32}\) The early history of group velocity has been reviewed by Havelock.\(^{33}\)

H. Lamb\(^{34}\) credits A. Schuster with noting in 1904 that a negative group velocity, \textit{i.e.}, a group velocity of opposite sign to that of the phase velocity, is possible due to anomalous dispersion. Von Laue made a similar comment in 1905.\(^{35}\) Lamb gave two examples of strings subject to external potentials that exhibit negative group velocities. These early considerations assumed that in case of a wave with positive group and phase velocities incident on the anomalous medium, energy would be transported into the

\(^{29}\)S.E. Harris, \textit{Electromagnetically Induced Transparency}, Physics Today \textbf{50}(7), 36-42 (1997).


medium with a positive group velocity, and so there would be waves with negative phase velocity inside the medium. Such negative phase velocity waves are formally consistent with Snell’s law\(^{36}\) (since \(\theta_i = \sin^{-1}[(n_i/n_t) \sin \theta_t]\) can be in either the first or second quadrant), but they seemed physically implausible and the topic was largely dropped.

Present interest in negative group velocity a based on anomalous dispersion in a gain medium, where the sign of the phase velocity is the same for incident and transmitted waves, and energy flows inside the gain medium in the opposite direction to the incident energy flow in vacuum.

The propagation of electromagnetic waves at frequencies near those of spectral lines of a medium was first extensively discussed by Sommerfeld and Brillouin,\(^{37}\) with emphasis on the distinction between signal velocity and group velocity when the latter exceeds \(c\). The solution presented here is based on the work of Garrett and McCumber,\(^{38}\) as extended by Chiao et al.\(^{39}\) A discussion of negative group velocity in electronic circuits has been given by Mitchell and Chiao.\(^{40}\)

a) **Negative Group Velocity**

In a medium of index of refraction \(n(\omega)\), the dispersion relation can be written

\[
k = \frac{\omega n}{c},
\]

where \(k\) is the wave number. The group velocity is then given by

\[
v_g = Re \left[ \frac{d\omega}{dk} \right] = \frac{1}{Re[d(\omega n)/d\omega]} = \frac{c}{n + \omega Re[dn/d\omega]}.
\]

We see from eq. (164) that if the index of refraction decreases rapidly enough with frequency, the group velocity can be negative. It is well known that the index of refraction decreases rapidly with frequency near an absorption line, where “anomalous” wave propagation effects can occur. However, the absorption makes it difficult to study these effects. The insight of Garrett and McCumber and of Chiao et al.\(^{41}\) is that demonstrations of negative group velocity are possible in

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\(^{41}\)See also, A.M. Steinberg and R.Y. Chiao, *Dispersionless, highly superluminal propagation in a medium with a gain doublet*, Phys. Rev. A 49, 2071-2075 (1994);
media with inverted populations, so that gain rather than absorption occurs at
the frequencies of interest. This was dramatically realized in the experiment of
Wang et al. by use of a closely spaced pair of gain lines, as perhaps first suggested
by Steinberg and Chiao.

We use a classical oscillator model for the index of refraction. The index
\( n \) is the square root of the dielectric constant \( \epsilon \), which is in turn related to the atomic
polarizability \( \alpha \) according to

\[
D = \epsilon E = E + 4\pi P = E(1 + 4\pi N\alpha),
\]

(in Gaussian units) where \( D \) is the electric displacement, \( E \) is the electric field
and \( P \) is the polarization density. Then, the index of refraction of a dilute gas is

\[
n = \sqrt{\epsilon} \approx 1 + 2\pi N\alpha.
\]

The polarizability \( \alpha \) is obtained from the electric dipole moment \( p = ex = \alpha E \)
induced by electric field \( E \). In the case of a single spectral line of frequency \( \omega_j \),
we say that an electron is bound to the (fixed) nucleus by a spring of constant
\( K = m\omega_j^2 \), and that the motion is subject to the damping force \(-m\gamma_j \dot{x}\), where
the dot indicates differentiation with respect to time. The equation of motion in
the presence of an electromagnetic wave of frequency \( \omega \) is

\[
\ddot{x} + \gamma_j \dot{x} + \omega_j^2 x = \frac{eE}{m} = \frac{eE_0}{m} e^{i\omega t}.
\]

Hence,

\[
x = \frac{eE}{m} \frac{1}{\omega_j^2 - \omega^2 - i\gamma_j \omega} = \frac{eE}{m} \frac{\omega_j^2 - \omega^2 + i\gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2},
\]

and the polarizability is

\[
\alpha = \frac{e^2}{m} \frac{\omega_j^2 - \omega^2 + i\gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}.
\]

In the present problem we have two spectral lines, \( \omega_{1,2} = \omega_0 \pm \Delta/2 \), both of
oscillator strength \(-1\) to indicate that the populations of both lines are inverted,
with damping constants \( \gamma_1 = \gamma_2 = \gamma \). In this case, the polarizability is given by

\[
\alpha = -\frac{e^2}{m} \frac{(\omega_0 - \Delta/2)^2 - \omega^2 + i\gamma \omega}{((\omega_0 - \Delta/2)^2 - \omega^2)^2 + \gamma^2 \omega^2} - \frac{e^2}{m} \frac{(\omega_0 + \Delta/2)^2 - \omega^2 + i\gamma \omega}{((\omega_0 + \Delta/2)^2 - \omega^2)^2 + \gamma^2 \omega^2}
\approx -\frac{e^2}{m} \frac{\omega_0^2 - \Delta\omega_0 - \omega^2 + i\gamma \omega}{(\omega_0^2 - \Delta\omega_0 - \omega^2)^2 + \gamma^2 \omega^2} - \frac{e^2}{m} \frac{\omega_0^2 + 2\Delta\omega_0 - \omega^2 + i\gamma \omega}{(\omega_0^2 + \Delta\omega_0 - \omega^2)^2 + \gamma^2 \omega^2}.
\]

R.Y. Chiao, Population Inversion and Superluminality, in Amazing Light, ed. by R.Y. Chiao (Springer-
Verlag, New York, 1996), pp. 91-108;
R.Y. Chiao and A.M. Steinberg, Tunneling Times and Superluminality, in Progress in Optics, Vol. 37, ed.
where the approximation is obtained by the neglect of terms in $\Delta^2$ compared to those in $\Delta \omega_0$.

