I. PROBLEM

Consider a classical model of matter in which spectral lines are associated with oscillators. In particular, consider a gas with two closely spaced spectral lines, \( \omega_{1,2} = \omega_0 \pm \Delta \), where \( \Delta \ll \omega_0 \). Each line has oscillator strength \( 1/Z \), where \( Z \) is the atomic number of a gas atom, and each has the same damping constant (and spectral width) \( \gamma \). For simplicity, you may suppose that \( \Delta = \gamma \).

Ordinarily, the gas would exhibit strong absorption of light in the vicinity of the spectral lines. But suppose that a laser of frequency \( \omega_2 \) ‘‘pumps’’ the second oscillator into an inverted population. Classically, this is described by assigning a negative damping constant to this oscillator: \( \gamma_2 = -\gamma \).

Deduce an expression for the group velocity of a pulse of light centered on frequency \( \omega_0 \) in this medium. Show also that frequencies very near \( \omega_0 \) propagate without attenuation.

In a recent experiment, \(^1\) the group velocity of light was reduced to 38 mph (17 m/s) by this technique in a sodium vapor of density \( N = 5 \times 10^{12} \) atoms/cm\(^3\) using a pair of lines for which \( 2\Delta \approx 10^{12} \) s.

II. SOLUTION

In a medium of index of refraction \( n(\omega) \), the dispersion relation can be written

\[
k = \frac{\omega n}{c},
\]

where \( k \) is the wave number and \( c \) is the speed of light. The group velocity is then given by

\[
v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = \frac{c}{n + \omega \frac{dn}{d\omega}}.
\]
\[
\alpha = \frac{1}{Z} \frac{e^2}{m} \frac{(\omega_0 - \gamma)^2 - \omega^2 - i \gamma \omega}{(\omega_0 - \gamma)^2 - \omega^2} + \frac{1}{Z} \frac{e^2}{m} \frac{(\omega_0 + \gamma)^2 - \omega^2 + i \gamma \omega}{(\omega_0 + \gamma)^2 - \omega^2} + \frac{1}{Z} \frac{e^2}{m} \frac{\omega_0^2 - 2 \gamma \omega_0 - \omega^2 - i \gamma \omega}{(\omega_0^2 - 2 \gamma \omega_0 - \omega^2)^2 + \gamma^2 \omega^2} + \frac{1}{Z} \frac{e^2}{m} \frac{\omega_0^2 + 2 \gamma \omega_0 - \omega^2 + i \gamma \omega}{(\omega_0^2 + 2 \gamma \omega_0 - \omega^2)^2 + \gamma^2 \omega^2},
\]

where the approximation is obtained by the neglect of terms in \(\gamma^2\) compared to those in \(\gamma \omega_0\).

We now consider the issue of attenuation of a pulse of frequency \(\omega\). Since \(k = \omega n/c = \omega(1 + 2\pi N\alpha)/c\), the spatial dependence \(e^{ikz}\) of a pulse propagating in the \(z\) direction includes attenuation if the imaginary part of the index \(n\) is nonzero. However, the population inversion described by \(\alpha(\omega_0)\) leads to \(\text{Im} \alpha(\omega_0) = 0\). Hence, there is no attenuation of a probe pulse at frequency \(\omega_0\).

In the present model, the pulse is attenuated at frequencies less than \(\omega_0\), but grows (lases) at frequencies greater than \(\omega_0\). In the experiment of Hau et al.,\(^1\) lasing did not occur because line 2 actually corresponded to a transition between the upper level of line 1 and a third, excited level. (In a sense, the quantum mechanical level structure with one high and two low energy levels is the inverse of that assumed in the classical model here, i.e., one low and two high levels.) Therefore, pumping at frequency \(\omega_2\) did not produce an inverted population that could lead to lasing; but it did lead to an effective sign reversal of the damping constant \(\gamma_2\) for a narrow range of frequencies near \(\omega_0\).

To obtain the group velocity at frequency \(\omega_0\), we need the derivative

\[
\frac{d \text{Re}(n)}{d\omega} \bigg|_{\omega_0} = 2 \pi N \frac{d \text{Re}(\alpha)}{d\omega} \bigg|_{\omega_0} = \frac{24\pi Ne^2}{5Zm\gamma'\omega_0},
\]

Since \(\alpha(\omega_0) = 0\), we have \(n(\omega_0) = 1\), and the phase velocity at \(\omega_0\) is exactly \(c\). The group velocity (2) is

\[
v_g = \frac{c}{1 + \frac{24\pi Ne^2}{25Zm\gamma'\omega_0}} \approx \frac{25Z\gamma^2}{e^2} \frac{c}{\pi Nr_0c^2},
\]

where \(r_0 = e^2/mc^2 \approx 3 \times 10^{-13}\) cm is the classical electron radius. The group velocity is lower in a denser medium.

In the experiment of Hau et al., the medium was sodium vapor (Z = 11), cooled to less than 1 \(\mu\)K to increase the density. An additional increase in density by a factor of 5 was obtained when the vapor formed a Bose condensate. Plugging in the experimental parameters, \(N = 5 \times 10^{12}/\text{cm}^3\) and \(\gamma = 5\times 10^9/\text{s}\), we find

\[
v_g \approx \frac{11 \cdot (5 \times 10^6)^2}{3 \cdot 5 \times 10^{12} \cdot 3 \times 10^{-13} \cdot 3 \times 10^3} \approx 2000 \text{ cm/s},
\]

compared to the measured value of 1700 cm/s.


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