Geometry of Viewing Mercury Drops
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In the MERIT experiment, the mercury jet, and any droplets ejected from it by the proton beam interaction, were viewed via shadow photography from a distance $D = 9.15$ cm from the center of the jet.

The jet may have had an elliptical cross section, so we describe the surface of the jet by the expression
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \] (1)

Of course, if the jet were circular with radius $a$, then $b = a$.

Can we get any indication of whether the jet were circular or elliptical using only our shadow photography measurements?

These measurements describe the projection $y_m(t)$ onto the $y$ (vertical) axis of a ray from the observer that passes through a droplet at position $(x_d(t), y_d(t))$.

An interesting question is whether the droplets leave the surface of the jet in a direction perpendicular to the surface, or at some other angle. Here, I assume that they leave perpendicularly, as shown above.

Suppose a droplet leaves the surface with velocity $v_0$ at time $t_0$ from point $(x_0, y_0)$. Then, at time $t > t_0$, it has traveled distance
\[ d = v_0(t - t_0), \] (2)
assuming that the velocity stays constant. The position of the drop is
\[ x_d = x_0 + d \sin \theta, \quad y_d = y_0 + d \cos \theta. \] (3)

The surface of the elliptical jet is
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{or} \quad x = \frac{a}{b} \sqrt{b^2 - y^2}, \quad \text{or} \quad y = \frac{b}{a} \sqrt{a^2 - x^2}. \] (4)
The tangent to the surface of the jet at the point \( (x_0, y_0) \) obeys

\[
\text{slope} = \frac{dx}{dy} = -\frac{a}{b} \frac{y_0}{x_0}.
\]  

(5)

The droplet moves perpendicular to the slope, so

\[
\tan \theta = -\frac{1}{\text{slope}} = -\frac{dy}{dx} = \frac{b x_0}{a y_0}, \quad \sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{x_0}{a}, \quad \cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{y_0}{b}.
\]  

(6)

Using these in eq. (3), we have

\[
x_d = (a + d) \sin \theta, \quad y_d = (b + d) \cos \theta,
\]  

(7)

When the drop is viewed via shadow photography from a point at distance \( D \) from the center of the jet, the position of the drop, \( y_m \), as projected onto the \( y \) axis is

\[
y_m = \frac{y_d D}{D - x_d} + c \approx y_d \left(1 + \frac{x_d}{D}\right) + c
\]

\[
\approx [b \cos \theta + c] + v_0(t - t_0) \cos \theta + \frac{[a + v_0(t - t_0)][b + v_0(t - t_0)]}{2D} \sin 2\theta,
\]  

(8)

where \( c \) is the \( y \)-coordinate of the center of the jet.

The apparent velocity of the droplet along the \( y \) axis is

\[
v_m = \frac{dy_m}{dt} \approx v_0 \left[\cos \theta + \frac{a + b + 2v_0(t - t_0)}{2D} \sin 2\theta\right].
\]  

(9)

If the droplet is moving towards the observer, \( 0 < \theta < 180^\circ \), then the apparent velocity \( v_m \) increasing slightly with time (if air resistance can be ignored). Do we have any evidence in our data for this small effect?

The earliest time \( t_{0m} \) that a droplet can be seen via shadow photography is when \( y_m \approx b \),\(^1\) so that

\[
t_{0m} \approx t_0 + \frac{b(1 - \cos \theta)}{v_0 \cos \theta} \approx t_0 + \frac{b(1 - v_m/v_0)}{v_m},
\]  

(10)

and

\[
v_m \approx \frac{v_0}{1 + v_0(t_{0m} - t_0)/b}.
\]  

(11)

In the first approximation, \( v_m \) depends on \( t_{0m} \) only through the height \( b \) of the jet.

An example of the relation between \( v_m \) and \( t_{0m} \) is shown in the figure on the next page.

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\(^1\)In case of a circular jet, the minimum measureable \( y_m \) is \( b/\sqrt{1 - b^2/D^2} \approx b(1 + b^2/2D^2) \). We ignore the second-order correction.
We now face a statistical question: Are the droplets distributed uniformly in angle $\theta$, or perhaps uniformly in angle $\phi$, or perhaps they are equally probable to be emitted from any point on the surface?

1. Uniform in $\theta$.

$$P(\theta)\,d\theta = \frac{d\theta}{2\pi}. \quad (12)$$

2. Uniform in $\phi$.

$$\tan \phi = \frac{x}{y}, \quad \sin \phi = \frac{x}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{y}{\sqrt{x^2 + y^2}}. \quad (13)$$

and from eq. (6),

$$x = a \sin \theta, \quad y = b \cos \theta. \quad (14)$$

After some algebra,

$$d\phi = \frac{ab}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta. \quad (15)$$

Hence,

$$P(\theta)\,d\theta = P(\phi)\,d\phi = \frac{d\phi}{2\pi} = \frac{ab}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \frac{d\theta}{2\pi}. \quad (16)$$

3. Uniform in position $s$ around the circumference $C$ of the ellipse.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta. \quad (17)$$

Hence,

$$P(\theta)\,d\theta = P(s)\,ds = \frac{ds}{C} \approx \frac{2\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}{3(a + b) - \sqrt{(3a + b)(a + 3b)}} \frac{d\theta}{2\pi}, \quad (18)$$

using Ramanujan’s (very good!) approximation for the circumference of an ellipse.