For a probe beam at frequency $\omega$, the index of refraction (166) has the form

$$n(\omega) \approx 1 - \frac{\omega_p^2}{2} \left[ \frac{\omega_0^2 - \Delta \omega_0 - \omega^2 + i \gamma \omega}{(\omega_0^2 - \Delta \omega_0 - \omega^2)^2 + \gamma^2 \omega^2} + \frac{\omega_0^2 + \Delta \omega_0 - \omega^2 + i \gamma \omega}{(\omega_0^2 + \Delta \omega_0 - \omega^2)^2 + \gamma^2 \omega^2} \right], \quad (171)$$

where $\omega_p$ is the plasma frequency given by eq. (13). This illustrated in Figure 3.

![Figure 3: The real and imaginary parts of the index of refraction in a medium with two spectral lines that have been pumped to inverted populations. The lines are separated by angular frequency $\Delta$ and have widths $\gamma = 0.4 \Delta$.](image)

The index at the central frequency $\omega_0$ is

$$n(\omega_0) \approx 1 - i \frac{\omega_p^2 \gamma}{(\Delta^2 + \gamma^2) \omega_0} \approx 1 - \frac{\omega_p^2 \gamma}{\Delta^2 \omega_0}, \quad (172)$$

where the second approximation holds when $\gamma \ll \Delta$. The electric field of a continuous probe wave then propagates according to

$$E(z, t) = e^{i(kz - \omega_0 t)} = e^{i \omega_0(n(\omega_0)z/c - t)} \approx e^{i(\Delta^2 c/\gamma \omega_p^2)z/c - t}. \quad (173)$$

From this we see that at frequency $\omega_0$ the phase velocity is $c$, and the medium has an amplitude gain length $\Delta^2 c/\gamma \omega_p^2$.

To obtain the group velocity (164) at frequency $\omega_0$, we need the derivative

$$\left. \frac{d(\omega n)}{d\omega} \right|_{\omega_0} \approx 1 - \frac{2 \omega_p^2 (\Delta^2 - \gamma^2)}{(\Delta^2 + \gamma^2)^2}, \quad (174)$$

where we have neglected terms in $\Delta$ and $\gamma$ compared to $\omega_0$. From eq. (164), we see that the group velocity can be negative if

$$\frac{\Delta^2}{\omega_p^2} - \frac{\gamma^2}{\omega_p^2} \geq \frac{1}{2} \left( \frac{\Delta^2}{\omega_p^2} + \frac{\gamma^2}{\omega_p^2} \right)^2. \quad (175)$$
The boundary of the allowed region (175) in \((\Delta^2, \gamma^2)\) space is a parabola whose axis is along the line \(\gamma^2 = -\Delta^2\), as shown in Fig. 4. For the physical region \(\gamma^2 \geq 0\), the boundary is given by

\[
\frac{\gamma^2}{\omega_p^2} = \sqrt{1 + \frac{4\Delta^2}{\omega_p^2} - 1 - \frac{\Delta^2}{\omega_p^2}}.
\] (176)

Thus, to have a negative group velocity, we must have

\[
\Delta \leq \sqrt{2}\omega_p,
\] (177)

which limit is achieved when \(\gamma = 0\); the maximum value of \(\gamma\) is \(0.5\omega_p\) when \(\Delta = 0.866\omega_p\).

![Figure 4: The allowed region (175) in \((\Delta^2, \gamma^2)\) space such that the group velocity is negative.](image)

Near the boundary of the negative group velocity region, \(|v_g|\) exceeds \(c\), which alerts us to concerns of superluminal behavior. However, as will be seen in the following sections, the effect of a negative group velocity is more dramatic when \(|v_g|\) is small rather than large.

The region of recent experimental interest is \(\gamma \ll \Delta \ll \omega_p\), for which eqs. (164) and (174) predict that

\[
v_g \approx -\frac{c}{30}\frac{\Delta^2}{\omega_p^2}.
\] (178)

A value of \(v_g \approx -c/310\) as in the experiment of Wang corresponds to \(\Delta/\omega_p \approx 1/12\). In this case, the gain length \(\Delta^2 c/\gamma\omega_p^2\) was approximately 40 cm.

For later use we record the second derivative,

\[
\frac{d^2(\omega n)}{d\omega^2}\bigg|_{\omega_0} \approx 8i\frac{\omega_p^2\gamma(3\Delta^2 - \gamma^2)}{(\Delta^2 + \gamma^2)^3} \approx 24i\frac{\omega_p^2\gamma}{\Delta^2 \Delta^2}.
\] (179)

where the second approximation holds if \(\gamma \ll \Delta\).
b) Propagation of a Monochromatic Plane Wave

To illustrate the optical properties of a medium with negative group velocity, we consider the propagation of an electromagnetic wave through it. The medium extends from \( z = 0 \) to \( a \), and is surrounded by vacuum. Because the index of refraction (171) is near unity in the frequency range of interest, we ignore reflections at the boundaries of the medium.

A monochromatic plane wave of frequency \( \omega \) and incident from \( z < 0 \) propagates with phase velocity \( c \) in vacuum. Its electric field can be written

\[
E_\omega(z, t) = E_0 e^{i\omega z/c} e^{-i\omega t} \quad (z < 0).
\]  

(180)

Inside the medium this wave propagates with phase velocity \( c/n(\omega) \) according to

\[
E_\omega(z, t) = E_0 e^{i\omega nz/c} e^{-i\omega t} \quad (0 < z < a),
\]  

(181)

where the amplitude is unchanged since we neglect the small reflection at the boundary \( z = 0 \). When the wave emerges into vacuum at \( z = a \), the phase velocity is again \( c \), but it has accumulated a phase lag of \( (\omega/c)(n - 1)a \), and so appears as

\[
E_\omega(z, t) = E_0 e^{i\omega a(n-1)/c} e^{i\omega z/c} e^{-i\omega t} = E_0 e^{i\omega an/c} e^{-i\omega(t-z/a)/c} \quad (a < z).
\]  

(182)

It is noteworthy that a monochromatic wave for \( z > a \) has the same form as that inside the medium if we make the frequency-independent substitutions

\[
z \rightarrow a, \quad \text{and} \quad t \rightarrow t - \frac{z-a}{c}.
\]  

(183)

Since an arbitrary waveform can be expressed in terms of monochromatic plane waves via Fourier analysis, we can use these substitutions to convert any wave in the region \( 0 < z < a \) to its continuation in the region \( a < z \).

A general relation can be deduced in the case where the second and higher derivatives of \( \omega n(\omega) \) are very small. We can then write

\[
\omega n(\omega) \approx \omega_0 n(\omega_0) + \frac{c}{v_g} (\omega - \omega_0),
\]  

(184)

where \( v_g \) is the group velocity for a pulse with central frequency \( \omega_0 \). Using this in eq. (181), we have

\[
E_\omega(z, t) \approx E_0 e^{i\omega_0 z(n(\omega_0)/c-1/v_g)} e^{i\omega z/v_g} e^{-i\omega t} \quad (0 < z < a).
\]  

(185)

In this approximation, the Fourier component \( E_\omega(z) \) at frequency \( \omega \) of a wave inside the gain medium is related to that of the incident wave by replacing the frequency dependence \( e^{i\omega z/c} \) by \( e^{i\omega z/v_g} \), i.e., by replacing \( z/c \) by \( z/v_g \), and multiplying by the frequency-independent phase factor \( e^{i\omega_0 z(n(\omega_0)/c-1/v_g)} \). Then, using transformation (183), the wave that emerges into vacuum beyond the medium is

\[
E_\omega(z, t) \approx E_0 e^{i\omega_0 a(n(\omega_0)/c-1/v_g)} e^{i\omega(z/c-a(1/c-1/v_g))} e^{-i\omega t} \quad (a < z).
\]  

(186)
The wave beyond the medium is related to the incident wave by multiplying by a frequency-independent phase, and by replacing $z/c$ by $z/c - a(1/c - 1/v_g)$ in the frequency-dependent part of the phase.

The effect of the medium on the wave as described by eqs. (185)-(186) has been called “rephasing”.

c) **Fourier Analysis and “Rephasing”**

The transformations between the monochromatic incident wave (180) and its continuation in and beyond the medium, (185) and (186), imply that an incident wave

$$E(z, t) = f(z/c - t) = \int_{-\infty}^{\infty} E_\omega(z)e^{-i\omega t}d\omega \quad (z < 0),$$

whose Fourier components are given by

$$E_\omega(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(z, t)e^{i\omega t}dt,$$

propagates as

$$E(z, t) \approx \begin{cases} f(z/c - t) & (z < 0), \\ e^{i\omega_0 z(n(\omega_0)c - 1/v_g)}f(z/v_g - t) & (0 < z < a), \\ e^{i\omega_0 a(n(\omega_0)c - 1/v_g)}f(z/c - t - a(1/c - 1/v_g)) & (a < z). \end{cases}$$

An interpretation of eq. (189) in terms of “rephasing” is as follows. Fourier analysis tells us that the maximum amplitude of a pulse made of waves of many frequencies, each of the form $E_\omega(z, t) = E_0(\omega)e^{i\phi(\omega)} = E_0(\omega)e^{i(k(\omega)z - \omega t + \phi(\omega))}$ with $E_0 \geq 0$, is determined by adding the amplitudes $E_0(\omega)$. This maximum is achieved only if there exists points $(z, t)$ such that all phases $\phi(\omega)$ have the same value.

For example, we consider a pulse in the region $z < 0$ whose maximum occurs when the phases of all component frequencies vanish, as shown at the left of Fig. 5. Referring to eq. (180), we see that the peak occurs when $z = ct$. As usual, we say that the group velocity of this wave is $c$ in vacuum.

Inside the medium, eq. (185) describes the phases of the components, which all have a common frequency-independent phase $\omega_0 z(n(\omega_0)c - 1/v_g)$ at a given $z$, as well as a frequency-dependent part $\omega(z/v_g - t)$. The peak of the pulse occurs when all the frequency-dependent phases vanish; the overall frequency-independent phase does not affect the pulse size. Thus, the peak of the pulse propagates within the medium according to $z = v_g t$. The velocity of the peak is $v_g$, the group velocity of the medium, which can be negative.

The “rephasing” (185) within the medium changes the wavelengths of the component waves. Typically the wavelength increases, and by greater amounts at longer wavelengths. A longer time is required before the phases of the waves all becomes the same at some point $z$ inside the medium, so in a normal medium the velocity of the peak appears to be slowed down. But in a negative group velocity medium,
wavelengths short compared to \( \lambda_0 \) are lengthened, long waves are shortened, and the velocity of the peak appears to be reversed.

By a similar argument, eq. (186) tells us that in the vacuum region beyond the medium the peak of the pulse propagates according to \( z = ct + a(1/c - 1/v_g) \). The group velocity is again \( c \), but the “rephasing” within the medium results in a shift of the position of the peak by amount \( a(1/c - 1/v_g) \). In a normal medium where \( 0 < v_g \leq c \) the shift is negative; the pulse appears to have been delayed during its passage through the medium. But after a negative group velocity medium, the pulse appears to have advanced!

This advance is possible because in the Fourier view, each component wave extends over all space, even if the pulse appears to be restricted. The unusual “rephasing” in a negative group velocity medium shifts the phases of the frequency components of the wave train in the region ahead of the nominal peak such that the phases all coincide, and a peak is observed, at times earlier than expected at points beyond the medium.

As shown in Fig. 5 and further illustrated in the examples below, the “rephasing” can result in the simultaneous appearance of peaks in all three regions.

d) Propagation of a Sharp Wave Front
To assess the effect of a medium with negative group velocity on the propagation of a signal, we first consider a waveform with a sharp front, as recommended by Sommerfeld and Brillouin.

As an extreme but convenient example, we take the incident pulse to be a Dirac delta function, \( E(z, t) = E_0 \delta(z/c - t) \). Inserting this in eq. (189), which is based on the linear approximation (184), we find

\[
E(z, t) \approx \begin{cases} 
E_0 \delta(z/c - t) & (z < 0), \\
E_0 e^{i\omega_0 z/n(\omega_0)/c-1/v_g} \delta(z/v_g - t) & (0 < z < a), \\
E_0 e^{i\omega_0 a/n(\omega_0)/c-1/v_g} \delta(z/c - t - a/(1/c - 1/v_g)) & (a < z),
\end{cases}
\]  

(190)

According to eq. (190), the delta-function pulse emerges from the medium at \( z = a \) at time \( t = a/v_g \). If the group velocity is negative, the pulse emerges from the medium before it enters at \( t = 0! \)

A sample history of (Gaussian) pulse propagation is illustrated in Fig. 6. Inside the negative group velocity medium, an (anti)pulse propagates backwards in space from \( z = a \) at time \( t = a/v_g < 0 \) to \( z = 0 \) at time \( t = 0 \), at which point it appears to annihilate the incident pulse.

This behavior is analogous to barrier penetration by a relativistic electron\(^{42}\) in which an electron can emerge from the far side of the barrier earlier than it hits the near side, if the electron emission at the far side is accompanied by positron emission, and the positron propagates within the barrier so as to annihilate the incident electron at the near side. In the Wheeler-Feynman view, this process involves only a single electron which propagates backwards in time when inside the barrier. In this spirit, we might say that pulses propagate backwards in time (but forward in space) inside a negative group velocity medium.

The Fourier components of the delta function are independent of frequency, so the advanced appearance of the sharp wavefront as described by eq. (190) can occur only for a gain medium such that the index of refraction varies linearly at all frequencies. If such a medium existed with negative slope \( dn/d\omega \), then eq. (190) would constitute superluminal signal propagation.

However, from Fig. 3 we see that a linear approximation to the index of refraction is reasonable in the negative group velocity medium only for \( |\omega - \omega_0| \lesssim \Delta/2 \). The sharpest wavefront that can be supported within this bandwidth has characteristic risetime \( \tau \approx 1/\Delta \).

For the experiment of Wang et al. where \( \Delta/2\pi \approx 10^6 \) Hz, an analysis based on eq. (184) would be valid only for pulses with \( \tau \gtrsim 0.1 \) \( \mu s \). Wang et al. used a pulse with \( \tau \approx 1 \) \( \mu s \), close to the minimum value for which eq. (184) is a reasonable approximation.

Since a negative group velocity can only be experienced over a limited bandwidth, very sharp wavefronts must be excluded from discussion of signal propagation.

However, it is well known that great care must be taken when discussing the signal velocity if the waveform is not sharp.

e) **Propagation of a Gaussian Pulse**

We now consider a Gaussian pulse of temporal length $\tau$ centered on frequency $\omega_0$ (the carrier frequency), for which the incident waveform is

$$E(z, t) = E_0 e^{-\left(z/c-t\right)^2/2\tau^2} e^{i\omega_0 z/c} e^{-i\omega_0 t} \quad (z < 0), \quad (191)$$

Inserting this in eq. (189) we find

$$E(z, t) = \begin{cases} 
E_0 e^{-\left(z/c-t\right)^2/2\tau^2} e^{i\omega_0 z/c} & (z < 0), \\
E_0 e^{-\left(z/v_g-t\right)^2/2\tau^2} e^{i\omega_0 (z/c-t)} & (0 < z < a), (192) \\
E_0 e^{i\omega_0 a (n(\omega_0)-1)/c} e^{-\left(z/a(1/c-1/v_g)-t\right)^2/2\tau^2} e^{i\omega_0 (z/c-t)} & (a < z).
\end{cases}$$

The factor $e^{i\omega_0 a (n(\omega_0)-1)/c}$ in eq. (192) for $a < z$ becomes $e^{\omega_0^2 \gamma a/\Delta^2 c}$ using eq. (172), and represents a small gain due to traversing the negative group velocity medium. In the experiment of Wang et al. this factor was only 1.16.

We have already noted in the previous section that the linear approximation to $\omega n(\omega)$ is only good over a frequency interval about $\omega_0$ of order $\Delta$, and so eq. (192) for the pulse after the gain medium applies only for pulsewidths

$$\tau \gtrsim \frac{1}{\Delta}. \quad (193)$$

Further constraints on the validity of eq. (192) can obtained using the expansion of $\omega n(\omega)$ to second order. For this, we repeat the derivation of eq. (192) in slightly more detail. The incident Gaussian pulse (191) has the Fourier decomposition (188)

$$E_{\omega}(z) = \frac{\tau}{\sqrt{2\pi}} E_0 e^{-\tau^2 (\omega - \omega_0)^2/2} e^{i\omega z/c} \quad (z < 0). \quad (194)$$

We again extrapolate the Fourier component at frequency $\omega$ into the region $z > 0$ using eq. (181), which yields

$$E_{\omega}(z) = \frac{\tau}{\sqrt{2\pi}} E_0 e^{-\tau^2 (\omega - \omega_0)^2/2} e^{i\omega z/c} \quad (0 < z < a). \quad (195)$$

We now approximate the factor $\omega n(\omega)$ by its Taylor expansion through second order:

$$\omega n(\omega) \approx \omega_0 n(\omega_0) + \frac{c}{v_g} (\omega - \omega_0) + \frac{1}{2} \frac{d^2(\omega n)}{d\omega^2} \bigg|_{\omega_0} (\omega - \omega_0)^2. \quad (196)$$

With this, we find from eqs. (187) and (195) that

$$E(z, t) = \frac{E_0}{A} e^{-\left(z/v_g-t\right)^2/2A^2 \tau^2} e^{i\omega_0 n(\omega_0) z/c} e^{-i\omega_0 t} \quad (0 < z < a), \quad (197)$$
where

\[ A^2(z) = 1 - i \frac{z}{ct^2} \frac{d^2(\omega n)}{d\omega^2} \bigg|_{\omega_0}. \]  

(198)

The waveform for \( z > a \) is obtained from that for \( 0 < z < a \) by the substitutions (183) with the result

\[ E(z, t) = \frac{E_0}{A} e^{i\omega_0 a(n(\omega_0)-1)/c} e^{-(z/c-a(1/c-1/v_g)-t)^2/2A^2\tau^2} e^{i\omega_0 z/c} e^{-i\omega_0 t} \]  

(a < z),

(199)

where \( A \) is evaluated at \( z = a \) here. As expected, the forms (197) and (199) revert to those of eq. (192) when \( d^2(\omega n(\omega_0))/d\omega^2 = 0 \).

So long as the factor \( A(a) \) is not greatly different from unity, the pulse emerges from the medium essentially undistorted, which requires

\[ \frac{a}{c\tau} \ll \frac{1}{24} \frac{\Delta^2}{\omega_p^2} \Delta \tau, \]

(200)

using eqs. (179) and (198). In the experiment of Wang et al., this condition is that \( a/c\tau \ll 1/120 \), which was well satisfied with \( a = 6 \) cm and \( c\tau = 300 \) m.

As in the case of the delta function, the centroid of a Gaussian pulse emerges from a negative group velocity medium at time

\[ t = \frac{a}{v_g} < 0, \]

(201)

which is earlier than the time \( t = 0 \) when the centroid enters the medium. In the experiment of Wang et al., the time advance of the pulse was \( a/|v_g| \approx 300a/c \approx 6 \times 10^{-8} \) s \( \approx 0.06\tau \).

If one attempts to observe the negative group velocity pulse inside the medium, the incident wave would be perturbed and the backwards-moving pulse would not be detected. Rather, one must deduce the effect of the negative group velocity medium by observation of the pulse that emerges into the region \( z > a \) beyond that medium, where the significance of the time advance (201) is the main issue.

The time advance caused by a negative group velocity medium is larger when \( |v_g| \) is smaller. It is possible that \( |v_g| > c \), but this gives a smaller time advance than when the negative group velocity is such that \( |v_g| < c \). Hence, there is no special concern as to the meaning of negative group velocity when \( |v_g| > c \).

The maximum possible time advance \( t_{\text{max}} \) by this technique can be estimated from eqs. (178), (200) and (201) as

\[ \frac{t_{\text{max}}}{\tau} \approx \frac{1}{12} \frac{\Delta}{\gamma} \Delta \tau \approx 1. \]

(202)

The pulse can advance by at most a few risetimes due to passage through the negative group velocity medium.

While this aspect of the pulse propagation appears to be superluminal, it does not imply superluminal signal propagation.
In accounting for signal propagation time, the time needed to generate the signal must be included as well. A pulse with a finite frequency bandwidth $\Delta$ takes at least time $\tau \approx 1/\Delta$ to be generated, and so is delayed by a time of order its risetime $\tau$ compared to the case of an idealized sharp wavefront. Thus, the advance of a pulse front in a negative group velocity medium by $\lesssim \tau$ can at most compensate for the original delay in generating that pulse. The signal velocity, as defined by the path length between the source and detector divided by the overall time from onset of signal generation to signal detection, remains bounded by $c$.

As has been emphasized by Garrett and McCumber and by Chiao, the time advance of a pulse emerging from a gain medium is possible because the forward tail of a smooth pulse gives advance warning of the later arrival of the peak. The leading edge of the pulse can be amplified by the gain medium, which gives the appearance of superluminal pulse velocities. However, the medium is merely using information stored in the early part of the pulse during its (lengthy) time of generation to bring the apparent velocity of the pulse closer to $c$.

The effect of the negative group velocity medium can be dramatized in a calculation based on eq. (192) in which the pulse width is narrower than the gain region (in violation of condition (200)), as shown in Fig. 6. Here, the gain region is $0 < z < 50$, the group velocity is taken to be $-c/2$, and $c$ is defined to be unity. The behavior illustrated in Fig. 6 is perhaps less surprising when the pulse amplitude is plotted on a logarithmic scale, as in Fig. 7. Although the overall gain of the system is near unity, the leading edge of the pulse is amplified by about 70 orders of magnitude in this example (the implausibility of which underscores that condition (200) cannot be evaded), while the trailing edge of the pulse is attenuated by the same amount. The gain medium has temporarily loaned some of its energy to the pulse permitting the leading edge of the pulse to appear to advance faster than the speed of light.

Our discussion of the pulse has been based on a classical analysis of interference, but, as remarked by Dirac\textsuperscript{43} classical optical interference describes the behavior of the wave functions of individual photons, not of interference between photons. Therefore, we expect that the behavior discussed above will soon be demonstrated for a “pulse” consisting of a single photon with a Gaussian wave packet.

Figure 6: Ten “snapshots” of a Gaussian pulse as it traverses a negative group velocity region \((0 < z < 50)\), according to eq. (192). The group velocity in the gain medium is \(v_g = -c/2\), and \(c\) has been set to 1.
Figure 7: The same as Fig. 6, but with the electric field plotted on a logarithmic scale from 1 to $10^{-65}$. 
8. **General Remarks about Electrostatic Waves**

We first verify that Maxwell’s equations imply that when an electric field $E$ has no time dependence, then $\nabla \times E = 0$.

If $\partial E/\partial t = 0$, then the magnetic field $B$ obeys $\partial^2 B/\partial t^2 = 0$, as follows on taking the time derivative of Faraday’s law, $c\nabla \times E = -\partial B/\partial t$ in Gaussian units. In principle, this is consistent with a magnetic field that varies linearly with time, $B(r, t) = B_0(r) + B_1(r)t$. However, this leads to arbitrarily large magnetic fields at early and late times, and is excluded on physical grounds. Hence, $\partial E/\partial t = 0$ implies that $\partial B/\partial t = 0$ also, and $\nabla \times E = 0$ according to Faraday’s law.

We next consider some general properties of a longitudinal plane electric wave, taken to have the form

$$E = E_x \hat{x} e^{i(kx - \omega t)}. \quad (203)$$

This obeys $\nabla \times E = 0$, and so can be derived from an electric potential, namely

$$E = -\nabla V \quad \text{where} \quad V = \frac{iE_x}{k} e^{i(kx - \omega t)}. \quad (204)$$

The electric wave (203) has no associated magnetic wave, since Faraday’s law tells us that

$$0 = \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad (205)$$

and any magnetic field in the problem must be static.

It is well known that electromagnetic waves in vacuum are transverse. A longitudinal electric wave can only exist in a medium that can support a nonzero polarization density $P$ (volume density of electric dipole moments). The polarization density implies an effective charge density $\rho$ given by

$$\rho = -\nabla \cdot P \quad (206)$$

which is consistent with the first Maxwell equation,

$$\nabla \cdot E = 4\pi \rho, \quad (207)$$

only if

$$P = -\frac{E}{4\pi}, \quad (208)$$

in which case the electric displacement $D$ of the longitudinal wave vanishes,

$$D = E + 4\pi P = 0. \quad (209)$$

Hence, the (relative) dielectric constant $\epsilon$ also vanishes.

Strictly speaking, eq. (208) could read $P = -E/4\pi + P'$, for any field $P'$ that obeys $\nabla \cdot P' = 0$. However, since any magnetic field in the problem is static, the fourth Maxwell equation tells us that

$$\nabla \times B = \frac{4\pi}{c} \left( J + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right). \quad (210)$$
has no time dependence. Recalling that the polarization current is related by

$$ J = \frac{\partial P}{\partial t}, $$

we again find relation (208) with the possible addition of a static field $P'$ that is associated with a truly electrostatic field $E'$. In sum, a longitudinal electric wave described by eqs. (203), (208) and (209) can coexist with background electrostatic and magnetostatic fields of the usual type.

Maxwell's equations alone provide no relation between the wave number $k$ and the wave frequency $\omega$ of the longitudinal wave, and hence the wave phase velocity $\omega/k$ is arbitrary. This suggests that purely longitudinal electric waves are best considered as limiting cases of more general waves, for which additional physical relations provide additional information as to the character of the waves.

### 8.2. Gauge Invariance

Since the electric wave (203) has no associated magnetic field, we can define its vector potential $A$ to be zero, which is certainly consistent with the Coulomb gauge condition $\nabla \cdot A = 0$. Suppose, however, we prefer to work in the Lorentz gauge, for which

$$ \nabla \cdot A = -\frac{1}{c} \frac{\partial V}{\partial t}. $$

Then, the vector potential will be nonzero, and the electric field is related by

$$ E = -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t} = E_x \hat{x} e^{i(kx - \omega t)}. $$

Clearly the potentials have the forms

$$ A = A_x \hat{x} e^{i(kx - \omega t)}, \quad V = V_0 e^{i(kx - \omega t)}, $$

which are consistent with $B = \nabla \times A = 0$. From the Lorentz gauge condition (212) we have

$$ kA_x = \frac{\omega}{c} V_0, $$

and from eq. (213) we find

$$ E_x = i kV_0 + \frac{i}{c} \frac{\omega}{k} A_x. $$

Hence,

$$ A = -i \frac{\omega c}{\omega^2 + k^2 c^2} E_x \hat{x} e^{i(kx - \omega t)}, \quad V = -i \frac{k c^2}{\omega^2 + k^2 c^2} E_x e^{i(kx - \omega t)}. $$

We could also derive the wave (203) from the potentials

$$ A = -i \frac{c}{\omega} E_x \hat{x} e^{i(kx - \omega t)}, \quad V = 0. $$
Thus, an “electrostatic wave” is not necessarily associated with an “electrostatic” scalar potential.

8.3. Energy Considerations

A common expression for the electric field energy density is $E \cdot D/8\pi$. However, this vanishes for longitudinal electric waves, according to eq. (209). Further, since the longitudinal electric wave can exist with zero magnetic field, there is no Poynting vector $S = (c/4\pi)E \times H$ or momentum density $p_{\text{field}} = D \times B/4\pi c$, according to the usual prescriptions.

Let us recall the origins of the standard lore. Namely, the rate of work done by the field $E$ on current density $J$ is

$$J \cdot E = \frac{\partial P}{\partial t} \cdot E = -\frac{1}{4\pi} \frac{\partial}{\partial t} E = -\frac{\partial E^2/8\pi}{\partial t},$$

using eqs. (208) and (211). This work is done at the expense of the electric field energy density $u_{\text{field}}$, which we therefore identify as

$$u_{\text{field}} = \frac{E^2}{8\pi} = \frac{E_x^2}{8\pi} \cos^2(kx - \omega t),$$

for the longitudinal wave (203). We readily interpret this energy density as moving in the $+x$ direction at the phase velocity $v_p = \omega/k$, even though the derivation of eq. (219) did not lead to a Poynting vector.

We should also note that energy is stored in the medium in the form of kinetic energy of the electrons (and, in general, ions as well) that contribute to the polarization,

$$P = Ne(x - x_0) = -\frac{E}{4\pi}.$$  

Thus, the velocity of an electron is given by

$$v = v_0 - \frac{\dot{E}}{4\pi Ne} = v_0 - \frac{\omega E_x \dot{x}}{4\pi Ne} \sin(kx - \omega t).$$

In squaring this to get the kinetic energy, we neglect the term in $v_0 \cdot \dot{x}$, assuming its average to be zero as holds for a medium that is at rest on average (and also holds for a plasma in a tokamak when $x$ is taken as the radial coordinate in a small volume). Then, we find the mechanical energy density to be

$$u_{\text{mech}} = \frac{1}{2} Nmv^2 = \frac{1}{2} Nmv_0^2 + \frac{E_x^2}{8\pi} \frac{\omega^2 m}{4\pi Ne^2} \sin^2(kx - \omega t) = u_{\text{mech},0} + \frac{\omega^2 E_x^2}{\omega_p^2 8\pi} \sin^2(kx - \omega t).$$

We again can interpret the additional term as an energy density that flows in the $+x$ direction at the phase velocity.

The total, time-averaged energy density associated with the longitudinal wave is

$$\langle u_{\text{wave}} \rangle = \frac{\omega^2 + \omega_p^2 E_x^2}{2\omega_p^2 8\pi}.$$
If the wave frequency is less than the plasma frequency, as is the case for examples of Bernstein waves discussed in the next section, the longitudinal electric field energy density is larger than that of the mechanical energy density of the wave.

8.4. Longitudinal Waves in a Magnetized Plasma

Turning now to the problem of plane waves in a magnetized plasma, we consider waves whose propagation vector $\mathbf{k}$ is transverse to the external magnetic field $\mathbf{B}_0$, and seek a solution where electric field vector $\mathbf{E}$ is parallel to $\mathbf{k}$.

We adopt a rectangular coordinate system in which the external magnetic field $\mathbf{B}_0$ is along the $+z$ axis and the plane electric wave propagates along the $+x$ axis:

$$\mathbf{E} = E_x \hat{x} \cos(kx - \omega t).$$ (225)

The unperturbed ($E = 0$) motion of an electron is on a helix of radius

$$r_B = \frac{v_\perp}{\omega_B},$$ (226)

where $v_\perp = \sqrt{2KT/m}$ for all electrons in our simplified analysis. Hence, we can write the general (nonrelativistic) motion as

$$x = x_0 + r_B \cos(\omega_B t + \phi_0) + \delta x,$$ (227)
$$y = y_0 + r_B \sin(\omega_B t + \phi_0) + \delta y,$$ (228)
$$z = z_0 + v_z t + \delta z,$$ (229)

noting that the circular motion of a negatively charged electron is counterclockwise in the $x$-$y$ plane for an external magnetic field along the $+z$ axis. For an electron in the collisionless plasma, we consider the Lorentz force only from the wave electric field and the external magnetic field, $-e(\mathbf{E} + \mathbf{v}/c \times \mathbf{B}_0)$. The equations of motion are then

$$m[-\omega^2 r_B \cos(\omega_B t + \phi_0) + \delta \ddot{x}] = -eE_x \cos(kx - \omega t)$$
$$-\frac{eB_0}{c} \omega_B r_B \cos(\omega_B t + \phi_0) + \delta \dot{y},$$ (230)

$$m[-\omega^2 r_B \sin(\omega_B t + \phi_0) + \delta \dot{y} = -\frac{eB_0}{c} \omega_B r_B \sin(\omega_B t + \phi_0) - \delta \ddot{x}]$$ (231)

$$m \ddot{z} = 0.$$ (232)

Recalling eq. (14) for the cyclotron frequency, the equations of motion reduce to

$$\delta \ddot{x} + \omega_B \delta \dot{y} = -\frac{eE_x}{m} \cos(kx - \omega t),$$ (233)
$$\delta \dot{y} - \omega_B \delta \ddot{x} = 0,$$ (234)
$$\delta \ddot{z} = 0.$$ (235)

Equation (235) has the trivial solution $\delta z = 0$, while eq. (234) integrates to

$$\delta \dot{y} = \omega_B \delta x.$$ (236)
With this, the remaining equation of motion becomes
\[ \delta \ddot{x} + \omega_B^2 \delta x = -\frac{eE_x}{m} \cos(kx - \omega t), \] (237)

To proceed, we must expand the factor \( \cos(kx - \omega t) \), which we do as follows:
\[
\cos(kx - \omega t) = \cos(kx_0 - \omega t + kr_B \cos(\omega_B t + \phi_0)) \\
\approx \cos(kx_0 - \omega t) \cos(kr_B \cos(\omega_B t + \phi_0)) \\
- kr_B \cos(\omega_B t + \phi_0) \sin(kx_0 - \omega t) \\
\approx \cos(kx_0 - \omega t) \left( 1 - \frac{1}{2} k^2 r_B^2 \cos^2(\omega_B t + \phi_0) \right) \\
\approx \cos(kx_0 - \omega t) \left( 1 - \frac{1}{4} k^2 r_B^2 \right) \\
= \cos(kx_0 - \omega t) \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right). \] (238)

In the above we have supposed that \( \delta x \ll r_B \) in going from the first line to the second, that \( r_B \ll x_0 \) in going from the second line to the third, that \( kr_B \ll 1 \) and \( \langle \cos(\omega_B t + \phi_0) \sin(kx_0 - \omega t) \rangle = 0 \) in going from the third line to the fourth, and that \( \langle \cos^2(\omega_B t + \phi_0) \rangle = 1/2 \) in going from the fourth line to the fifth. Perhaps the most doubtful assumption is that \( kr_B \ll 1 \).

The approximate equations of motion is now
\[ \delta \ddot{x} + \omega_B^2 \delta x = -\frac{eE_x}{m} \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right) \cos(kx_0 - \omega t). \] (239)

The solution to this is
\[ \delta x = -\frac{e}{m(\omega_B^2 - \omega^2)} \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right) E_x \cos(kx_0 - \omega t). \] (240)

The resulting electric dipole moment density \( P \) is
\[ P = -Ne \delta x \hat{x} = -\frac{N e}{m(\omega_B^2 - \omega^2)} \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right) E = \frac{\omega_P^2}{4\pi(\omega_B^2 - \omega^2)} \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right) E, \] (241)

where \( \omega_P \) is the (electron) plasma frequency (15).

For a longitudinal wave, the electric displacement must vanish according to eq. (209), so we find
\[ 0 = D = E + 4\pi P = \left[ 1 + \frac{\omega_P^2}{\omega_B^2 - \omega^2} \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right) \right] E, \] (242)

which requires that
\[ \omega^2 = \omega_B^2 + \omega_P^2 \left( 1 - \frac{k^2 v_\perp^2}{4\omega_B^2} \right) = \omega_B^2 + \omega_P^2 \left( 1 - \frac{k^2 KT}{2m\omega_B^2} \right). \] (243)
This result corresponds to keeping only the first term of Bernstein’s series expansion, eq. (50).

In the limit of a cold plasma, where \( v_\perp = 0 \), the frequency of the longitudinal wave is \( \sqrt{\omega_B^2 + \omega_p^2} \), which is the so-called upper hybrid resonance frequency. (This result is well-known to follow from the assumption of a cold plasma.)

In our model, the effect of nonzero temperature is to lower the frequency of the longitudinal wave, bringing it closer to the cyclotron frequency, \( \omega_B \). The effect is greater for shorter wavelengths (larger wave number \( k \)). Our approximation implies that for wavelengths small compared to \( r_\perp \), the characteristic radius of the electron cyclotron motion at temperature \( T \), the frequency of the wave approaches zero. However, our approximation becomes doubtful for \( kr_\perp \gg 1 \). Bernstein finds that the wave frequency is restricted to a band around \( \omega_B \), which result is only hinted at by our analysis.

If we evaluate the dispersion relation (241) at the cyclotron frequency, \( \omega = \omega_B \), then we find the following representative values for parameters of a Bernstein wave:

\[
k = \frac{2\omega_B}{v_\perp} = \frac{2}{r_\perp}, \quad \lambda = \pi r_\perp, \quad \text{and} \quad v_p = \frac{\omega_B}{k} = \frac{v_\perp}{2} \ll c.
\] (244)

While our analysis does not constrain the phase velocity, \( v_p = \omega/k \), of the longitudinal wave, we do find a relation between \( v_p \) and the group velocity, \( v_g = d\omega/dk \):

\[
v_g = \frac{d\omega}{dk} = -\frac{\omega_p^2}{\omega_B^2} \frac{KT}{2mv_p}.
\] (245)

The longitudinal electric waves are negative group velocity waves! In Prob. 7 we resolved a paradox associated with this latter phenomenon, where we found that a negative group velocity can have any magnitude without contradicting the insight of Einstein that signals must propagate at velocities less than or equal to \( c \). Hence, the lack of a constraint on \( v_p \) is not a fundamental flaw in the analysis.